Aalto-yliopisto
Teknillinen korkeakoulu

Stochastic Programs Without Duality Gaps

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Outline

Introduction

- We study dynamic stochastic optimization problems which arise in many applications in operations research and mathematical finance.

Pennanen. *Convex duality in stochastic programming and mathematical finance*. Mathematics of Operations Research (to appear)
Pennanen, Perkkiö. *Stochastic Programs Without Duality Gaps*. 2011 (submitted)
Pennanen, Perkkiö. *Convex duality in stochastic optimization over processes of bounded variation*. (manuscript)
Pennanen, Perkkiö. *Stochastic problems of Bolza over predictable processes of bounded variation*. (manuscript)

Normal integrands

- Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space.
- A function $f : \mathbb{R}^n \times \Omega \to \overline{\mathbb{R}}$ is a **normal integrand** if $f(\cdot, \omega)$ is lower semicontinuous for all ω, and f is $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F}$-measurable.
- The **integral functional** $I_f : L^0(\Omega; \mathbb{R}^n) \to \overline{\mathbb{R}}$ defined by

$$I_f(x) = \mathbb{E} f(x(\omega), \omega)$$

  is well-defined, where the expectation is defined as $+\infty$ unless the positive part is integrable.
- If $f(\cdot, \omega)$ is convex, I_f is convex.
- Rockafellar, *Integral functionals, normal integrands and measurable selections*, 1976

Normal integrands

Examples of normal integrands

- **Carathéodory function**: $f(\cdot, \omega)$ is continuous almost surely and $f(x, \cdot)$ is measurable for all x.
- **Indicator function**:

$$f(x, \omega) = \begin{cases} 0 & \text{if } x \in D(\omega) \\ +\infty & \text{otherwise} \end{cases}$$

  of a closed measurable set D.
- $f = f_1 + f_2$ where f_1 and f_2 are normal integrands.

Stochastic optimization

For given integers n_t, let

$$\mathcal{N} = \{(x_t)_{t=0}^T \mid x_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^{n_t})\}$$

be the space of **strategies**. We consider the problem

$$\underset{x \in \mathcal{N}}{\text{minimize}} \quad \mathbb{E}[f(x(\omega), u(\omega), \omega)],$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \overline{\mathbb{R}}$ is a convex normal integrand, $n = n_0 + \ldots + n_T$ and $u \in L^0(\Omega, \mathcal{F}, \mathbb{R}^m)$.

The aim is to study the **value function**

$$\varphi(u) = \inf_{x \in \mathcal{N}} I_f(x, u).$$

Example: Superhedging in liquid markets

Let $S = (S_t)_{t=0}^T$ be an $\mathbb{R}^d$-valued adapted stochastic process, $n_t = d$, $m = 1$ and

$$f(x, u, \omega) = \begin{cases} 0 \quad \text{if} & \sum_{t=0}^{T-1} x_t \cdot \Delta S_t(\omega) \geq u, \\ +\infty & \text{otherwise.} \end{cases}$$

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Let $u \in L^0(\Omega, \mathcal{F}, P; \mathbb{R})$. Now

$$\varphi(u) = \inf_{x \in \mathcal{N}} I_f(x,u) = \begin{cases} 0 & \text{if } u \in \mathcal{C} \\ \infty & \text{otherwise,} \end{cases}$$

where $\mathcal{C} = \{u \mid \exists x \in \mathcal{N} : \sum_{t=0}^{T-1} x_t \cdot \Delta S_t \geq u \ P\text{-a.s.}\}$ is the set of **claim processes** which can be **hedged** with zero cost.

Example: Variance optimal hedging

Let $S = (S_t)_{t=0}^T$ be an $\mathbb{R}^d$-valued $\mathbb{F}$-adapted stochastic process, $u \in L^0(\Omega, \mathcal{F}, P; \mathbb{R})$ and

$$f(x, u, \omega) = (V_0 + \sum_{t=0}^{T-1} z_t \cdot \Delta S_{t+1}(\omega) - u)^2,$$

where $x_0 = (z_0, V_0)$, $x_t = z_t$ for $t = 1, \ldots, T$.

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$$\varphi(u) = \inf_{x \in \mathcal{N}} E(V_0 + \sum_{t=0}^{T-1} z_t \cdot \Delta S_{t+1} - u)^2,$$

where V_0 is as an initial value of a self-financing trading strategy and z_t is the portfolio of risky assets held over period $[t, t+1]$. Föllmer and Schied, *Stochastic Finance*, Section 10.3

Examples

- Superhedging in illiquid markets
- Optimal consumption in illiquid markets
- Liquidation problems in some market impact models.

More examples:

Pennanen. *Convex duality in stochastic programming and mathematical finance*. Mathematics of Operations Research (to appear)

Convex duality

- We want to derive dual expressions for

$$\varphi(u) = \inf_{x \in \mathcal{N}} I_f(x, u).$$

- Let $\mathcal{U} = L^p(\Omega; \mathbb{R}^m)$ and $\mathcal{Y} = L^q(\Omega; \mathbb{R}^m)$ where $1/p + 1/q = 1$, $1 \leq p \leq \infty$, and

$$\langle u, y \rangle = \mathbb{E} u(\omega) \cdot y(\omega).$$

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$$\langle u, y\rangle = \mathbb{E}u(\omega)\cdot y(\omega).$$

- The **conjugate** of φ on $\mathcal{Y}$ is the convex function defined by

$$\varphi^*(y) = \sup_{u\in\mathcal{U}}\{\langle u, y - \varphi(u)\}.$$

- If φ is convex, then $\varphi^{**} = \operatorname{cl}\varphi$, where

$$\operatorname{cl}\varphi = \begin{cases} \operatorname{lsc}\varphi & \text{if } (\operatorname{lsc}\varphi)(u) > -\infty\ \forall u\in\mathcal{U},\\ -\infty & \text{otherwise.}\end{cases}$$

Convex duality

- The **Lagrangian** associated with f is the extended real-valued function on $\mathcal{N} \times \mathcal{Y}$ defined by

$$L(x,y) = \inf_{u \in \mathcal{U}} \{ I_f(x,u) - \langle u, y \rangle \}.$$

  The Lagrangian is convex in x and concave in y.

