Complexity of solving tropical linear systems and conjecture on a tropical effective Nullstellensatz

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Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus$, $\otimes$.
If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.
If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.
Examples • $\mathbb{Z}^+ := \{0 \le a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;
• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;
• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \le j \le n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree trdeg* $= i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.
Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j\{Q_j\}$;
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Tropical linear systems

If T is an ordered semi-group then tropical linear function over T can be written as $\min_{1\le i\le n}\{a_i + x_i\}$.

Tropical linear system

$$\min_{1\le j\le n}\{a_{i,j} + x_j\},\ 1 \le i \le m \tag{1}$$

(or $(m \times n)$-matrix $A = (a_{i,j})$) has a *tropical solution* $x = (x_1 \ldots, x_n)$ if for every row $1 \le i \le m$ there are two columns $1 \le k < l \le n$ such that

$$a_{i,k} + x_k = a_{i,l} + x_l = \min_{1\le j\le n}\{a_{i,j} + x_j\}$$

Coefficients $a_{i,j} \in \mathbb{Z}_\infty := \mathbb{Z} \cup \{\infty\}$. Not all $x_j = \infty$. For $a_{i,j} \in \mathbb{Z}$ we assume $0 \le a_{i,j} \le M$.

$n \times n$ matrix $(a_{i,j})$ is **tropically non-singular** if $\min_{\pi\in S_n}\{a_{1,\pi(1)} + \cdots + a_{n,\pi(n)}\}$ is attained for a unique permutation π

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Complexity of solving tropical linear systems

Theorem

One can solve a tropical linear system (1) within complexity polynomial in n, m, M. (Akian-Gaubert-Guterman; G.)

Moreover, the algorithm either finds a solution over $\mathbb{Z}_\infty$ or produces an $n \times n$ tropically nonsingular submatrix of A.

Corollary

The problem of solvability of tropical linear systems is in the complexity class $NP \cap coNP$.

Remark

My algorithm has also a complexity bound polynomial in $2^{nm}, \log M$ (as well as an obvious algorithm which invokes linear programming).

Open question. Are tropical linear systems solvable within polynomial (in $n, m, \log M$) complexity (i. e. in the complexity class P)? Is it true for my algorithm?

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Tropical and Kapranov ranks

Tropical rank $trk(A)$ of matrix A is the maximal size of its tropically nonsingular square submatrices.

A lifting of A is a matrix $F = (f_{i,j})$ over the field of Newton-Puiseux series $K = R((t^{1/\infty}))$ for a field R such that the order $ord_t(f_{i,j}) = a_{i,j}$ where $f_{i,j} = b_1 \cdot t^{q_1} + b_2 \cdot t^{q_2} + \cdots$ with rational exponents $a_{i,j} = q_1 < q_2 < \cdots$ having common denominator, or $f_{i,j} = 0$ when $a_{i,j} = \infty$.

Kapranov rank $Krk_R(A) =$ minimum of ranks (over K) of liftings of A.

$trk(A) \leq Krk_R(A)$ and not always equal (Develin-Santos-Sturmfels)

Complexity of computing ranks

• For $n \times n$ matrix B testing $trk(B) = n$ ($\Leftrightarrow B$ is tropically nonsingular) has polynomial complexity (Hungarian method);
• $trk(A) = r$ is NP-hard, $trk(A) \geq r$ is NP-complete (Kim-Roush);
• Solvability of polynomial equations over R is reducible to $Krk_R(A) = 3$ (Kim-Roush).
Example $R = \mathbb{Q}$ or $R = GF[p](t)$.

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Complexity of computing ranks

• For $n \times n$ matrix B testing $trk(B) = n$ ($\Leftrightarrow B$ is tropically nonsingular) has polynomial complexity (Hungarian method);
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Example $R = \mathbb{Q}$ or $R = GF[p](t)$.

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Solvability of a tropical linear system and rank(s)

The theorem on complexity of solving tropical linear systems implies

Corollary

The following statements are equivalent

1) a tropical linear system (1) with $m \times n$ matrix A has a solution;

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Remark

- *The corollary holds for matrices over $\mathbb{R}_\infty$.*
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Testing uniqueness of a tropical solution

Proposition

One can test uniqueness (in the tropical projective space) of a solution of a tropical linear system (1) within complexity polynomial in n, m, M.

Open question. Is it possible to compute the dimension of a tropical linear space within complexity polynomial in n, m, M?

Solving tropical nonhomogeneous linear systems

Proposition

One can test solvability of a tropical nonhomogeneous linear system

$$\min_{1 \le j \le n} \{a_{i,j} + x_j, a_i\}, \ 1 \le i \le m$$

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Solvability of tropical **polynomial** systems is NP-complete (Theobald).

