Estimation of a Pseudo-Periodic Function Observed in the Stationary Noise

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Suppose we observe the random process $Y(t)$ on the fixed finite interval $[a, b]$:

$$dY(t) = s(t)dt + \varepsilon dW(t) \quad (1)$$

where s is the unknown function, $W(t)$ is the standard Wiener process.
The function $s \in L^2_{[a,b]}$ and

$$s = \sum_{j=1}^{\infty} \theta_j(s)\varphi_j$$

is its expansion in some orthonormal basis $\{\varphi_j,\ j = 1, 2, \ldots\}$ of the space $L^2_{[a,b]}$.

If $\widetilde{\theta}_j,\ 1 \le j \le N$ are some estimators of $\theta_j(s)$, constructed on observations (1) then the function

$$\widetilde{s} = \sum_{j=1}^{N} \widetilde{\theta}_j \varphi_j,$$

is the estimator of the unknown function s.
This method of estimation was initiated by N. N. Centsov (1962) with

$$\widetilde{\theta}_j = y_j = \int_a^b \overline{\varphi_j(t)}\, dY(t) = \theta_j(s) + \varepsilon \int_a^b \overline{\varphi_j(t)}\, dW(t)$$

The estimator $\widetilde{s}$ is unbiased for the function

$$s_N = \sum_{j=1}^{N} \theta_j(s)\varphi_j,$$

which is an orthoprojection of s on the subspace L_N, generated by the piece $\{\varphi_1, \ldots, \varphi_N\}$ of the orthonormal basis, and we have to neglect the value of

$$\varrho_N = \sum_{j=N+1}^{\infty} \theta_j(s)\varphi_j.$$

We indicate a compact subset $L \subset L^2_{[a,b]}$, the function s is supposed to belong to.
The quality of the estimation procedure given by an estimator $\widetilde{s}$, can be naturally measured by the value of quadratic risk

$$R\left(\widetilde{s}\,;L\right) = \sup_{s \in L} \mathbf{E} \left\|\widetilde{s} - s\right\|^2_{L^2_{[a,b]}} .$$

The minimax risk is

$$R^*\left(L\right) = \inf_{\widetilde{s}} R\left(\widetilde{s}\,;L\right) .$$

The equality

$$y_j = \theta_j(s) + \varepsilon x_j,\ x_j = \int_a^b \overline{\varphi_j(t)}\, dW(t),$$

turns our problem into the equivalent problem of estimation of the unknown vector

$$\theta = (\theta_1, \theta_2, \ldots) \in l_2$$

on the observations

$$y_j = \theta_j + \varepsilon x_j,\ j = 1, 2, \ldots, \tag{2}$$

where $x_1, \ldots, x_n, \ldots$ are iid $\mathcal{N}(0,1)$ and $\varepsilon > 0$ is known, $\theta \in \Theta$, where Θ is the compact subset of the space l_2.

In this problem

$$R\left(\widetilde{\theta}\,;\Theta\right) = \sup_{\theta\in\Theta} \mathbf{E}\left\|\widetilde{\theta}-\theta\right\|_{l_2}^2$$

is the risk of an estimator $\widetilde{\theta}$ of the vector $\theta\in\Theta$ and

$$R_*\left(\Theta\right) = \inf_{\widetilde{\theta}} R\left(\widetilde{\theta}\,;\Theta\right)$$

is the minimax risk.

We give the following example. Suppose the unknown function $s \in S(\beta, \tau, C) \subset L^2_{[-\pi/\tau, \pi/\tau]}$, $\beta > 1/2$, $\tau > 0$:

$$S = \left\{ g(t) = \sum_{-\infty<k<\infty} \widehat{g}_k e^{ik\tau t} : \sum_{-\infty<k<\infty} \left|\widehat{g}_k\right|^2 (1 + |\tau k|)^{2\beta} \leq C \right\}.$$

Ibragimov and Has'minskiy (1977, 1978) obtained the asymptotic behavior of the minimax risk $R^*(S(\beta, \tau, C))$ in this problem with $\varepsilon \to 0$ up to the order:

$$\lambda_2(\beta, \tau, C)\, \varepsilon^{\frac{2\beta}{2\beta+1}} \leq R^*(S(\beta, \tau, C)) \leq \lambda_1(\beta, \tau, C)\, \varepsilon^{\frac{2\beta}{2\beta+1}}. \quad (3)$$

In 1980 Pinsker obtained the exact asymptotic.

