On the variance of sample size

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Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with discrete distribution. We consider the set $\mathcal{S}_n$ of the first n samples and let the r.v. $K_n = |\mathcal{S}_n|$ be its size.
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Since values of X_j's are of no importance for us, without loss of generality we may arrange them so that $\mathbb{P}[X_j = x_i] = p_i$ and

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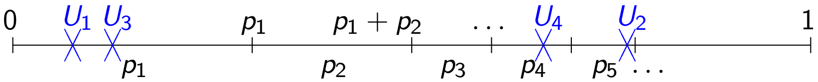
One can also admit that the underlying distribution has a continuous part, but all samples from continuous distribution are different a.s. and it is simple to analyze. So we always suppose that the distribution of X_j's is purely discrete and its support is infinite. In this case $K_n \to \infty$ as $n \to \infty$, so it is possible (and interesting) to investigate its behaviour in the limit.

It is convenient to model this construction by a sequence of the i.i.d. random variables uniformly distributed on the unit interval $[0, 1]$ divided into subintervals of lengths $p_1, p_2, \dots$:



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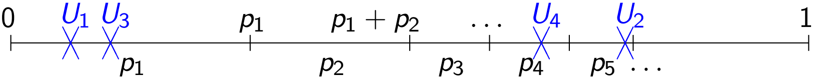

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- $M_{i,n}$—the number of samples that get into i-th subinterval among the first n samples;
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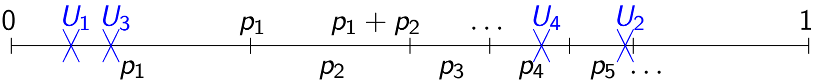

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Obviously,

$$\sum_i M_{i,n} = n, \qquad \sum_i I_{i,n} = K_n.$$

Poissonization is a standard procedure in such problems. It can be described in several alternative but equivalent ways:

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We use indices for the fixed n version and brackets notation for the Poissonized version (K_n vs $K(n)$ etc).

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- No need for the normalization $\sum_i p_i = 1$: by a linear time change one can renormalize this sum, so just its finiteness is needed.
- Additive structure:
  - (p_i') and (p_i'')—two sequences with finite sums;
  - $(p_i) = (p_i') \cup (p_i'')$—union as multisets;
  - $K'(n)$, $K''(n)$ and $K(n)$—the corresponding numbers of different samples in the Poissonized settings

  then

$$K(n) \stackrel{d}{=} K'(n) + K''(n), \qquad K'(n),\ K''(n) \text{ independent.}$$

We are interested in the mean and variance of the number of different values in the first n samples. Let us introduce

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The formulas become particularly simple after Poissonization:

$$\Phi(n) := \mathbb{E}[K(n)] = \sum\nolimits_i (1 - e^{-np_i}),$$
$$V(n) := \mathrm{Var}\, K(n) = \sum\nolimits_i (e^{-np_i} - e^{-2np_i}) = \Phi(n) - \Phi(2n).$$

We shall not use formulas for Φ_n and V_n but one can write them down.

Since there are infinitely many possible values, $K_n \to \infty$ a.s. as $n \to \infty$, and so does its mean: $\Phi_n \to \infty$. It is also known that $K_n/\mathbb{E}[K_n] \to 1$ as $n \to \infty$ in probability (Bahadur, 1960) and even a.s. (Karlin, 1967).

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The next question is whether the variance V_n increase to infinity or not. This question is particularly interesting because it is known that if $V_n \to \infty$, $n \to \infty$, then $\frac{K_n - \Phi_n}{\sqrt{V_n}} \Rightarrow \mathcal{N}$, the standard normal distribution (Karlin, 1967; Dutko, 1984; LLT by Hwang and Janson, 2006).

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This is easy to see for the Poissonized version of the problem. Then $K(n)$ is a sum of independent r.v.'s with Bernoulli distributions and the asymptotic normality follows, say, by application of Lindeberg's theorem. De-Poissonization requires some work which was done by Dutko in 1984.

So the interesting case is when V_n does not tend to ∞. Two alternatives are possible: either V_n oscillates unboundedly or it remains bounded as $n \to \infty$. Introduce

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We propose the following criterion for boundedness of V_n:

Theorem 1. The boundedness of $V(t)$ is equivalent to the existence of an integer k such that

$$\limsup_j \frac{p_{j+k}}{p_j} \leq \frac{1}{2}.$$

Moreover, this inequality implies $\overline{v} \leq k$. If for any k

$$\liminf_j \frac{p_{j+k}}{p_j} \geq \frac{1}{2}$$

then $\underline{v} = \infty$.

If V_n remains bounded, it is interesting whether it converges to a limit as $n \to \infty$. It can be also checked in terms of "lagged ratio":

Theorem 2. The limit $\lim_n V_n = v$ exists if and only if

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This has an unexpected corollary:

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This can be extended by considering the case when the distribution has just finitely many atoms. In this (not very interesting) case $K_n \to \mathrm{const}$ a.s. so its variance converges to zero.

