# AMERICAN OPTION PRICING IN STOCHASTIC VOLATILITY MODELS

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We consider the Heston model when the stock dynamics is given by a system of stochastic equations

$$dS(\vartheta) = S(\vartheta)[rd\vartheta + \sqrt{v(\vartheta)}d\hat{w}_1(\vartheta)], \quad S(t) = s > 0, \tag{1}$$

$$dv(\vartheta) = \kappa_1(\theta_1 - v(\vartheta))d\vartheta + \sigma_1\sqrt{v(\vartheta)}dw_2(\vartheta), \quad v(t) = v > 0. \tag{2}$$

Here $\hat{w}_1(\vartheta)$, $w_2(\vartheta) \in R^1$ are $\mathcal{F}_t$-measurable Wiener processes, defined on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_t$, $E[d\hat{w}_1(\vartheta)dw_2(\vartheta)] = \rho d\vartheta$, while $r$, $\kappa_1$, $\theta_1$ $\sigma_1$ and correlation coefficient $\rho$ are given constants, $|\rho| < 1$.
For simplicity we assume that $P$ is a martingale measure.

The price $F(t, s, v)$ of an American option with a contract function $\Phi(s)$ is given by

$$F(t, s, v) = \sup_{\tau \in \mathcal{T}_{[t,T]}} E_{t,s,v}\{e^{-r(\tau - t)}\Phi(S(\tau))\}, \tag{3}$$

where $\mathcal{T}_{[t,T]}$ is a set of stopping time $\tau \in [t, T]$ with respect to $\mathcal{F}_t$.
In particular a contract function for a put option has the form $\Phi(s) = [K - s]_+ = \max(K - s, 0)$, constant $K > 0$.
An alternative definition

$$F(t, s, v) = \sup_{S^*(\tau,v), \tau \in [t,T]} E_{t,s,v}\{e^{-r(\tau_{S^*} - t)}[K - S(\tau_{S^*})]_+\}, \tag{4}$$

where $\tau_{S^*}$ is the first exit time when the process $S$ hits the optimal execution boundary $S^*(\vartheta, v)$, $\vartheta \in [t, T]$ allows to include explicitly an unknown function $S^*(\vartheta, v)$ to be defined in a process of solution.

This explains the possibility to describe $F(t, s, v)$ for a put option as a solution of a free boundary value problem for a parabolic equation

$$F = K - s, \quad \frac{\partial F}{\partial t} + \mathcal{L}_1 F - rF < 0, \quad 0 \leq s \leq S^*(t, v), \qquad (5)$$

$$F > K - s, \quad \frac{\partial F}{\partial t} + \mathcal{L}_1 F - rF = 0, \quad S^*(t, v) < s < \infty, \qquad (6)$$

where $\mathcal{L}_1$ is a linear operator acting as follows

$$\mathcal{L}_1 F = rs\frac{\partial F}{\partial s} + \kappa_1(\theta_1 - v)\frac{\partial F}{\partial v} + \frac{1}{2}s^2 v\frac{\partial^2 F}{\partial s^2} + \frac{1}{2}\sigma_1^2 v\frac{\partial^2 F}{\partial v^2} + \rho s\sigma_1 v\frac{\partial^2 F}{\partial s \partial v}. \qquad (7)$$

Boundary conditions at $s = S^*(t, v)$ are stated as a continuity condition for $F(t, s, v)$ and its derivative in $s$ and

$$F(t, S^*(t, v), v) = [K - S^*(t, v)]_+, \quad \frac{\partial F}{\partial s}(t, S^*(t, v), v) = -1. \qquad (8)$$

A substitution

$$s = Ke^{x+\alpha y}, \quad Kf(t,x,y) = F(t,s,v), \quad K\varphi(x,y) = \Phi(s), \quad (9)$$

leading to dimensionless variables allows to reduce the problem (5)-(6) to a problem with constant (in $x$) coefficients and eliminate mixed derivatives.

