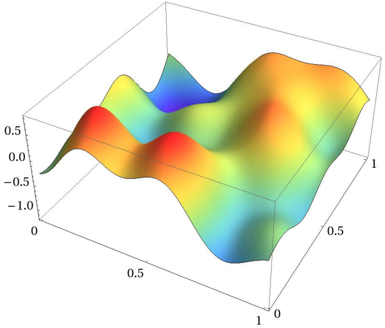

Extrapolation of α-stable random fields

Elena Shmileva

Joint Work with E.SPODAREV, W.KARCHER, ULM University

Contents

Set up

Consider $Y(x), x \in \mathbf{R}^d$ a real-valued random field on $\mathbf{R}^d$.

Let $\{x_i\}_{i=1}^N$, $x_i \in \mathbf{R}^d$, $i = 1..N$ be a set of distinguished locations.

Assume that $\{Y(x_i)\}_{i=1}^N$ are known/ estimated from some given data set.

Problem: to find a linear predictor of Y at arbitrary location $x \in \mathbf{R}^d$, i.e. $Y(x)$. We are looking for this predictor in the following linear form:

$$\widehat{Y}(x) = \sum_{i=1}^{N} \lambda_i(x) Y(x_i), \ \forall \ x \in \mathbf{R}^d.$$

Exactness:

$$Y(x_i) = \widehat{Y}(x_i), \ \forall \ i = 1..N,$$

this hold if and only if

$$\lambda_i(x_i) = 1, \ \forall \ i = 1..N, \quad and \quad \lambda_i(x_j) = 0, \ \forall \ i \neq j$$

The problem is to find the set of functions $\{\lambda_i(x), x \in \mathbf{R}^d\}_{i=1}^N$ s.t. the predictor $\{\widehat{Y}(x), x \in \mathbf{R}^d\}$ is "in some sense close" to the random field $\{Y(x), x \in \mathbf{R}^d\}$.

Construction of predictors depends on the properties of the r.f. Y that we have/assume.

If $Y : \mathbf{E}(Y(x))^2 < \infty \quad \rightarrow \quad$ **Kriging.**

Kriging

Master's thesis on Mining Engineering by D.G.Krige, 1951

Mean Square Error minimization:

for each $x \in \mathbf{R}^d$ to find $\lambda = (\lambda_1, ...\lambda_N)$ such that

$$\mathbf{E}(Y(x) - \sum_{i=1}^{N} \lambda_i(x) Y(x_i))^2 \to \min_{\lambda=(\lambda_1,...\lambda_N)} \tag{1}$$

Under conditions:

1. $\mathbf{E}Y(x) = 0$ for all $x \in \mathbf{R}^d$
2. Weak stationarity $Cov(Y(x), Y(s)) = C(\|x - s\|)$, where $\| \cdot \|$ - Euclidian norm in $\mathbf{R}^d$.

(1) $\longleftrightarrow$ system of N linear equations:

$$\sum_{i=1}^{N} \lambda_i(x) Cov(Y(x_i), Y(x_j)) = Cov(Y(x), Y(x_j)), \; j = 1..N$$

Cov. matrix - positive definite, so solution exists and is unique.

Also, $\lambda_i(x_i) = 1, \; \forall \; i = 1..N$, and $\lambda_i(x_j) = 0, \; \forall \; i \neq j$, so this is interpolation.

Extrapolation of α-stable r. fields, $\alpha \neq 2$

If $\{Y(x_i)\}_{i=1}^N$ is too scattered, $\mathbf{E}(Y(x))^2 = \infty$, then the field Y should have heavy tailed marginals.

What kind of heavy tailed distributions do we know?

X - α-stable, $\alpha \in (0, 2)$.
Properties:

1. $P\{|X| > x\} = C \cdot x^{-\alpha}(1 + o(1)),\ as\ x \to \infty$
2. $\mathbf{E}|X|^p < \infty$, $p \in (0, \alpha)$

α-stable r.vector $X = (X_1, ..., X_N)$ is uniquely determined by $(\Gamma(ds), \mu)$, where $\Gamma(ds)$ is a finite measure on the unite sphere $S^{N-1} \subset \mathbf{R}^N$, $\mu \in \mathbf{R}^N$.

$\Gamma(ds)$ - spectral measure
μ - shift parameter

Characteristic function of r.v. X:
$\varphi(z) = \exp\{-\int_{S^{N-1}} |(z,s)|^\alpha (1 - i \cdot sign(z,s) \cdot \tan \pi\alpha/2)\Gamma(ds)$
$+i(z,\mu)\}$
for all $z \in \mathbf{R}^N$, $\alpha \neq 1$.

Examples:

1. Let $N = 2$. Consider $X = (X_1, X_2)$
   X_1 independent of $X_2 \iff \Gamma(ds)$ is concentrated on the intersection with the axis
2. X_1 and X_2 linearly dependent $\iff$ *supp*$(\Gamma(ds))$ is subset of intersection of S^{N-1} and a hyper-plane
3. $X = (X_1, ...X_N)$ is symmetric ($S\alpha S$) $\iff \Gamma(ds)$ is symmetric and $\mu = 0$

NO covariance and NO correlation coefficient.

