Curves of Marginal Stability in Supersymmetric CP($N-1$) theory with $\mathcal{Z}_N$ twisted masses

Pavel A. Bolokhov

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In collaboration with M.Shifman and A.Yung

The BPS Spectrum and Curves of Marginal Stability in Supersymmetric CP($N-1$) theory with $\mathcal{Z}_N$ twisted masses

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MOTIVATION

Two dimensional sigma models show remarkable similarity with four dimensional supersymmetric gauge theories

The spectrum of CP$(N-1)$ theory is known to match the one of $\mathcal{N}=2$ SQCD at the root of the baryonic Higgs branch

Now we know that both theories in fact describe the same states — the monopoles confined on the non-Abelian strings

The problem of finding the spectrum of the theory with twisted masses is also connected to finding the curves of marginal stability (CMS)

In supersymmetric theories the curves of marginal stability represent a boundary on which the spectrum of stable states gets re-arranged

For example, they form a boundary between the region where the strong coupling spectrum is stable and the one where the weak coupling spectrum exists.

These curves replace the phase transition lines for supersymmetric theories.

We consider a theory where the masses are taken on a circle,

$$m_l \quad = \quad m_0 \cdot e^{2\pi i l/N} \,, \qquad\qquad l = 0, 1, ..., N-1 \,,$$

such that a $\mathcal{Z}_N$ remnant of the $\mathrm{U}(1)_{\mathrm{R}}$ remains unbroken.

CMS in two-dimensional sigma models are built in the plane/space of mass parameters.

However, in our case, there is really only one parameter m_0.

The CMS are found in the plane of m_0, or, equivalently, m_0^N.

The conditions for the CMS require one first to determine the spectrum of the theory.

The spectrum of BPS kinks is determined by the mass formula

$$m_{\rm BPS} \;=\; \Big| \mathcal{W}_{\rm eff}(\mathtt{vac}_2) \;-\; \mathcal{W}_{\rm eff}(\mathtt{vac}_1) \Big| \,,$$

where $\mathcal{W}_{\rm eff}$ is the exact superpotential of the CP($N-1$) theory.

We analyze this superpotential in detail.

$$\mathrm{CP}(N-1)$$

The theory we consider is the $\mathcal{N}=(2,2)$ CP$(N-1)$ with twisted masses.

It has a gauged formulation, with the bosonic part

$$\mathcal{L} = \frac{1}{e_0^2}\left(\frac{1}{4}F_{\mu\nu}^2 + \left|\partial_\mu\sigma\right|^2 + \frac{1}{2}D^2\right) + \left|\nabla_\mu n^i\right|^2$$

$$+ \; i\,D\left(|n_i|^2 - r\right) + 2\sum_i\left|\sigma - \frac{m_i}{\sqrt{2}}\right|^2|n^i|^2 \,.$$

The sigma model limit corresponds to going to the strong coupling $e_0^2 \to \infty$.

When the auxiliary fields are resolved, the only variables are the CP($N-1$) ones n_l and the scalar σ.

In the classical theory there are N vacua

$$n_l = (\,0,\ ...,\ 1, ...,\ 0\,)\,, \qquad k = 0, ..., N-1\,,$$
$$\sigma = m_k\,.$$

In quantum theory, at strong coupling, the vacua change, although there are still N of them.

The theory has an exact superpotential of Veneziano-Yankielowicz type,

$$\mathcal{W}_{\rm eff}(\hat{\sigma}) \;=\; -\, i\,\tau\hat{\sigma} \;+\; \frac{1}{2\pi}\sum_j (\hat{\sigma}-m_j)\left\{\ln\frac{\hat{\sigma}-m_j}{\mu} \;-\; 1\right\} .$$

It is obtained by integrating out n^l fields.

Here $\hat{\sigma}$ is a twisted superfield containing σ, and μ is the UV cut-off.

μ is related to the strong coupling scale Λ and the coupling constant $r \;\equiv\; 2\beta$ as

$$\mu \;=\; \Lambda\, e^{2\pi r/N}$$

The vacua of this theory, similarly to the masses, sit on a circle

$$\sigma_p \;=\; \sqrt[N]{\Lambda^N + m_0^N} \cdot e^{2\pi i p/N}\,, \qquad p \;=\; 0, \ldots, N-1\,.$$

The vacuum values of the superpotential are

$$\mathcal{W}_{\text{eff}}(\sigma_p) \;=\; -\,\frac{1}{2\pi}\left\{ N\,\sigma_p \;+\; \sum_j m_j\,\ln\frac{\sigma_p - m_j}{\Lambda}\right\}.$$