**Lemma.** *Assume that stochastic processes* $S(\vartheta)$, $v(\vartheta)$ $(0 \le t \le \vartheta \le T)$ *are governed by the system*

$$dS(\vartheta) = S(\vartheta)[rd\vartheta + \sqrt{v(\vartheta)}d\hat{w}_1(\vartheta)], \quad S(t) = s, \qquad (10)$$

$$dv(\vartheta) = \kappa_1(\theta_1 - v(\vartheta))d\vartheta + \sigma_1\sqrt{v(\vartheta)}dw_2(\vartheta), \quad v(t) = v, \qquad (11)$$

*where* $\hat{w}_1(t)$, $w_2(t)$ *are correlated Wiener processes and* $E[d\hat{w}_1(t)dw_2(t)] = \rho dt$. *Then processes* $X(\vartheta)$, $y(\vartheta)$ *defined by* $S(\vartheta) = \exp(X(\vartheta) + \alpha y(\vartheta))$, $v(\vartheta) = (1 + \alpha^2\sigma^2)y(\vartheta)$ *satisfy stochastic equations*

$$dX(\vartheta) = a(y(\vartheta))d\vartheta + \sqrt{y(\vartheta)}dw_1(\vartheta), \quad X(t) = x, \qquad (12)$$

$$dy(\vartheta) = \kappa(\theta - y(\vartheta))d\vartheta + \sigma\sqrt{y(\vartheta)}dw_2(\vartheta), \quad y(t) = y, \qquad (13)$$

*where* $w_1(t), w_2(t)$ *are independent Wiener processes,*

$$a(y) = r - \alpha\kappa(\theta - y) - \frac{y}{2(1-\rho^2)}, \qquad (14)$$

$$\alpha = \frac{\rho}{\sigma\sqrt{1-\rho^2}}, \quad \sigma = \sigma_1\sqrt{1-\rho^2}, \quad \kappa = \kappa_1, \quad \theta = \theta_1(1-\rho^2). \qquad (15)$$

The above considerations show that (5) and (6) are reduced to

$$\frac{\partial f}{\partial t} + \mathcal{L}f - rf = 0, \quad (t,x,y) \in \mathcal{C}, \tag{16}$$

$$\frac{\partial f}{\partial t} + \mathcal{L}f - rf < 0, \quad (t,x,y) \in \mathcal{E}, \tag{17}$$

where $\mathcal{C}, \mathcal{E}$ are given by

$$\begin{cases} f(t,x,y) = 1 - e^{x+\alpha y}, & \text{if } h(t,y) \geq x, \text{ that is } x \in \mathcal{E}, \\ f(t,x,y) > 1 - e^{x+\alpha y}, & \text{if } h(t,y) < x, \text{ that is } x \in \mathcal{C}, \end{cases} \tag{18}$$

$h(t,y)$ is an unknown function to be defined in the process of solution the problem and $Ke^{h(\vartheta,y)+\alpha y(\vartheta)} = S^*(\vartheta, v)$.

$$\mathcal{L}f = a(y)\frac{\partial f}{\partial x} + \frac{y}{2}\frac{\partial^2 f}{\partial x^2} + \kappa(\theta - y)\frac{\partial f}{\partial y} + \frac{\sigma^2 y}{2}\frac{\partial^2 f}{\partial y^2}, \tag{19}$$

and $a(y)$ is given by (14).

# Perpetual American put option

A price $f(x, y)$ of a perpetual American put option with a contract function $\varphi(x, y)$ in dimensionless coordinates is determined by a relation

$$f(x, y) = \sup_{\tau \in \mathcal{T}} E_{x,y}[e^{-r\tau}\varphi(x(\tau), y(\tau))]. \tag{20}$$

In this case the function $f(t, x, y) \equiv f(x, y)$ does not depend on $t$ and satisfies the boundary problem

$$(\mathcal{L} - r)f(x, y) = 0, \quad x > h(y), \tag{21}$$

$$f(x, y) = (1 - e^{x+\alpha y})_+, \quad x \leq h(y), \tag{22}$$

$$f(h, y) = 1 - e^{h+\alpha y}, \quad \frac{\partial}{\partial x} f(x, y)|_{x=h} = -e^{h+\alpha y}, \tag{23}$$

where operator $\mathcal{L}$ has the form (19).

