Current progress in Majorana theory

In my lecture I will discuss the current state of the *Majorana theory.*

The Monster book. In April 2009 the boook [A.A. Ivanov, *The Monster Group and Majorana Involutions*, Cambridge Univ. Press, Cambridge 2009] has been published. This was a result of over 20 years of research. The main original aim was to provide the first complete construction and uniqueness proofs for the largest and most famous among the 26 sporadic simple groups, known as the Monster group. The proof was have culminated the proof by Ivanov and Norton the so-called Y-conjecture during the Durham symposium on 'Groups and Geometries' in July 1990. At that time the Y-conjecture became a theorem which John Conway called NICE (where 'N' is for Norton, 'I' is for Ivanov, 'C' is for Conway and 'E' is for everyone else involved). The proof was the most spectacular example of the so-called 'Geometric Presentations of Groups'. The classical version of this is precisely the Steinberg presentations for the groups of Lie type, under this title an invited lecture at the Kyoto 1990 International Congress of mathematicians was given by A.A.Ivanov and this is the title of a conference to be held in Birmingham this summer.

Halfway through the actual writing up of the book (October 2005– May 2008). The exceptional importance of the so-called Monster algebra (also known as the Conway–Griess–Norton algebra) has been further appreciated. This came through a result of a Japanese mathematician S. Sakuma, who has classified all the subalgebras in the Monster algebra generated by pairs of axial vectors corresponding to the $2A$ (also known as the Baby Monster) involutions. There are nine isomorphism types of these subalgebras identified and studied by J. Conway and S. Norton. The outstanding importance of Sakuma's result was that the subalgebras were classified under very mild assumptions involving the fusion rules of the eigenspaces of the axial vectors. The far reaching importance of these properties brought about the need of special name for them that is how the term 'Majorana theory' emerged and for the first time was announce at the Oberwolfach conference on 'Groups and Geometries' in April of 2008.

During the 2008 Oberwoffact conference the Abel prises of J. Thompson and J. Tits were also celebrated. At the special session our prominent colleagues shares their memories of the most exciting period of 1970's when the finite simple group were classified and when most spectacular sporadic simple groups. Among them there was Berns Fischer the 'father' of the Monster group. Right after the lecture martin Liebeck suggested to make record of Fischer's story and this is how the last chapter of the Monster group emerged, which is probably the most attractive one (especially for the general mathematical readers).

Majorana representations of groups. After the Majorana setting was axiomatised it became clear that one should start with classifying the Majorana representations of small groups. At such circumstances one usually starts looking at the A_5's. There are two classes of A_5-subgroups in the Monster whose involutions are of type $2A$. Some preliminary estimates led to a conjecture that the corresponding axial vectors in the Monster algebra generate 26- and 21-dimensional subalgebras, respectively. A conjecture was posed in Chapter 8 of the Monster book that A_5 possess only two Majorana representations corresponding (in a sense which has been made explicit and rigorous) to the subalgebras of the Monster algebras. At that time this was more like a dream and no-one could have believed that in less than two years these conjucture will be corrected (the second representations turned out to be just 20-dimensional) and fully proved. The current status of the classification project of the Majoorana representations of the small groups is the following:

(i) the representations of the dihedral group have been classified in the original paper by S. Sakuma.

(ii) the representations of S_4 are completely classified and the result os published: [A.A. Ivanov, D.V. Pasechnik, Á. Seress, and S. Shpectorov, Majorana representations of the symmetric group of degree 4, *J. Algebra* **324** (2010), 2432-2463.]

(iii) the representations of A_5 are completely classified and the manuscript [A.A. Ivanov and Á. Seress, Majorana Representations of A_5] *Math. Z.* (submitted)

(iv) the representation of $L_3(2)$ are completely classified and the manuscript [A.A. Ivanov and S. Shpectorov, Majorana Representations of $L_3(2)$] *Adv. Geom.* (to appear)

(v) an important class of the representations of A_6 and A_7 has been characterized in [A.A. Ivanov, On Majorana representations od A_6 and A_7] *Comm. Math. Physics.*

(vi) the Majorana representations of $L_2(11)$ is the research project of Sophie Docelle [second year Ph D student at Mathematics Department, Imperial College].

(vii) the Majorana representations of $L_3(3)$ is the research project of Alonso Castillo [first year Ph D student at Mathematics Department, Imperial College].

Certainly larger groups are under consideration including A_8, which so far appeared to be a much harder case.

Refinement of the knowledge of the Monster algebra. Simon Norton from Cambridge possesses an incredible amount of information about the Monster group and its algebra. A very small part of this information is published (usually without any proofs and with justification looked obscure for an outsider). The Majorana theory provides a tool to put this information is a systematic and checkable form. The true success of this project can be seen in correction of some information revealed by Norton. These can be illustrated by the A_5-algebras: one of them is 20 (rather than 21)-dimensional and the crucial relations in the other one needs certain signs to be alternated. So to say, the Majorana theory brings us beyond Norton's expertise of the Monster.

Classifying subconfigurations. Besides classifying the Majorana representations of specific groups, another promising direction in developing the Majorana theory is to study some specific configurations of the Majorana axes and identification of the subalgebras they generate. The first such problem would be the classification of the subalgebras generated by triples of Majorana axes containing pairs generating $2A$-algebras (there are 36 such configurations in the Monster as given by S. Norton). Michael Aschbacher is particular keen on this direction of developing and hopefully at some stage we could have a close cooperation with him on this project.