How to hedge Asian options in fractional Black-Scholes model

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Outline of the talk

1. Introduction
2. Main results
3. Conclusions

1. Introduction

- Asian options
- Fractional Brownian motion
- Fractional Black-Scholes
- Pathwise stochastic integration
- Hedging problem
- Case of European options

Introduction

- Change of variables formulas [Itô formulas] for both arithmetic and geometric averages of geometric fractional Brownian motion.
- Valid for all convex functions, not only for smooth ones.
- Can be used for obtaining hedges (but not prices) for Asian options in fractional Black-Scholes model.
- Explicit hedges in some cases where hedges are not known explicitely even in the ordinary Black-Scholes model.

Asian options

Let $S(t)$ be the price of the underlying asset. Asian options depend on the time average of the underlying.

The payoff of the arithmetic Asian option is

$$f\left(\frac{1}{T}\int_0^T S(s)ds\right)$$

and the payoff of the geometric Asian option

$$f\left(\exp\left(\frac{1}{T}\int_0^T \log S(s)ds\right)\right).$$

Asian options (continued)

- Arithmetic Asian options are important in practise.
  - Used for example in commodity markets.
- Geometric Asian options are easier to consider analytically e.g. in ordinary Black-Scholes model.
- The problem with arithmetic Asian options is that sum of lognormals is not lognormal.
- Here we overcome this problem by using pathwise methods.

Fractional Brownian motion

- Fractional Brownian motion (fBM) B^H with Hurst index $H \in (0,1)$ is a Gaussian process satisfying

$$\mathbb{E}B^H(t) = B^H(0)$$

and having the following covariance structure

$$\mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

- If $H = \frac{1}{2}$, we are in the case of ordinary BM.

FBM (continued)

- For $H > \frac{1}{2}$ the process has long range dependence property and for $H < \frac{1}{2}$ the increments are negatively correlated.
- FBM is self-similar with parameter H.
- FBM is not semi-martingale nor Markov process (unless $H = \frac{1}{2}$).
- B^H has Hölder continuous sample paths of any order $\delta \in (0, H)$.
- For $H > \frac{1}{2}$, fBM B^H has zero quadratic variation over a sequence of subdivisions where the mesh goes to zero.

Fractional Black-Scholes

- The price of the underlying is modeled as $S(t) = \exp B^H(t)$, where $\frac{1}{2} < H < 1$.
- S has Hölder continuous sample paths of any order $\delta \in (0, H)$.
- S has zero quadratic variation property.
- The hedging results remain true if we add any deterministic drift to B^H as long as the path properties (Hölder continuity and quadratic variation) are not changed.

Pathwise integration

The stochastic integrals considered here are pathwise:

- Riemann-Stieltjes integrals (RS)
- generalized Lebesgue-Stieltjes integrals (gLS)

If not mentioned otherwise the integrals are gLS.

Hedging problem

- Given claim $F(S) = f((S(s)|s \in [0, T]))$.
- Find adapted $(\mathcal{H}(s))_{s\in[0,T]}$ such that $F(S) = A + \int_0^T \mathcal{H}(s)dS(s)$.
  - Integral should be economically justified.
- Separate problem of mathematical finance compared to pricing.

Case of European options

Theorem

Let f be convex function. Then

$$f(S(T)) = f(S(0)) + \int_0^T f'_-(S(t))S(t)dB^H(t)$$

in generalized Lebesgue-Stieltjes sense.

- Note that $dS(t) = S(t)dB^H(t)$.
- (Azmoodeh-Mishura-Valkeila 2009).

2. Main results

- Replication of the averages
- Options depending on geometric average
- Options depending on arithmetic average
- Fractional Bachelier model

Replication of geometric average

$$G(t) = \exp\left(\frac{1}{T}\int_0^t \log S(s)ds\right) S(t)^{\frac{T-t}{T}}.$$

Proposition

For all $t \in [0, T]$ it holds almost surely that

$$G(t) = S(0) + \int_0^t \frac{T-s}{T} G(s) dB^H(s),$$

in Riemann-Stieltjes sense.