After discretization in $y$-space we reduce equation (21) to a free boundary value problem for a system of parabolic equations

$$\frac{1}{2}y_j\frac{\partial^2}{\partial x^2}f_j(x)+a(y_j)\frac{\partial}{\partial x}f_j(x)-q_j(x)f_j(x)+\sum_{k\neq j}\lambda_{jk}f_k(x)=0,\quad x>h_j, \tag{24}$$

$$f_j(x)=(1-e^{x+\alpha y_j}),\quad x\leq h_j. \tag{25}$$

where $q_j=r+\Lambda_j$, $\Lambda_j=\sum_{k\neq j}\lambda_{jk}$, or

$$(q_j-L_j)f_j=\sum_{k\neq j}\lambda_{jk}f_k(x), \tag{26}$$

$$L_j=a(y_j)\partial_x+\frac{y_j}{2}\partial_x^2, \tag{27}$$

$$\lambda_{jk}=0,\ \text{if } |j-k|>1, \tag{28}$$

$$\lambda_{jk} = \frac{\sigma^2}{8\Delta_v^2} + \frac{1}{\Delta_v}\left[\left(\frac{\kappa\theta}{2} - \frac{\sigma^2}{8}\right)\frac{1}{\sqrt{y}_j} - \frac{\kappa}{2}\sqrt{y_j}\right]_+, \quad \text{if } k = j+1, \tag{29}$$

$$\lambda_{jk} = \frac{\sigma^2}{8\Delta_v^2} + \frac{1}{\Delta_v}\left[-\left(\frac{\kappa\theta}{2} - \frac{\sigma^2}{8}\right)\frac{1}{y_j} + \frac{\kappa}{2}\sqrt{y_j}\right]_+, \quad \text{if} \quad k = j-1. \tag{30}$$

with boundary condition

$$f_j(x) = (1 - e^{x+\alpha y_j}), \quad x \leq h_j. \tag{31}$$

Introduce a function $\tilde{f}_j = f_j - G_j$, $G_j(x) = 1 - e^{x+\alpha y_j}$
We construct a solution of (26) by a system of successive approximations.
Set $\tilde{f}_k^0(x) = 0$ and for $n = 1, 2, \dots$ $\tilde{f}_j^n(x)$ satisfy equations

$$(q_j - L_j)\tilde{f}_j^n(x) = F_j^{n-1}(x), \quad x > h_j^n, \tag{32}$$

$$\tilde{f}_j^n(x) = 0, \quad x \le h_j^n, \tag{33}$$

where

$$F_j^{n-1}(x) = \sum_{k \ne j} \lambda_{jk} \tilde{f}_k^{n-1}(x) + \tilde{g}_j(x) \tag{34}$$

and
$\tilde{g}_j(x) = \sum_{k \ne j} \lambda_{jk} G_k(x) - (q_j - L_j)G_j(x) = \sum_{k \ne j} \lambda_{jk} G_k - g_j(x)$
To obtain numerical results for the price of the option we apply a method based on the Wiener-Hopf factorization developed in [1]-[4].

**Theorem.** *For $n = 1, 2, \ldots$*
*1) the functions $F_j^{n-1}(x) = \sum_{k \neq j} \lambda_{jk} \tilde{f}_k^{n-1}(x) + \tilde{g}_j$,*
*$v_j^{n-1}(x) = \mathcal{E}_j^+ F_j^{n-1}(x)$ are non-decreasing and have a unique zero at the point $x = h_j^n$, hence $h_j^n$ is a root of the equation*
*$v_j^{n-1}(x) = 0$,*
*2) $\tau_j$ is an optimal stopping time,*
*3) $\tilde{f}_j^n(x) = q_j^{-1} \mathcal{E}_j^- I_{(h_j^n, \infty)} v_j^{n-1}(x)$,*
*4) $f_j^n(x) = \tilde{f}_j^n(x) + G_j$,*
*5) the function $\tilde{f}_j^n(x)$ is non-dereasing and vanishes when $x < h_j^n$,*
*where*

$$\mathcal{E}_j^+ u(x) = \beta_j^+ \int_0^\infty e^{-\beta_j^+ y} u(x+y) dy,$$

$$\mathcal{E}_j^- u(x) = -\beta_j^- \int_{-\infty}^0 e^{-\beta_j^- y} u(x+y) dy,$$