Suppose we observe the random process $Y(t)$ on the large interval $[-T, T]$:

$$dY(t) = s(t)\,dt + X(t)\,dt, \tag{4}$$

where $s \in \mathcal{L}_*(\Lambda)$ is an unknown function we wish to estimate, $X(t)$ is a generalized Gaussian stationary process with the zero mean and the spectral density f.

It means that we possess the random variables

$$Y[\varphi] = \int_{-\infty}^{\infty} s(t)\,\overline{\varphi(t)}\,dt + X[\varphi],\ \varphi \in \mathcal{D}(T),$$

where

$$\mathcal{D}(T) = \{\varphi : \varphi \in \mathcal{D},\ \textbf{supp}\ \varphi \subset [-T, T]\},$$

D is the space of the infinitely differentiable finite functions.

X is a linear operator from the space $\mathcal{D} = \mathcal{D}\left(R^1\right)$ to the Gaussian subset Γ of the space $L^2(dP)$ and

1. For the functional $\mathcal{R}\left(\varphi_1, \varphi_2\right) = \mathbf{E}\, X[\varphi_1]\overline{X[\varphi_2]}$ the equality

$$\mathcal{R}\left(\varphi_1, \varphi_2\right) = \mathcal{R}\left(\eta_t\varphi_1, \eta_t\varphi_2\right)$$

holds for any real t and any functions $\varphi_1, \varphi_2 \in \mathcal{D}$; η_t is a shift operator: $\eta_t\varphi(z) = \varphi(t+z)$.

2.

$$\mathbf{E}\, X[\varphi] = 0,\ \varphi \in \mathcal{D}.$$

3.

$$\mathcal{R}\left(\varphi_1, \varphi_2\right) = \mathbf{E}\, X[\varphi_1]\overline{X[\varphi_2]} = \int\limits_{-\infty}^{\infty} \widehat{\varphi_1}(u)\,\overline{\widehat{\varphi_2}\,(u)} f(u)\, du,\ \varphi_1, \varphi_2 \in \mathcal{D},$$

where $\widehat{\varphi}$ is the Fourier transform of φ.

The unknown function s is supposed to belong to the compact subset $\mathcal{L}_*(\Lambda) = \mathcal{L}_*(\Lambda,\, \beta,\, C)$ of the Banach space $\mathcal{L}(\Lambda)$ of the pseudo-periodic functions with the spectral set Λ, which means that

$$\mathcal{L}_*(\Lambda) = \left\{ s(t) = \sum_{u \in \Lambda} a_u\, e^{iut} \,:\, \sum_{u \in \Lambda} |a_u|^2 (|u| + 1)^{2\beta} \le C,\, \beta > 1/2 \right\}.$$

Λ is a countable set and

$$\tau = \inf_{u \neq v;\, u,v \in \Lambda} |u - v| > 0.$$

The corresponding Banach norm $\|\cdot\|_{\mathcal{L}}$:

$$\|s\|_{\mathcal{L}}^2 = \int_x^{x+1} |s(t)|^2 \, dt.$$

We assume at first that $f \in A_2(\mathcal{M})$. The class $A_2(\mathcal{M})$ consists of the nonnegative functions g for which

$$\sup_I \frac{1}{|I|} \int_I g(t)dt \frac{1}{|I|} \int_I \frac{1}{g(t)} dt \le \mathcal{M}. \tag{5}$$

Second $f \in B_\gamma$, where $B_\gamma = B_\gamma(c_1, c_2, T_0, \beta, \Lambda), \gamma > -1$ consists of the nonnegative functions $g = g(t)$ for which

$$0 < c_2 \le M^{-1-\gamma} \sum_{u \in \Lambda, |u| \le M} 2T \int_{|u-t| \le \frac{1}{T}} g(t)\, dt \le c_1 < \infty \tag{6}$$

for any $T > T_0$, $M = T^{\frac{1}{1+\gamma+2\beta}}$.