The BPS mass formula becomes

$$m_{\text{BPS}} \;=\; \left|\,\mathcal{W}_{\text{eff}}(\sigma_1) \;-\; \mathcal{W}_{\text{eff}}(\sigma_0)\,\right|$$

The quasi-classical limit corresponds to

$$\mathcal{W}_{\text{eff}}(\sigma_1) \;-\; \mathcal{W}_{\text{eff}}(\sigma_0) \;\sim\; \frac{N}{2\pi}\,\Delta m \cdot \ln\frac{|m_0|}{\Lambda} \;=\; r \cdot \Delta m\,.$$

The problem with the Veneziano-Yankielowicz superpotential is that it is not “precisely” exact.

It only is good for finding the spectrum.

The logarithms are multi-valued functions.

The superpotential therefore gives too many states — $\mathcal{Z}^N$, if taken naively,

$$\sum_j m_j \, \ln \frac{\sigma_p - m_j}{\Lambda}.$$

Only certain branches of the logarithms in $m_{\rm BPS}$ are physical.

We need to deal with the ambiguity.

Let us work within one complex sheet of the logarithm, while absorbing the ambiguity into

$$i\,\vec{N}\cdot\vec{m}\,,$$

where $\vec{N}$ is a set of integers $\vec{N} \;=\; (n_0,\;\ldots,\;n_{N-1})$.

Instead of working with the original formula for the central charge — which would have two such sets of integers

$$m_{\text{BPS}} \;=\; \left|\,\mathcal{W}_{\text{eff}}(\sigma_1) \;-\; \mathcal{W}_{\text{eff}}(\sigma_0)\,\right|$$

— we transform the formula for the superpotential $\mathcal{W}_{\text{eff}}(\sigma_1)$.

Remind, all these expressions are functions of m_0.

The answer is as follows,

$$m_{\text{BPS}} = U_0(m_0) + i\,\vec{N}\cdot\vec{m}\,,$$

with an explicit function

$$U_0(m_0) = -\frac{1}{2\pi}\left(e^{2\pi i/N} - 1\right)\left\{ N\sqrt[N]{m_0^N + \Lambda^N} + \right.$$
$$\left. + \sum_j m_j \ln \frac{\sqrt[N]{m_0^N + \Lambda^N} - m_j}{\Lambda} \right\}.$$

$U_0(m_0)$ is a single-valued function in a region, which is wide enough for determining the spectrum

The spectrum itself is described by the set of integers $\vec{N}$.

The central charge formula takes the canonical form, analogous to that of the four-dimensional gauge theories

$$\mathcal{Z} \;=\; \text{“}\mathcal{W}_{\text{eff}}(\sigma_1) \;-\; \mathcal{W}_{\text{eff}}(\sigma_0)\text{''} \;+\; i\,\vec{N}\cdot\vec{m}.$$

There is a topological (“magnetic”) contribution and a Noether (“electric”) contribution proportional to the masses

They are characterized by a topological

$$\vec{T} \;=\; (\,-1, \quad 1, \quad ..., \quad 0\,)$$

and Noether

$$\vec{N} \;=\; (\,n_0, \quad n_1, \quad ..., \quad n_{N-1}\,)$$

charges:

$$\mathcal{Z} \;=\; \vec{T}\cdot\vec{\mathcal{W}}_{\text{eff}} \;+\; i\,\vec{N}\cdot\vec{m}.$$

In some 1998, Dorey by quantization of solitons showed that the theory has a tower of dyonic kinks

