Discrete complex analysis on a convex quadrangulation of the plane

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3d Northern Triangular Seminar,
St. Petersburg, April 11, 2011

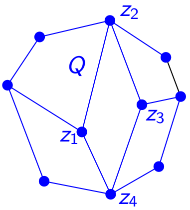
Overview

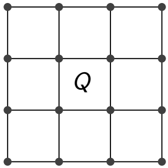
Discrete analytic functions

A graph $Q \subset \mathbb{C}$ is a *quadrilateral lattice* $\Leftrightarrow$ each bounded face is a convex quadrilateral.
A function $f\colon Q \to \mathbb{C}$ is *discrete analytic* $\Leftrightarrow$

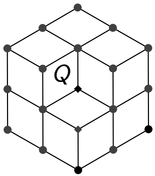
$$\frac{f(z_1)-f(z_3)}{z_1-z_3} = \frac{f(z_2)-f(z_4)}{z_2-z_4}$$

for each face $z_1 z_2 z_3 z_4$ with the vertices listed clockwise.

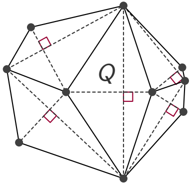




square lattice
Isaacs (1940s)



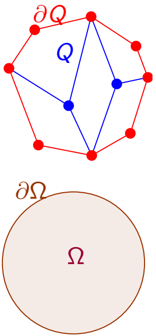
rhombic lattice
Duffin (1960s)



orthogonal lattice
Mercat (2000s)

The Dirichlet problem

The *boundary* ∂Q is the boundary of the outer face of Q (a closed curve hereafter). A *discrete harmonic* function $f\colon Q \to \mathbb{R}$ is the real part of a discrete analytic function. The *Dirichlet problem* on Q is to find a discrete harmonic function $f_{Q,u}\colon Q \to \mathbb{R}$ having given boundary values $u\colon \partial Q \to \mathbb{R}$. The *Dirichlet problem* in a domain Ω is to find a continuous function $f_{\Omega,u}\colon \mathrm{Cl}\Omega \to \mathbb{R}$ having given boundary values $u\colon \partial\Omega \to \mathbb{R}$ and such that $\Delta f_{\Omega,u} = 0$ in Ω.



Uniqueness Theorem (S., 2011). *The Dirichlet problem on any finite quadrilateral lattice has a unique solution.*

Convergence theorem

A sequence $\{Q_n\}$ is *nondegenerate* $\Leftrightarrow \exists\,\mathrm{const} > 0$:

- Angle(side of a face, diagonal of a face)$> \mathrm{const}$,
- $\dfrac{\mathrm{MinSize}}{\mathrm{MaxSize}} := \dfrac{\text{minimal edge length}}{\text{maximal edge length}} > \mathrm{const}.$

Convergence Theorem (S., 2011). *Let $\Omega \subset \mathbb{C}$ be a domain bounded by a smooth closed curve $\partial\Omega$ without self-intersections. Let $u\colon \mathbb{C} \to \mathbb{R}$ be a smooth function. Take a nondegenerate sequence of finite orthogonal lattices $\{Q_n\} \subset \mathrm{Cl}\Omega$ such that $\mathrm{MaxSize}(Q_n)$, $\mathrm{Dist}(\partial Q_n, \partial\Omega) \to 0$.* ***Then the solution $f_{Q_n,u}\colon Q_n^0 \to \mathbb{R}$ of the Dirichlet problem on Q_n converges to the solution $f_{\Omega,u}\colon \Omega \to \mathbb{R}$ of the Dirichlet problem in Ω*** *uniformly on compact sets.*

Remark. *Square lattices*: Courant–Friedrichs–Lewy, 1926.
Rhombic lattices: Chelkak–Smirnov, 2008.

Energy

The *energy* of a function $u\colon \Omega \to \mathbb{R}$ is $E_\Omega(u) := \int_\Omega |\nabla u|^2 dA$.
Denote $u_k := u(z_k)$ and $c_{13} := i\frac{z_2-z_4}{z_1-z_3}$.
The *energy* of a function $u\colon Q \to \mathbb{R}$ is

$$E(u) := \sum \frac{|c_{13}|^2 (u_1 - u_3)^2 - 2\mathrm{Im}\, c_{13}(u_1 - u_3)(u_2 - u_4) + (u_2 - u_4)^2}{\mathrm{Re}\, c_{13}},$$

where the sum is over all the faces $z_1 z_2 z_3 z_4$ with the vertices listed clockwise.

Lemma. *A discrete harmonic function has minimal energy among all the functions with the same boundary values.*

Idea of the proof. A function u is discrete harmonic $\Leftrightarrow$ $\Delta^Q u = 0$ in $Q - \partial Q$, where $(\Delta^Q u)(z_k) := \partial E(u)/\partial u_k$.

Corollary: Uniqueness Theorem.

Scaling limit

B_n/W_n := *black*/*white* vertices of the bipartite graph Q_n.

Lemma 1. *The energies $E(f_{Q_n,u})$ are bounded.*

Lemma 2. *In any compact set $K \subset \Omega$ the sequence $f_{Q_n,u} \colon B_n \cap K \to \mathbb{R}$ is uniformly bounded and equicontinuous.*

Lemma 3. *Some subsequence of the sequence $f_{Q_n,u} \colon B_n \to \mathbb{R}$ converges to a harmonic function $f \colon \Omega \to \mathbb{R}$ uniformly on compact sets.*

Lemma 4. *The limit function $f \colon \Omega \to \mathbb{R}$ is the solution of the Dirichlet problem on Ω.*

Proofs. Lemmas 1, 3, 4: via methods of Duffin, Chelkak–Smirnov, Courant–Friedrichs–Lewy, respectively.
Lemma 2: new approach, known methods do not work.