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"Dual" (classical) Nullstellensatz

For polynomials $g_1, \dots, g_s \in \mathbb{C}[X_1, \dots, X_k]$ consider an infinite Cayley matrix C with the columns indexed by monomials X^I and the rows by shifts $X^J \cdot g_i$

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Nullstellensatz tells about ideal $\langle g_1, \dots, g_s \rangle$, while the (infinite) dual Nullstellensatz forgets the ideal, and therefore, gives a hope to hold in the tropical setting

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Conjecture on tropical (dual) Nullstellensatz

Assume w.l.o.g. that for tropical polynomials $h = \bigoplus_J (a_J \otimes X^{\otimes J})$ in k variables which we consider, function $J \rightarrow a_J$ is concave on $\mathbb{R}^k$. This assumption does not change tropical varieties.

For tropical polynomials $h_1, \dots, h_s$ consider (infinite in all 4 directions) Cayley matrix H with the rows indexed by $X^{\otimes I} \otimes h_i$ for $I \in \mathbb{Z}^k$.

Conjecture. $h_1, \dots, h_s$ have a tropical solution iff infinite tropical linear system $H \otimes (\dots, z_0, \dots)$ has a solution with $z_0 \neq \infty$.

Conjecture. Similar for a finite submatrix H_0 of H (generated by a set of rows of H) with the size bounded via k and $trdeg(h_i)$.

Theorem

Univariate ($k = 1$) tropical polynomials $h_1, \dots, h_s$ have a solution iff tropical linear system $H_0 \otimes (z_0, \dots, z_N)$ has a solution with $z_0 \neq \infty$ where H_0 is (finite) submatrix of H generated by its rows $X^{\otimes I} \otimes h_i$ for $0 \leq I \leq 4 \cdot (trdeg(h_1) + \dots + trdeg(h_s))$.

For two tropical polynomials ($s = 2$) the bound $trdeg(h_1) + trdeg(h_2)$ holds using the classical resultant and Kapranov's theorem (Tabera).

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Conjecture on tropical (dual) Nullstellensatz

Assume w.l.o.g. that for tropical polynomials $h = \bigoplus_J (a_J \otimes X^{\otimes J})$ in k variables which we consider, function $J \to a_J$ is concave on $\mathbb{R}^k$. This assumption does not change tropical varieties.

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Conjecture. $h_1, \dots, h_s$ have a tropical solution iff infinite tropical linear system $H \otimes (\dots, z_0, \dots)$ has a solution with $z_0 \neq \infty$.

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Solution $Y := \{(J, y_J)\} \subset \mathbb{R}^{k+1}$ of a tropical linear system $H \otimes Y$ treat also as a graph on $\mathbb{R}^k$.

The conjecture is equivalent to the following.

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Proof of the tropical dual Nullstellensatz for $k = 1$

Fix a tropical polynomial h_i. Points of intersection $(G_i^{(l)} + (0, b_{l,i})) \cap Y$ call *extremal*, their union for $l \in \mathbb{Z}$ denote $E_i \subset \mathbb{R}^2$.

Lemma

E_i are vertices of a convex polygon lying below Y.

Edges of E_i are of two sorts. Either an edge (*r-principal*) is parallel to r-th edge e_r of G_i or an edge (*intermediate*) is a parallel shift of a "diagonal" connecting two vertices of G_i not lying in a single edge of G_i.

Lemma

1) for each r r-principal edges form an interval (perhaps, infinite) with the distance between any pair of adjacent extremal points less or equal to the length of e_r;
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Corollary

In the convex polygon $\cap_{1 \le i \le s} E_i$ the sum of lengths of the intermediate edges is less than $3 \cdot \sum_{1 \le i \le s} trdeg(h_i)$ and the sum of lengths of the principal (not all coinciding for different $E_i, 1 \le i \le s$) edges is less than $\sum_{1 \le i \le s} trdeg(h_i)$.

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Algorithm for solving tropical linear systems: finite coefficients

First assume that the coefficients of system (1) are finite:
$0 \leq a_{i,j} \leq M$, $1 \leq i \leq n$, $1 \leq j \leq m$.
Induction on m. Suppose that (tropical) vector $x := (x_1, \ldots, x_n)$ fulfils $m-1$ equations (except, perhaps, the first one).

The algorithm modifies x and either produces a solution of (1) or finds $n \times n$ tropically nonsingular submatrix of A (in the latter case (1) has no solution).
After each step of modification a vector is produced (we keep for it the same notation x) such that it still fulfils $m-1$ equations, and $m \times n$ matrix $B := (a_{i,j} + x_j)$ (after suitable permutations of rows and columns) has a form below.

If $a_{i,j} + x_j = \min_{1 \leq l \leq n}\{a_{i,l} + x_l\}$ mark entry i, j with $*$. The first row contains a single $*$ (otherwise, x is a solution of (1)) and every other row contains at least two $*$.