Let p_j form the geometric sequence with the common ratio $1/2$, that is $p_j = 1/2^j$.
Then the Poissonized variance can be calculated as follows:

$$V(t) = \lim_{m\to\infty} \sum_{j=1}^{m} (e^{-t/2^j} - e^{-2t/2^j}) = \lim_{m\to\infty} -e^{-2t/2^1} + e^{-t/2^m} = 1 - e^{-t}$$

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Hence $V(t) \to 1$ as $t \to \infty$ in accordance with Thm. 2 because the lagged ratio $p_{j+1}/p_j = 1/2$.

Let the common ratio be $1/4$, that is $p_j = 3/4^j$.
Then the Poissonized variance can be calculated as follows:

$$V(t) = \lim_{m\to\infty} v_m(t); \qquad v_m(t) = \sum_{j=1}^{m} (e^{-3t/4^j} - e^{-6t/4^j}).$$

Partial sums $v_m(t)$ satisfy the recursion

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Take t and $m = m(t)$ sufficiently large, then $e^{-6t/4^{m+1}} \simeq 1$, e^{-3t} is small. If $t_j = \frac{4^j}{3}\ln 2$ then j-th summand $(e^{-3t/4^j} - e^{-6t/4^j}) = 1/4$ is maximal. Summation of 5 summands around it gives $v_m(t_j) > 0.501$ $(m \geq j+2)$. Hence $v_m(2t_j) \simeq v_{m+1}(2t_j) < 0.499$ and $V(t)$ oscillates. Actual amplitude of the oscillation is about 0.028 in this case.

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$$V_n = \log_{1/q} 2 + \delta_V(\log_{1/q} n) + o(1), \qquad n \to \infty,$$

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Thm. 2 extends this to "asymptotically geometric probabilities".
Karlin (1967) erroneously claimed that the variance converges for any geometric probabilities. Our motivation for study this question was, in particular, in the necessity to puzzle out this contradiction. It turns out that Karlin's sufficient condition for the convergence of V_n is in fact necessary and sufficient, and produces the correct criteria $\log_{1/q} 2 \in \mathbb{Z}$.

Suppose that the following regular variation assumption holds: for $y > 0$

$$\max\{j : p_j \geq 1/y\} \sim y^{\gamma}\ell(y), \qquad y \to \infty,$$

where $0 < \gamma \leq 1$ and ℓ is a slowly varying function.
(This case was considered by Karlin (1967)).
Then the inequality $p_{j+k(j)}/p_j \leq 2/3$ implies $k(j) \to \infty$ as $j \to \infty$.
So $\liminf \frac{p_{j+k}}{p_j} \geq \frac{1}{2}$ for any fixed k and Thm. 1 implies that $V_n \to \infty$ and K_n has asymptotically normal distribution.

We prove the statements for Poissonized version of the process, and then show how de-Poissonization can be done.
It is convenient to introduce the counting measure

$$\nu(dx) = \sum\nolimits_j \delta_{p_j}(dx)$$

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It turns out that bounds on "lagged ratio" p_{j+k}/p_j and variance of $K(t)$ can be expressed in terms of $\Delta\nu(x)$ for small x, providing an easy way to establish connections between them.

Lemma 1. For a fixed integer $k \geq 1$ the bound

$$\Delta\nu(x) \leq k$$

for sufficiently small $x > 0$ holds if and only if

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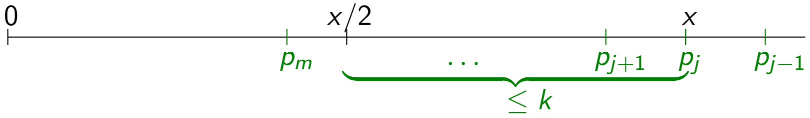
Proof.

$\Delta\nu(x) \leq k \;\Rightarrow\; 2p_{j+k} \leq p_j$. Take $x = p_j$:



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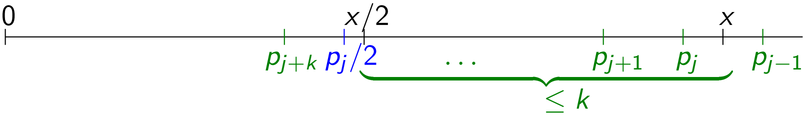

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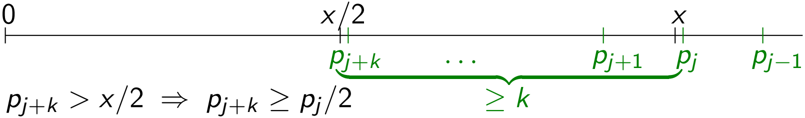
Proof.