*$\beta_j^+$, $\beta_j^-$ are positive and negative roots of the characteristic equation*

$$\frac{y_j}{2} \beta^2 + a(y_j) \beta - q_j = 0.$$

**Algorithm.** Calculate

$$m_{0,j}(x) = \mathcal{E}_j^+(g_j(x)) = q_j(\mathcal{E}_j^-)^{-1}G_j(x) = q_j\left(1 - \frac{\beta_j^- - 1}{\beta_j^-}e^{x+\alpha y_j}\right).$$

Set $f_j^0(x) = G_j(x)$. Then for $n = 1, 2, \ldots$
1) moving from $x_1 = x_{\max}$ down with the chosen step $\Delta_x$, calculate the values of fuctions

$m_{1,j}^n(x) = \mathcal{E}_j^+\left(\sum_{k\neq j}\lambda_{jk}f_k^{n-1}(x)\right)$ and
$v_j^n(x) = m_{1,j}^n(x) - m_{0,j}(x)$.

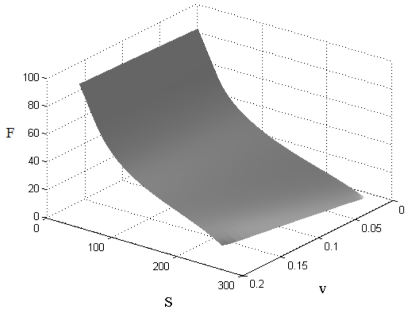
2) functions $v_j^n(x)$ are increasing, we stop calculation as soon as $v_j^{n-1}(x) < 0$, or $x_i < x_{\min}$, and set $h_j^n = x_{i-1}$, $i_j^- = i - 1$.
3) moving from $x_{i_j^-} = h_j^n$ up with step $\Delta_x$ calculate the values of functions

$$f_{0,j}^n(x) = q_j^{-1}\mathcal{E}_j^- I_{(-\infty,h_j^n]}m_{0,j}(x) = e^{\beta_j^-(x-h_j^n)}(1 - e^{\alpha y_j}e^{h_j^n}),$$

$$f_{1,j}^n(x) = q_j^{-1}\mathcal{E}_j^- I_{(h_j^n,\infty)}m_{1,j}^n(x),$$

$$f_j^n(x) = f_{1,j}^n(x) + f_{0,j}^n(x) \text{ when } x \geq x_{i_j^-} = h_j^n,$$

and $f_j^n(x) = G_j(x)$ for $x \leq x_{i_j^-} = h_j^n$.



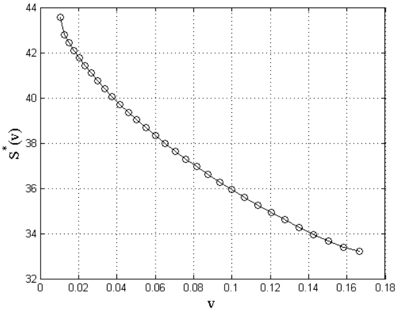




Figure: Perpetual American put option price $F(s, v)$ and optimal stopping boundary $S^*(v)$ in Heston model. Parameters $K = 100$, $\delta = 0,1$, $r = 0,05$, $\kappa_1 = 2$, $\theta_1 = 0,03$, $\sigma_1 = 0,2$, $\rho = -0,2$.

## American put option with finite maturity time

Then we consider American put option with finite maturity

$$\partial_t f(t,x,y) + (\mathcal{L} - r) f(t,x,y) = 0, \quad x > h(t,y), \tag{35}$$

$$f(t,x,y) > 1 - e^{x+\alpha y}, \quad x > h(t,y), \tag{36}$$

$$\partial_t f(t,x,y) + \mathcal{L} f(t,x,y) - r f(t,x,y) \le 0, \quad x \le h(t,y), \tag{37}$$

$$f(t,x,y) = (1 - e^{x+\alpha y})_+, \quad x \le h(t,y), \tag{38}$$

$$f(t,h,y) = 1 - e^{h+\alpha y}, \quad \frac{\partial}{\partial x} f(t,x,y)|_{x=h(t,y)} = -e^{h+\alpha y}, \tag{39}$$

where operator $\mathcal{L}$ has the form (19).
Given a maturity date $T$ we divide the period $[0,T]$ into $N$ subperiods $0 \le t_0 < t_1 < \cdots < t_k < \cdots < t_N = T$ and apply the Carr randomization technique.