- The **dual objective** is a concave function defined by

$$g(y) = \inf_{x \in \mathcal{N}} L(x,y).$$

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- We have $g = -\varphi^*$ which gives the **dual representation**

$$(\mathrm{cl}\,\varphi)(u) = \sup_{y \in \mathcal{Y}} \{ \langle u, y \rangle + g(y) \},$$

Convex duality

Why convex duality?

- Optimality conditions can be given in terms of the dual representation
- Duality techniques are used in numerical optimization
- Dual variables have interpretations: martingale measures, consistent price systems, shadow prices of information
  - Why martingale measures in mathematical finance?

Convex duality

Our framework differs from the duality framework of

R.T. Rockafellar, *Conjugate duality and optimization*, SIAM, 1974

by allowing a larger space of strategies (strategies are not restricted to be in a LCTVS)

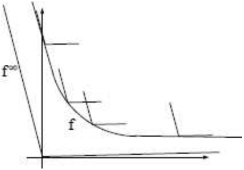
- Our extension allows to use certain techniques from mathematical finance to close the duality gap in some situations where topological arguments fail.
- We do not put emphasis on the existence of dual solutions in general. The mere absence of duality gap is behind some fundamental results in mathematical finance.

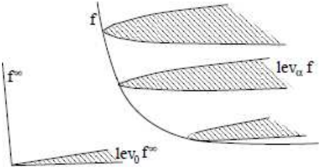
Duality result

Let $h : \mathbb{R}^n \times \Omega \to \overline{\mathbb{R}}$ be a convex normal integrand and assume $\operatorname{dom} h(\cdot, \omega)$ is nonempty, the **recession function** of h is

$$h^\infty(x, \omega) = \sup_{\lambda > 0} \frac{h(\lambda x + \bar{x}, \omega) - h(\bar{x}, \omega)}{\lambda},$$

which is independent of the choice of $\bar{x} \in \operatorname{dom} h(\cdot, \omega)$.





Duality result

Theorem

Assume that there is a $y \in \mathcal{Y}$ and an $m \in L^1(\Omega, \mathcal{F}, P)$ such that for P-almost every ω,

$$f(x, u, \omega) \geq u \cdot y(\omega) + m(\omega) \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m$$

and that $\{x \in \mathcal{N} \,|\, f^\infty(x(\omega), 0, \omega) \leq 0 \text{ a.s.}\}$ is a linear space. Then

$$\varphi(u) = \inf_{x \in \mathcal{N}} I_f(x, u)$$

is lower semicontinuous on $\mathcal{U}$ and the infimum is attained for every $u \in \mathcal{U}$.

Example: Superhedging in liquid markets

For

$$f(x,u,\omega) = \begin{cases} 0 & \text{if} \quad \sum_{t=0}^{T-1} x_t \cdot \Delta S_t(\omega) \geq u, \\ +\infty & \text{otherwise,} \end{cases}$$

we have

$$\{x \in \mathcal{N} \mid f^\infty(x(\omega), 0, \omega) \leq 0 \text{ a.s.}\} = \{x \in \mathcal{N} | \sum_{t=0}^{T-1} x_t \cdot \Delta S_t \geq 0\}.$$

Thus f satisfies the linearity condition of the theorem if and only if the price process S satisfies the *no-arbitrage* condition.

Example: Variance optimal hedging

For

$$f(x,u,\omega) = \left(V_0 + \sum_{t=0}^{T-1} z_t \cdot \Delta S_{t+1}(\omega) - u\right)^2$$

the linearity condition holds, since

$$\{x \in \mathcal{N} \mid f^\infty(x(\omega),0,\omega) \leq 0 \text{ a.s.}\} = \{x \in \mathcal{N} \mid V_0 + \sum_{t=0}^{T-1} z_t \cdot \Delta S_{t+1} = 0\}.$$

Thus the infimum in

$$\varphi(u) = \inf_{x \in \mathcal{N}} E\left(V_0 + \sum_{t=0}^{T-1} z_t \cdot \Delta S_{t+1} - u\right)^2$$

is attained for every $u \in \mathcal{U}$, and φ is lower semicontinuous.

Remarks on the duality result

- The proof uses a characterization of the linearity condition which is based on dynamic programming equations.
  - With certain assumptions the dynamic programming equations take the more familiar form of Bellman equations.
- $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$, where $\mathcal{U}$ is more general space than $L^p(\Omega; \mathbb{R}^n)$
  - Parameters and dual variabless can belong to more general LCTVS than Banach spaces.
  - The presented duality result extends to this setting provided that $\mathcal{U}$ is **decomposable**.

More general settings

- Let $f : X \times M \times \Omega \to \overline{\mathbb{R}}$, where X is the space of functions of bounded variation on $[0, T]$ and M is the space of Radon measures on $[0, T]$. Let $\mathcal{N}$ be the predictable processes of bounded variation and consider

$$\underset{x \in \mathcal{N}}{\text{minimize}} \quad \mathbb{E}[f(x(\omega), u(\omega), \omega)],$$

  where $u \in L^0(\Omega, \mathcal{F}; M)$.
  - Continuous times illiquid markets
  - Stochastic Lagrange variational problems

Pennanen, Perkkiö. *Convex duality in stochastic optimization over processes of bounded variation*. (manuscript)
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