Corollary: Convergence Theorem.

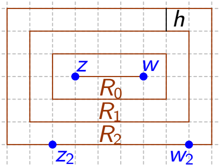
Equicontinuity

Strong Lemma 2. *Fix a compact set $K \subset \Omega$. Then*

$$|f_{Q_n,u}(z) - f_{Q_n,u}(w)| = \mathrm{O}\left(\ln^{-1/2}\left(1 + |z-w|^{-1}\right)\right)$$

uniformly with respect to any $z, w \in K \cap B_n$.

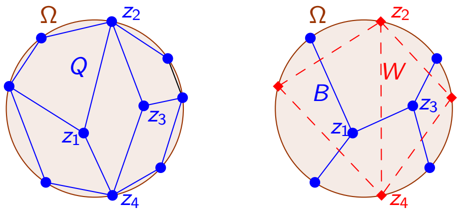
Proof (for a square lattice).
$h := \mathrm{MaxSize}(B_n)$, $\quad b := \mathrm{Dist}(K, \partial\Omega)$,
$R_m :=$ rectangle $2mh \times (2mh + |z-w|)$,
$f := f_{Q_n,u}$, assume $f(z) \geq f(w)$ wlog.



$m \leq b/4h \Rightarrow R_m \subset \mathrm{Int}\, Q_n$ for large $n \Rightarrow$
$\exists z_m, w_m \in \partial R_m : f(z_m) \geq f(z), f(w_m) \leq f(w) \Rightarrow$

$$E(f) \geq \sum_{m=0}^{[b/4h]} \frac{|f(z_m) - f(w_m)|^2}{8m + 2|z-w|/h} \geq \frac{1}{8}|f(z) - f(w)|^2 \ln\left(1 + \frac{b}{|z-w|}\right).$$

Physical interpretation



Physical interpretation (gives insight, not used formally):

- The graph B is an *alternating-current network* in a natural way
- *Admittance* $c(z_1z_3) := i\frac{z_2-z_4}{z_1-z_3} \Rightarrow \mathrm{Re}\, c(z_1z_3) > 0$
- *Voltage* $V(z_1z_3) := f(z_1) - f(z_3)$
- *Current* $I(z_1z_3) := if(z_2) - if(z_4)$
- *Energy* $E(f) := \mathrm{Re} \sum_{z_1z_3} V(z_1z_3)\bar{I}(z_1z_3)$.

Meaning of main results

Theorem 1 $\Leftrightarrow$ *Boundary voltage drops at the initial moment and boundary currents after one quarter of the period uniquely determine all the voltage drops and currents in an alternating-current network at all the moments of time.*

Question. What is the physical meaning of the boundary condition, which is **the Dirichlet one** at the initial moment and **the Neuman one** after one quarter of the period?

Theorem 2 $\Rightarrow$ *Distributed direct-current networks can be approximated by lumped direct-current networks.*

Question. What is the physical meaning of approximation of distributed **direct**-current networks by lumped **alternating**-current networks?

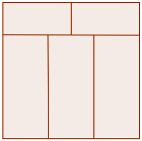
Application of alternating-current networks

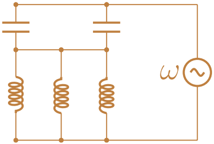
Theorem (Freiling–Laczkovich–Rinne–Szekeres, 1995).
For a number $c > 0$ the following conditions are equivalent:

- *a square can be tiled by similar rectangles of side ratio c;*
- *the number c is algebraic and all its algebraic conjugates have positive real parts.*

Physical proof (Prasolov–S., 2009).

tiling $\leftrightarrow$ LC network $\leftrightarrow$ admittance $A(\omega)$ $\leftrightarrow$ restriction





- $A(\omega/i) \in \mathbb{Q}(\omega)$
- $\mathrm{Re}(\omega/i) > 0 \Leftrightarrow \mathrm{Re}\, A(\omega/i) > 0$
- $A(c/i) = 1$

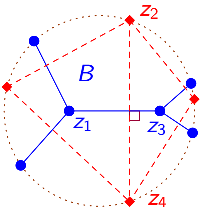
conjugates of c have positive real parts

$\uparrow$

Foster's reactance theorem

Probabilistic interpretation

A *random walk* on an orthogonal lattice is a random walk on the graph B with transition probabilities proportional to admittances $c(z_1 z_3) := i\frac{z_2 - z_4}{z_1 - z_3} > 0$.



Problem. Prove that the trajectories of a loop-erased random walk on an orthogonal lattice converge to SLE_2 curves in the scaling limit.

Remark. Rhombic lattices: Chelkak–Smirnov, 2008.

Generalizations

Problem. Generalize Convergence Theorem to:

1. **nonorthogonal** quadrilateral lattices;
2. sequences of lattices with **unbounded** ratio of maximal and minimal edge lengths (to involve *adaptive meshes* for computer science applications);
3. **discontinuous** boundary values (for convergence of discrete harmonic measure, the Green function, the Cauchy and the Poisson kernels);
4. domains with **rough** boundaries (for probabilistic applications);
5. **mixed** boundary conditions;
6. **infinite** lattices and unbounded domains;
7. **other** elliptic PDE.

Acknowledgements

THANKS!