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Continuation: producing a candidate for solution

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \\ B_5 & B_6 \end{pmatrix}$$

• a square matrix B_1 contains $*$ on the diagonal and no $*$ above the diagonal. Hence B_1 is tropically nonsingular.
• B_2, B_4 contain no $*$.
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Termination of the algorithm

Construct recursively a set L of the left columns by augmenting. As a base of recursion the first column belongs to L.

For current L if there exists a row with single $*$ in a column off L, join this column to L. These rows and columns form matrix B_1.

If L coincides with the set of all the columns then B_1 is $n \times n$ tropically nonsingular submatrix of B and therefore, (1) has no solution. This completes the description of the algorithm.

Tropical norm and complexity bound

To estimate the number of steps of the algorithm define a *tropical norm* of a vector (in the tropical projective space) $(y_1, \dots, y_n)$ as

$$\sum_{1 \le i \le n} (y_i - \min_{1 \le j \le n} \{y_j\}).$$

After every modification step the tropical norm of vector $(a_{1,1} + x_1, \dots, a_{1,n} + x_n)$ (corresponding to the first row) drops.

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Solving tropical linear systems over $\mathbb{Z}_\infty$

For the inductive (again on m) hypothesis assume that $(m-1) \times n$ matrix A' (obtained from A by removing its first row) has a block form (after permuting its rows and columns)

$$\begin{pmatrix} A_{1,1} & \infty & \cdots & \infty & \infty \\ A_{2,1} & A_{2,2} & \cdots & \infty & \infty \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{t-1,1} & A_{t-1,2} & \cdots & A_{t-1,t-1} & \infty \\ \hline A_{t,1} & A_{t,2} & \cdots & A_{t,t-1} & A_{t,t} \end{pmatrix}$$

where each entry of upper-triangular blocks equals ∞.

A finite vector $y = (y_1, \dots, y_n) =: (y^{(1)}, \dots, y^{(t)}) \in \mathbb{Z}^n$ is produced (where $y^{(1)}, \dots, y^{(t)}$ is its partition corresponding to the block structure) such that each diagonal block $A_{p,p}$, $1 \leq p \leq t-1$ has $*$ (with respect to vector $y^{(p)}$) everywhere on its diagonal and no $*$ above the diagonal. Matrix $A_{p,p}$ is of size $u_p \times v_p$ with $u_P \geq v_p$.
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Continuation: modifying candidate for a solution

To be closer to the finite case $\mathbb{Z}$ extend the lowest block $\overline{A_{t,1}}\ \overline{A_{t,2}} \cdots \overline{A_{t,t-1}}\ \overline{A_{t,t}}$ of A' by joining to it the first row of A as its first row. The resulting extension of matrix $\overline{A_{t,t}}$ denote by C.

Again as in the finite case assume (after a permutation of the columns) that a single $*$ (with respect to vector $y^{(t)}$) in the first row of C is located in the first column.

The algorithm modifies vector $y^{(t)}$ keeping it to be a solution of $\overline{A_{t,t}}$ and keeping the same notation for the modified vectors.
If $y^{(t)}$ is a solution of C then vector $(\infty, \dots, \infty, y^{(t)})$ is a solution of A (1) and the algorithm terminates the inductive step.

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Continuation of modifying a candidate: graph of possibly infinite coordinates

In addition, the algorithm considers an oriented graph with the nodes being the coordinates of vector $y^{(t)} =: (y_1^{(t)}, \dots, y_s^{(t)})$ and with an edge from node $y_j^{(t)}$ to $y_l^{(t)}$ when $y_j^{(t)} - y_l^{(t)} \leq M$ (remind that all finite coefficients of matrix A satisfy $0 \leq a_{i,j} \leq M$).

Denote by S the set of nodes of the graph reachable from the first node $y_1^{(t)}$.

Lemma

$L \subset S$ and in the course of the algorithm while modifying S, the next S is a subset of the previous one.

The algorithm modifies $y^{(t)}$ while $L \neq S$.

If $L = S$ then (after suitable permutations of the rows and columns)

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Termination of the algorithm

$$C = \begin{pmatrix} C_1 & \infty \\ C_2 & \infty \\ C_3 & C_4 \end{pmatrix}$$

- L are columns of a square matrix C_1;
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This completes the inductive step of the algorithm and constructing a new block structure of matrix A.
Vector $y^{(t)} =: (y^{(t)}, y^{(t+1)})$ (abusing the notations) and vector $(\infty, \dots, \infty, y^{(t+1)})$ is a solution of A (1).

The algorithm terminates if either all the columns or all the rows are exhausted. If all the columns are exhausted then (1) has no solution. Otherwise, if first all the rows are exhausted then $(\infty, \dots, \infty, y^{(t+1)})$ is a solution of (1).

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