The quality of an estimator $\widetilde{s}$ of the unknown function s is measured by the value of the risk:

$$R\left(\widetilde{s},\, \mathcal{L}_*(\Lambda)\right) = R\left(\widetilde{s},\, \mathcal{L}_*(\Lambda)\,,\, T\right) = \sup_{s \in \mathcal{L}_*(\Lambda)} \mathbf{E} \left\| s - \widetilde{s} \right\|_{\mathcal{L}}^2 .$$

We define the minimax risk:

$$R^*\left(\mathcal{L}_*(\Lambda)\,,\, T\right) = \inf_{\widetilde{s}} R\left(\widetilde{s},\, \mathcal{L}_*(\Lambda)\,,\, T\right) .$$

Consider the function system $\{\varphi_u = \varphi_u(t) = e^{iut},\ u \in \Lambda\}$ and the conjugate system $\left\{\varphi_u^* = \varphi_{u,\,T}^*,\ u \in \Lambda\right\}$:

$$(\varphi_u^*, \varphi_v)_T = \frac{1}{2T} \int\limits_{-T}^{T} \varphi_u^*(t)\overline{\varphi_v(t)}\, dt = \delta_{uv},\ u,\ v \in \Lambda.$$

Let $r = \lfloor \beta \rfloor$.
Consider the even function $k(x) = k\,(T,\ x)$ which is $r+1$ times continuously differentiable, monotonically decreasing when $x \in [T-2,\ T-1]$ and

$$k(x) = \begin{cases} 1 & x \in [0,\ T-2] \\ 0 & x > T-1 \end{cases}$$

For some $\gamma > -1$ we now define the estimator $\widetilde{s}_\gamma^*$:

$$\widetilde{s}_\gamma^* = \sum_{u \in \Lambda,\, |u| \le M} y_u^k \varphi_u^*, \; M = T^{\frac{1}{1+\gamma+2\beta}}, \tag{7}$$

where

$$y_u^k = \frac{1}{2T} Y\left[k\varphi_u\right], \; u \in \Lambda.$$

Theorem

Consider the problem of estimation of the unknown function s defined by conditions (4)-(6).
1. There exists such constant
$C_1 = C_1\left(c_1,\, T_0,\, \gamma,\, \beta,\, \tau,\, \mathcal{M},\, C\right) < \infty$ *that the inequality*

$$R\left(\widetilde{s}_\gamma^{*},\, \mathcal{L}_*\left(\Lambda\right),\, T\right) < C_1\, T^{-\frac{2\beta}{2\beta+\gamma+1}} \tag{8}$$

holds for any $T > T_*\left(T_0,\, \tau\right)$*;*
2. There exists such constant
$C_2 = C_2\left(c_1,\, c_2,\, T_0,\, \gamma,\, \beta,\, \tau,\, \mathcal{M},\, C\right) > 0$*, that the inequality*

$$R^{*}\left(\mathcal{L}_*\left(\Lambda\right),\, T\right) > C_2\, T^{-\frac{2\beta}{2\beta+\gamma+1}} \tag{9}$$

holds for any $T > T_*\left(T_0,\, \tau\right)$*.*

Remark

Applying (8) and (9) we have firstly:

$$R^*\left(\mathcal{L}_*\left(\Lambda\right),\ T\right) < C_1\ T^{-\frac{2\beta}{2\beta+\gamma+1}}, \tag{10}$$

and secondly:

$$1 \leq \frac{R\left(\widetilde{s}_\gamma^*,\ \mathcal{L}_*\left(\Lambda\right),\ T\right)}{R^*\left(\mathcal{L}_*\left(\Lambda\right),\ T\right)} < C_3 < \infty.$$