$\Delta\nu(x) \geq k \;\Rightarrow\; p_{j+k} \geq p_j/2$. Let $p_{j+1} \leq x < p_j$



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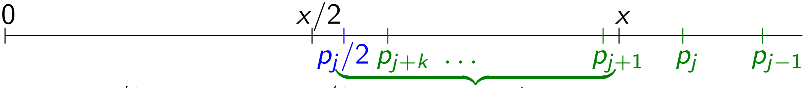

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Hence

$$t \int_0^\infty e^{-tx} \Delta\nu(x)\, dx = t \int_0^\infty e^{-tx} \sum\nolimits_j \mathbb{1}_{\{p_j \le x < 2p_j\}}\, dx$$
$$= t \sum\nolimits_j \int_{p_j}^{2p_j} e^{-tx} dx = \sum\nolimits_j (e^{-tp_j} - e^{-2tp_j}) = V(t).$$

Proof of Thm. 1:

First part: $\limsup_j p_{j+k}/p_j \leq 1/2 \;\Rightarrow\; \overline{v} \leq k$.

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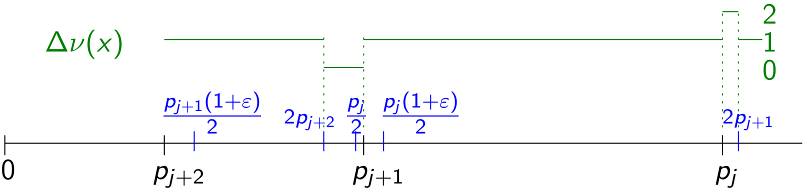

- Moreover, $\lambda(\{x \in [p_{j+1}, p_j] : \Delta\nu(x) = 2\}) \le \varepsilon p_j$.

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- Since $\varepsilon > 0$ is arbitrary, $\limsup_{x \downarrow 0} D(x)/x \le 1$.
- Then integration by parts gives

$$\begin{aligned} V(t) &= t \int_0^\infty e^{-tx} dD(x) \\ &= t^2 \int_0^\infty e^{-tx} D(x)\, dx \\ &= \int_0^\infty y e^{-y} \frac{D(y/t)}{y/t} dy \end{aligned}$$

and by Fatou's lemma

$$\overline{v} = \limsup_{t \to \infty} V(t) \le \int_0^\infty y e^{-y} \limsup_{t \to \infty} \frac{D(y/t)}{y/t} dy \le 1.$$

Second part: $\overline{v} \le M \;\Rightarrow\; \exists k : \;\limsup \frac{p_{j+k}}{p_j} \le \frac{1}{2}$.
Due to the special structure:

$$V(t) = \sum_j (e^{-p_j t} - e^{-2p_j t}) \ge \sum_{p_j \in (x/2, x]} (e^{-p_j t} - e^{-2p_j t})$$
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Minimum is separated from zero: if $z = e^{-xt/2}$ then

$$\min_{p \in [x/2,x]} (e^{-pt} - e^{-2pt}) = \left\{ \begin{array}{ll} z^2 - z^4 & 0 \le z \le \frac{\sqrt{5}-1}{2} \\ z - z^2 & \frac{\sqrt{5}-1}{2} \le z \le 1 \end{array} \right\} \ge \sqrt{5} - 2 > 0.$$

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Hence $\Delta\nu(x) \le \frac{2M}{\sqrt{5}-2}$ for small x, and the claim follows from Lemma 1.

Proof of Thm. 2:
Recall that $D(x) = \int_0^x \Delta\nu(x)dx$. Lemma 2 allows to write

$$V(t) = t\int_0^\infty e^{-tx}\Delta\nu(x)dx.$$

So $V(t) \to v \;\Leftrightarrow\; \int_0^\infty e^{-tx}dD(x) \sim v/t \;(t\to\infty)$.
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Similarly as above $p_{j+k}/p_j \to 2$, $j \to \infty$ iff

$$\lim_{x\to 0}\frac{|u \in (0,x] : \Delta\nu(x) \neq k|}{x} = 0.$$

This is equivalent to (*).

Let $K_r(t) = \sum_j \mathbb{1}_{M_j(t)=r}$ be the number of values that occur exactly r times in the Poissonized model.
Its mean $\Phi_r(t) = \mathbb{E}[K_r(t)] = \frac{t^r}{r!} \int_0^\infty x^r e^{-tx} \nu(dx)$.

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The following estimates hold:

$$\Phi_n - \Phi(n) = O\left(\frac{\Phi_2(n)}{n}\right), \qquad V_n - V(n) \quad = O\left(\frac{\Phi_1(n)^2 + \Phi_2(n)}{n}\right).$$

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The first follows from the inequality
$0 \le e^{-nx} - (1-x)^n \le nx^2 e^{-nx}$:

$$0 \le \Phi(n) - \Phi_n = \int_0^\infty (e^{-nx} - (1-x)^n)\nu(dx) \le \frac{2}{n}\Phi_2(n).$$

The second requires a bit more sophisticated but elementary analysis.
One can show that $\limsup \Phi_r(t) \le 2e\overline{v}$.

Thank you!