How to measure dependence?

Covariation (Samorodnitsky, Taqqu, 1994):

Let (X_1, X_2) be $S\alpha S$ and $\alpha \in (1, 2)$

$$[X_1, X_2]_\alpha = \int_{S^1} s_1 s_2^{<\alpha-1>} \Gamma(ds),$$

where $a^{<p>} := -|a|^p sign(a)$.

Why this is a good measure of dependence?

Properties of Covariation:

1. X_1, X_2 are independent, then $[X_1, X_2]_\alpha = 0$
2. Let (X_1, X_2, Y) SαS, then
   $[X_1 + X_2, Y]_\alpha = [X_1, Y]_\alpha + [X_2, Y]_\alpha$, but it is not true for the second argument.
3. $[X, Y]_\alpha \neq [Y, X]_\alpha$
4. for $p \in (1, \alpha)$

$$[X, Y]_\alpha = \frac{\sigma_Y^\alpha}{\mathbf{E}|Y|^p} \mathbf{E} X Y^{<p-1>},$$

here $\sigma_Y^\alpha = \int_{S^1} |s_2|^\alpha \Gamma(ds)$ scale parameter of Y
So, Covariation is connected with mixed moments, i.e. with dependence structure.

LSL (Least Scale Linear) estimator

What kind of extrapolation methods can we propose?
Let Y be a SαS field. Still $\widehat{Y}(x) = \sum_{i=1}^{N} \lambda_i(x) Y(x_i), \ \forall\, x \in \mathbf{R}^d$.

$L_p, \ p \in (1, \alpha)$ optimization:

$$\mathbf{E}| Y(x) - \sum_{i=1}^{N} \lambda_i(x) Y(x_i)|^p \to \min_{\lambda=(\lambda_1,\dots\lambda_N)} \tag{2}$$

Lemma: $Y(x) - \widehat{Y}(x)$ is SαS again.

Lemma: $(\mathbf{E}|Y|^p)^{1/p} = c_\alpha(p) \cdot \sigma_Y$

So, (2) $\longleftrightarrow$

$$\sigma_{Y(x)-\widehat{Y}(x)} \to \min_{\lambda=(\lambda_1,\dots\lambda_N)}$$

$$\frac{d\sigma_{Y(x)-\widehat{Y}(x)}}{d\lambda_i} = \alpha[Y(x_i), Y(x) - \widehat{Y}(x)]_\alpha$$

then

$$\left[Y(x_i), Y(x) - \sum_{j=1}^{N} \lambda_j(x) Y(x_j) \right]_\alpha = 0, \; i = 1..N \tag{3}$$

no additivity by the second argument, so system of nonlinear equations

If there is an integral representation of Y, then (3) is rewritten via Kernel function and can be solved numerically.

The solution exists and unique.
Exactness is evident.

COL (Covariation Orthogonal Linear) estimator

Let's exchange arguments

$$[Y(x), Y(x_i)]_\alpha - \sum_{j=1}^{N} \lambda_j(x) \left[Y(x_j), Y(x_i)\right]_\alpha = 0, \; i = 1..N$$

system of N linear equations. Simple to find a solution. Exactness is evident.

In many cases, for example for Gaussian, sub-Gaussian stable fields, COL and LSL coincide.

MCL (Maximization of Covariation Linear) estimator

$$\textit{for each } x \in \mathbf{R}^N \quad [\widehat{Y}(x), Y(x)]_\alpha \to max_{\lambda=(\lambda_1,\ldots\lambda_N)}$$

Optimization problem with a side condition:

$$\sum_{i=1}^{N} \lambda_i(x)[Y(x_i), Y(x)]_\alpha \to max_{\lambda=(\lambda_1,\ldots\lambda_N)}$$

$$\sigma_{\widehat{Y}(x)} = \sigma_{Y(x)}$$

Lagrange function:

$$L(\lambda,\gamma) = \sum_{i=1}^{N} \lambda_i [Y(x_i), Y(x)]_\alpha + \gamma(\sigma_{\widehat{Y}(x)} - \sigma_{Y(x)})$$

taking derivatives

$$[Y(x_i), Y(x)]_\alpha + \gamma \left[Y(x_i), \sum_{j=1}^{N} \lambda_j(x) Y(x_j) \right]_\alpha = 0$$

Putting $x = x_i$ and $\gamma = -1$, we get exactness.

There is uniqueness under some mild conditions.

Questions

- How to define $[\cdot,\cdot]_\alpha$ for $\alpha \in (0,1]$?
- How to define $[\cdot,\cdot]_\alpha$ for arbitrary α-stable r.v.'s with shift?
- How to compare all the methods?

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Thank you for your attention!!!