Suppose the spectral density $f \equiv \sigma > 0$ and the set

$$\Lambda = \{\tau\, n : \; n \in \mathbb{Z}\} \, .$$

All conditions of the theorem hold and it is clear that $\gamma = 0$. That is why according to (9) and (10), for $T > T_*$

$$C_2\, T^{-\frac{2\beta}{2\beta+1}} < R^* \left(\mathcal{L}_* \left(\Lambda\right),\; T\right) < C_1\, T^{-\frac{2\beta}{2\beta+1}} .$$

The following lemma is a consequence of the Wiener and Paley (1934) results.

Lemma

There is such $T_1 = T_1(\tau)$ that for any $T > T_1(\tau)$ the following norms

$$\|s\|^2_{\mathcal{L}} = \sup_x \int_x^{x+1} |s(t)|^2 \, dt, \; \|s\|^2_T = \frac{1}{2T} \int_{-T}^{T} |s(t)|^2 \, dt,$$

$$\|s\|^2_2 = \sum_{u \in \Lambda} |a(u)|^2$$

given on $\mathcal{L}(\Lambda)$ are equivalent up to the constants which do not depend on T.

Applying this fact and the Bari's theorem we can easily obtain

Lemma

Let the norm $\|\cdot\|_T$ *is given on the set* $\mathcal{L}(\Lambda)$. *The function system* $\{\varphi_u(t) = e^{iut},\, u \in \Lambda\}$ *is the Riss' basis uniformly by* $T > T_1(\tau)$ *in* $\mathcal{L}(\Lambda)$. *There exists the conjugate function system*

$$\{\varphi_u^* :\; (\varphi_w^*, \varphi_v) = \delta_{wv},\, w,\, v \in \Lambda;\, u \in \Lambda\}$$

which is also the Riss' basis uniformly by $T > T_1(\tau)$ *in* $\mathcal{L}(\Lambda)$.

We denote $P_{\mathcal{L}(\Lambda)}$ the orthoprojector on the space $\mathcal{L}(\Lambda)$ in $L^2_{[-T,\,T]}-$metric and

$$b_u = \frac{1}{2T}\int\limits_{-T}^{T} k(T;t)\overline{\varphi_u(t)}s(t)\,dt,\ u \in \Lambda$$

the coefficients in the expansion of the function $P_{\mathcal{L}(\Lambda)}\{ks\}$ in the function system $\{\varphi_u^*,\ u \in \Lambda\}$.
Hence we have

$$y_u^k = b_u + \sigma_u \mathbf{x}_u,\ u \in \Lambda,$$

where $\mathbf{x}_u,\ u \in \Lambda$ are the normal standard random variables

$$\sigma_u = \frac{\left\|\widehat{k\varphi_u}\right\|_f}{2T} = \frac{1}{2T}\sqrt{\int\limits_{-\infty}^{\infty}\left|\widehat{k\varphi_u}(t)\right|^2 f(t)\,dt},\ u \in \Lambda.$$

Applying Lemma 1 we have for any $T > T_1(\tau)$

$$R\left(\widetilde{s}_\gamma^*, \mathcal{L}_*(\Lambda), T\right) = \sup_{s \in \mathcal{L}_*(\Lambda)} \mathbf{E} \left\| s - \widetilde{s}_\gamma^* \right\|_{\mathcal{L}}^2 \leq K_1(\tau) \sup_{s \in \mathcal{L}_*(\Lambda)} \mathbf{E} \left\| s - \widetilde{s}_\gamma^* \right\|_T^2 .$$

Fix any function $s = \sum_{u \in \Lambda} a_u \varphi_u \in \mathcal{L}_*(\Lambda)$.