We use the time derivative $\partial_t f$ approximation by $\frac{f(t_{k+1},x,y)-f(t_k,x,y)}{\Delta_k}$ and apply the same discretization in $y$-space as in the case of a perpetual American option.
As a result the discretized form of (35)-(38) has the form

$$\frac{f_j^{k+1}(x)-f_j^k(x)}{\Delta_k}+\frac{1}{2}y_j\frac{\partial^2}{\partial x^2}f_j^n(x)+a(y_j)\frac{\partial}{\partial x}f_j^n(x)- \tag{40}$$

$$-q_j(x)f_j^n(x)+\sum_{k\neq j}\lambda_{jk}f_k^n(x)=0,\quad x>h_j^n,$$

$$f_j^n(x)=(1-e^{x+\alpha y_j}),\quad x\le h_j^n,\quad n<N, \tag{41}$$

$$f_j^N(x)=(1-e^{x+\alpha y_j})_+,\quad n=N. \tag{42}$$

Introduce $q_j^k=\Delta_k^{-1}+q_j$ and $G_j(x)=1-e^{\alpha y_j}e^x$ then

$$(q_j^k-L_j)f_j^k(x)=\Delta_k^{-1}f_j^{k+1}(x)+\sum_{j'\neq j}\lambda_{jj'}f_{j'}^k(x),\quad x>h_j^k, \tag{43}$$

$$f_j^k(x)=G_j(x),\quad x\le h_j^k, \tag{44}$$

and next we proceed as in the perpetual case.

**Theorem.** *Under some additional assumptions let for $k = N$ and for some $j$ a set of functions $\tilde{f}^k_{l*}$ with $j \neq l$ be given. Then for the same $j$ and $k = N-1$ we have:*
*1) function $u^k_j = \sum_{j' \neq j} \lambda_{jj'} f^k_{j*} + \Delta^{-1}_k f^{k+1}_{j*} - (q^k_j - L_j)G_j$ is a non-decreasing and*

$$u^k_j(-\infty) < 0 < u^k_j(\infty) = \infty, \tag{45}$$

*2) function $\tilde{m}^k_j = \mathcal{E}^{k+}_j u^k_j$ is an increasing continuous function and satisfies (45),*
*3) equation $\tilde{m}^k_j(x) = 0$ has a unique zero at the point $h^k_{j*}$,*
*4) the hitting time of $(-\infty, h^k_{j*}]$, $\tau^-_{h^k_{j*}}$ is the optimal stopping time for the process $x_j(t)$,*
*5) Carr's approximation to the option price $f^k_j$ in the state $j$ is given by $f^k_{j*} = (q^k_j)^{-1}\mathcal{E}^{k-}_j I_{(h^k_{j*},\infty)}\tilde{m}^k_j + G_j = \tilde{f}^k_{j*} + G_j$*
*6) function $\tilde{f}^k_{j*} = f^k_{j*} - G_j$ is a positive non-decreasing function and satifies (45). In addition $\tilde{f}^k_{j*}(\infty) = \infty$, $\tilde{f}^k_{j*}(x) = 0$ when $x < h^k_{j*}$ and $\tilde{f}^k_{j*}$ increases on $[h^k_{j*}, \infty)$.*

**Algorithm.** Denote by $f^k_{j*}, h^k_{j*}$ option price and optimal stopping boundary at $t_k$. At $k = N$ we set $f^N_{j*}(x) = 1 - e^{\alpha y_j} e^x$ and $h^N_{j*} = -\alpha y_j$. For a fixed $k < N$ calculate

$$m^k_{0j} = -\mathcal{E}^{k+}_j (q^k_j - \mathcal{L}_j) G_j$$

and set $f^k_{0,j} = G_j$, $h^k_{0,j} = h^N_{j*}$. Then for $n = 1, 2, \dots$
1) moving from $x_1 = x_{\max}$ down with the chosen step $\Delta_x$ calculate the values of functions

$$m^k_{j,n} = \mathcal{E}^{k+}_j \left( \sum_{j \neq j'} \lambda_{jj'} f^k_{j,n-1} + \Delta_k^{-1} f^{k+1}_{j*} \right),$$

$$\tilde{m}^k_{j,n} = m^k_{j,n} + m^k_{0,j},$$

2) functions $\tilde{m}^k_{j,n}$ are increasing, we stop calculation as soon as $\tilde{m}^k_{j,n}(x_i) < 0$, or $x_i < x_{\min}$ and set $h^k_{j,n} = x_{i-1}$, $i^-_j = i - 1$.