Replication of geometric average (continued)

Corollary

In particular

$$
\begin{aligned}
&\exp\left(\frac{1}{T}\int_0^T B^H(s)ds\right)\\
=&S(0)+\int_0^T \frac{T-s}{T}\exp\left(\frac{1}{T}\int_0^s B^H(u)du+\frac{T-s}{T}B^H(s)\right)dB^H(s).
\end{aligned}
$$

Replication of arithmetic average

Proposition

For all $t \in [0, T]$ it holds almost surely that

$$\frac{T-t}{T} S(t) + \frac{1}{T} \int_0^t S(s) ds = S(0) + \int_0^t \frac{T-s}{T} S(s) dB^H(s),$$

in Riemann-Stieltjes sense.

Corollary

In particular

$$\frac{1}{T} \int_0^T S(s) ds = S(0) + \int_0^T \frac{T-s}{T} S(s) dB^H(s).$$

Geometric Asian options

$$G(t) = \exp\left(\frac{1}{T}\int_0^t B^H(s)ds\right) S(t)^{\frac{T-t}{T}}.$$

Theorem

Let f be a convex function. Then it holds almost surely that

$$f\left(G(t)\right) = f(S(0)) + \int_0^t \frac{T-s}{T} f'_-\left(G(s)\right) G(s) dB^H(s),$$

where the stochastic integral in the right side is understood in generalized Lebesgue-Stieltjes sense.

Geometric Asian options (continued)

Corollary

In particular,

$$f\left(\exp\left(\frac{1}{T}\int_0^T B^H(s)ds\right)\right)$$
$$= f(S(0)) + \int_0^T \frac{T-s}{T} f'_-\left(G(s)\right) G(s) dB^H(s).$$

Arithmetic Asian options

Theorem

Let f be a convex function. Then it holds almost surely that

$$
\begin{aligned}
& f\left(\frac{T-t}{T} S(t) + \frac{1}{T}\int_0^t S(s)ds\right) \\
=& f(S(0)) + \int_0^t f'_-\left(\frac{T-s}{T} S(s) + \frac{1}{T}\int_0^s S(u)du\right)\frac{T-s}{T} S(s) dB^H(s),
\end{aligned}
$$

where the stochastic integral in the right side is understood in the sense of generalized Lebesgue-Stieltjes integral.

Arithmetic Asian options (continued)

Corollary

In particular,

$$
\begin{aligned}
& f\left(\frac{1}{T}\int_0^T S(s)ds\right) \\
=& f(S(0)) + \int_0^T f'_-\left(\frac{T-s}{T}S(s) + \frac{1}{T}\int_0^s S(u)du\right)\frac{T-s}{T}S(s)dB^H(s).
\end{aligned}
$$

Fractional Bachelier model

The case of arithmetic average can be written also when the geometric price process S is replaced by a fractional Brownian motion B^H with $H \in \left(\frac{1}{2}, 1\right)$. In that case we obtain for a convex function f that

$$
\begin{aligned}
& f\left(\frac{T-t}{T} B^H(t) + \frac{1}{T}\int_0^t B^H(s)ds\right) \\
=& f(B^H(0)) + \int_0^t \frac{T-s}{T} f'_-\left(\frac{T-s}{T} B^H(s) + \frac{1}{T}\int_0^s B^H(u)du\right) dB^H(s)
\end{aligned}
$$

almost surely as a generalized Lebesgue-Stieltjes integral.

3. Conclusions

- Extended the functional Itô formula of (Cont-Fournié 2010) for non-smooth convex functions in the special case of driving gfBm or fBm and functional depending on the average of the driving process.
- Obtained hedging strategies for Asian options in fractional Black-Scholes model. We were able to find hedges also for Asian options depending on the ordinary arithmetic average. Explicit hedges for such options are not known even in the case of Black-Scholes model.

Conclusions (continued)

- In the case of Asian options, fBm behaves as continuous function of bounded variation. However, this is not the case for all path-dependent options: see for example the case of lookback options (Azmoodeh-Tikanmäki-Valkeila 2010).
- Some related models behave differently. For example in exponential mixed Brownian motion and fractional Brownian motion market model the hedges of Asian options are the same as in ordinary Black-Scholes model, (Bender-Sottinen-Valkeila 2008).

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Thanks for your attention!