We have

$$\mathbf{E} \left\| s - \widetilde{s}_\gamma^* \right\|_T^2 \leq 2 \left\| s - P_{\mathcal{L}(\Lambda)} ks \right\|_T^2 + 2\mathbf{E} \left\| P_{\mathcal{L}(\Lambda)} ks - \widetilde{s}_\gamma^* \right\|^2 \leq$$

$$\leq 2 \left\| s - ks \right\|_T^2 + 2\mathbf{E} \left\| P_{\mathcal{L}(\Lambda)} ks - \widetilde{s}_\gamma^* \right\|_T^2 .$$

According to the construction of the function k we have

$$\|ks - s\|_T^2 \leq \frac{2}{T} \sup_x \int_x^{x+1} |s(t)|^2 \, dt.$$

By Lemma 1 using condition $s \in \mathcal{L}_* (\Lambda)$ we have

$$\sup_x \int_x^{x+1} |s(t)|^2 \, dt \leq K_2(\tau) \sum_{u \in \Lambda} |a_u|^2 \leq C \, K_2(\tau).$$

Hence

$$\|ks - s\|_T^2 \leq K_3 \, (C, \, \tau) \, \frac{1}{T}$$

and to obtain (8) it is sufficient to prove the inequality

$$\mathbf{E} \left\| P_{\mathcal{L}(\Lambda)} ks - \widetilde{s}_\gamma^* \right\|_T^2 \leq K_4 \, T^{-\frac{2\beta}{2\beta+\gamma+1}}. \tag{11}$$

for any large enough T.

We have

$$\mathbf{E}\left\|P_{\mathcal{L}(\Lambda)}ks-\widetilde{s}_{\gamma}^{*}\right\|_{T}^{2}=\mathbf{E}\left\|\sum_{u\in\Lambda,\,|u|>M}b_u\varphi_u^{*}-\sum_{u\in\Lambda,\,|u|\le M}\sigma_u\mathbf{x}_u\varphi_u^{*}\right\|_{T}^{2}\le$$

$$\le 2\left\|\sum_{u\in\Lambda,\,|u|>M}b_u\varphi_u^{*}\right\|_{T}^{2}+2\mathbf{E}\left\|\sum_{u\in\Lambda,\,|u|\le M}\sigma_u\mathbf{x}_u\varphi_u^{*}\right\|_{T}^{2}\le$$

$$\le K_5(\tau)\sum_{u\in\Lambda,\,|u|>M}|b_u|^2+K_5(\tau)\sum_{u\in\Lambda,\,|u|\le M}|\sigma_u|^2.$$

The last inequality is valid by Lemma 2. To finish we need two more lemmas.

Lemma

There exists such constant $K_6 = K_6(\tau,\ \beta)$ *that for any function* $s = \sum_{u\in\Lambda} a_u\,\varphi_u \in \mathcal{L}_*(\Lambda)$ *we have*

$$\sum_{u\in\Lambda} |b_u|^2 (1+|u|)^{2\beta} \le K_6 \sum_{u\in\Lambda} |a_u|^2 (1+|u|)^{2\beta}.$$

The coefficients $b_u,\ u\in\Lambda$ *are the same as above.*

Lemma

Let the function $f \in A_2(\mathcal{M})$. *Then for any* $T > 2$ *an any* u

$$L_2 \int\limits_{|t-u|\le 1/T} f(t)dt \le \frac{1}{T^2} \int\limits_{-\infty}^{\infty} \left|\widehat{k\varphi_u}(t)\right|^2 f(t)dt \le L_1 \int\limits_{|t-u|\le 1/T} f(t)dt,$$

where $0 < L_2(\mathcal{M}) \le L_1(\mathcal{M}) < \infty$.

By Lemma 3 we have

$$\sum_{u\in\Lambda,\,|u|>M} |b_u|^2 \le M^{-2\beta} \sum_{u\in\Lambda,\,|u|>M} |b_u|^2 \left(|u|+1\right)^{2\beta} \le$$

$$\le K_6 \sum_{u\in\Lambda} |a_u|^2 \left(1+|u|\right)^{2\beta} \le C\,K_6 M^{-2\beta},$$

and as $M = T^{-\frac{2\beta}{1+\gamma+2\beta}}$ we have for any large enough T

$$\sum_{u\in\Lambda,\,|u|>M} |b_u|^2 \le C\,K_6\,T^{-\frac{2\beta}{1+\gamma+2\beta}}. \tag{12}$$