3) moving from $x_{i_j^-} = h^k_{j,n}$ up with step $\Delta_x$ calculate the values of functions

$$f^k_{0jn} = (q^k_j)^{-1}\mathcal{E}^{k-}_j I_{(-\infty,h^k_{j,n}]}(-m^k_{0j}),$$

$$f^k_{1jn} = (q^k_j)^{-1}\mathcal{E}^{k-}_j I_{(h^k_{j,n},\infty)} m^k_{j,n},$$

$$f^k_{j,n} = f^k_{1j,n} + f^k_{0j,n}$$

and $f^k_{j,n}(x) = G_j(x)$ for $x \leq x_{i_j^-} = h^k_{j,n}$.

American put option price with finite maturity time $T = 0,5$ with parameters:
$r = 0,09,\ \rho = -0,2,\ \kappa_1 = 1,58,\ \theta_1 = 0,03,\ \sigma_1 = 0,2,\ K = 100$

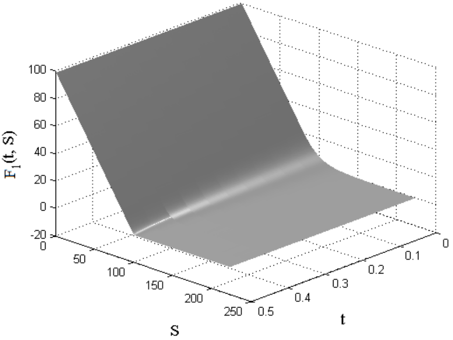



Figure: American put option price $F_1(t,s) = F(t,s,v)|_{v=0.09}$

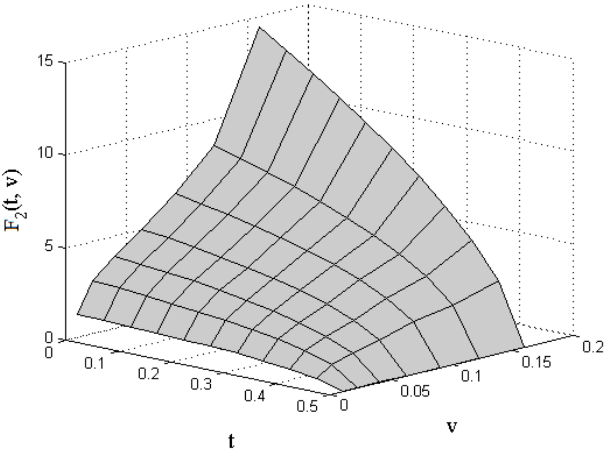



Figure: American put option price $F_2(t, v) = F(t, s, v)|_{s=100}$

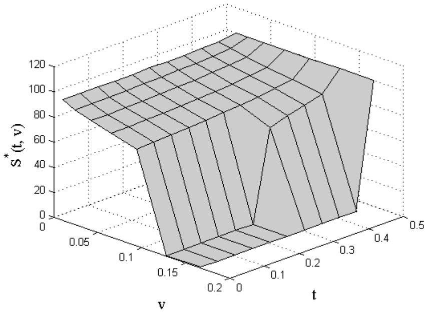



Figure: Free boundary $S^*(t,v)$

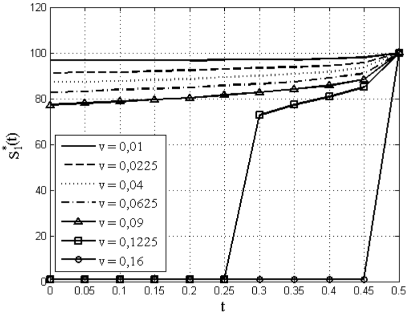



Figure: Free boundary $S_1^*(t)$ with different $v$

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