Further, by Lemma 4

$$\sum_{u\in\Lambda,\,|u|\le M} |\sigma_u|^2 \le K_7\,(\mathcal{M}) \sum_{u\in\Lambda,\,|u|\le M} \int_{|t-u|\le\frac{1}{T}} f(t)\,dt.$$

As $f\in B_\gamma$, for any $T>T_0$ we have
$M^{2\beta} \sum_{u\in\Lambda,\,|u|\le M} \int_{|t-u|\le\frac{1}{T}} f(t)\,dt \le c_1$, and therefore

$$\sum_{u\in\Lambda,\,|u|\le M} |\sigma_u|^2 \le K_8\,T^{-\frac{2\beta}{1+\gamma+2\beta}}. \tag{13}$$

(12) and (13) imply (8).

The first step of the proof of (9) is to pass from the problem of estimation of the function $s \in \mathcal{L}_* (\Lambda)$ to the problem of estimation of the vector which belongs to the correspondent compact subset of l_2. As we denoted above $y_u^k = \frac{1}{2T} Y [k\, \varphi_u]$ for any $u \in \Lambda$. So we can write

$$y_u^k = b_u + \sigma_u \mathbf{x}_u, \; u \in \Lambda. \tag{14}$$

It can be shown that there exists such constant $D_1 = D_1 (\tau, C)$ that for any $T > T_2 (\tau)$ the following inequality holds:

$$R^* (\mathcal{L}_* (\Lambda), T) \geq D_1 R_*(A_{C^*}, \sigma), \; \sigma = (\sigma_u)_{u\in\Lambda} .$$

Here $R_*(A_{C^*}, \sigma)$ is the minimax risk in the problem of estimation of the vector $(b_u)_{u\in\Lambda} \in A_{C^*}$ on observations (14), the set

$$A_{C^*} = \left\{ (b_u)_{u\in\Lambda} : \sum_{u\in\Lambda} |b_u|^2 (|u| + 1)^{2\beta} \leq C^* \right\}, \; C^* = C^* (C, \tau, \beta)$$

The random variables $\mathbf{x}_u$, $u \in \Lambda$ participating in (14) are not independent. That is why the next step is to pass to the problem of estimation of the vector $(b_u)_{u \in \Lambda} \in A_{C^*}$ on the same observations with the additional condition of independence of the random variables $\mathbf{x}_u$, $u \in \Lambda$.

For this purpose it can be shown that the equality

$$(\mathbf{x}_u,\, \mathbf{x}_v) = \left(\frac{\widehat{k\varphi_u}}{\left\| \widehat{k\varphi_u} \right\|_f}, \frac{\widehat{k\varphi_v}}{\left\| \widehat{k\varphi_v} \right\|_f} \right)_f$$

which is valid for any $u,\ v \in \Lambda$ and condition (5) imply the inequality

$$\inf_{a_v,\, v\in\Lambda,\, v\neq u} \mathbf{E} \left(\mathbf{x}_u - \sum_{v\in\Lambda,\, v\neq u,\, |v|\leq N} a_v \mathbf{x}_v \right)^2 \geq \varrho(\mathcal{M}) > 0$$

for any $u \in \Lambda$ and any positive N. By this inequality it can be shown that

$$R_*(A_{C^*},\, \sigma) \geq D_2\,(\mathcal{M})\, R_*^0(A_{C^*},\, \sigma)$$

where $R_*^0(A_{C^*},\, \sigma)$ is the risk in the same problem with the additional condition of independence of the random variables $\mathbf{x}_u,\ u \in \Lambda$.

For this risk the following estimate can be obtained:

$$R_*^0(A_{C^*}, \sigma) \geq D_3(\tau, \mathcal{M}, C, \beta) \sum_{u \in \Lambda, |u| \leq M} \sigma_u^2$$

which together with Lemma 4 implies (9).

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Thanks

Thank you for your attention!