$$\vec{N} \ = \ n\,\vec{T}, \qquad\qquad n \ \in \ \mathcal{Z}.$$

That is, the mass formula is

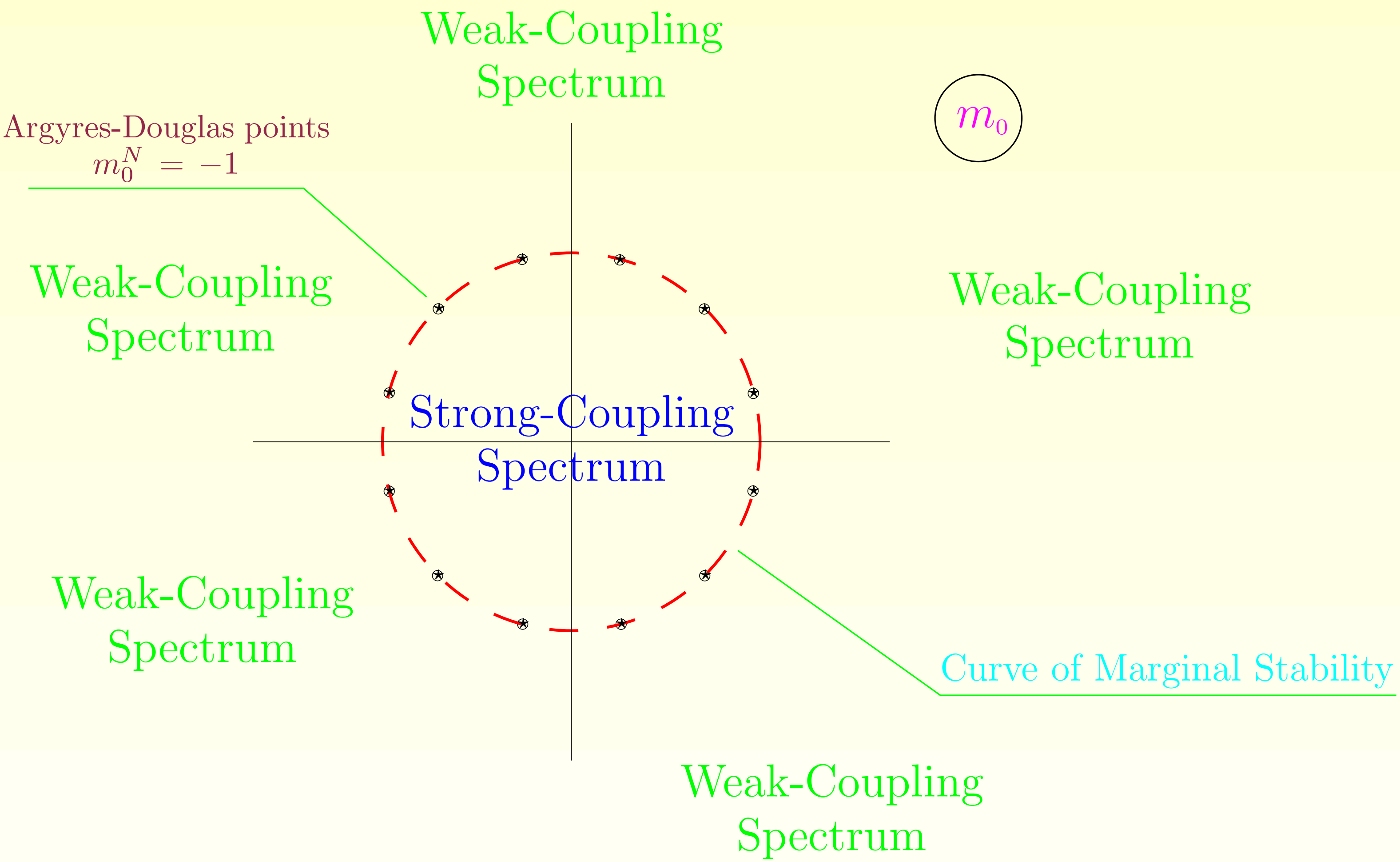
$$\mathcal{Z} \ = \ \text{“}\mathcal{W}_{\text{eff}}(\sigma_1) \ - \ \mathcal{W}_{\text{eff}}(\sigma_0)\text{”} \ + \ i\,n\,(m_1 \ - \ m_0)$$

That is, all states are characterized by just two integers $(T,\ n)$.

We argue that the form of the Noether charge

$$\vec{N} \ = \ (\,-n, \quad n, \quad 0, \quad ..., \quad 0\,)$$

is only valid as asymptotics at weak coupling and large excitation number n.


Weak-Coupling Spectrum
m_0
Argyres-Douglas points
$m_0^N = -1$
Weak-Coupling Spectrum
Weak-Coupling Spectrum
Strong-Coupling Spectrum
Weak-Coupling Spectrum
Curve of Marginal Stability
Weak-Coupling Spectrum

What happens in the circle?

What happens in the circle?

Mirror symmetry gives
the spectrum around the origin

Mirror Theory

[Vafa, Hori '00]

$$\mathcal{W}_{\text{mirror}}^{\text{CP}(N-1)} \;=\; -\,\frac{\Lambda}{2\pi}\left\{\sum_j X_j \;+\; \sum_j \frac{m_j}{\Lambda}\,\ln X_j\right\},$$

subject to

$$\prod_j X_j \;=\; 1\,.$$

Mirror Theory

[Vafa, Hori '00]

$$\mathcal{W}_{\text{mirror}}^{\text{CP}(N-1)} = -\frac{\Lambda}{2\pi}\left\{\sum_j X_j + \sum_j \frac{m_j}{\Lambda}\ln X_j\right\},$$

subject to

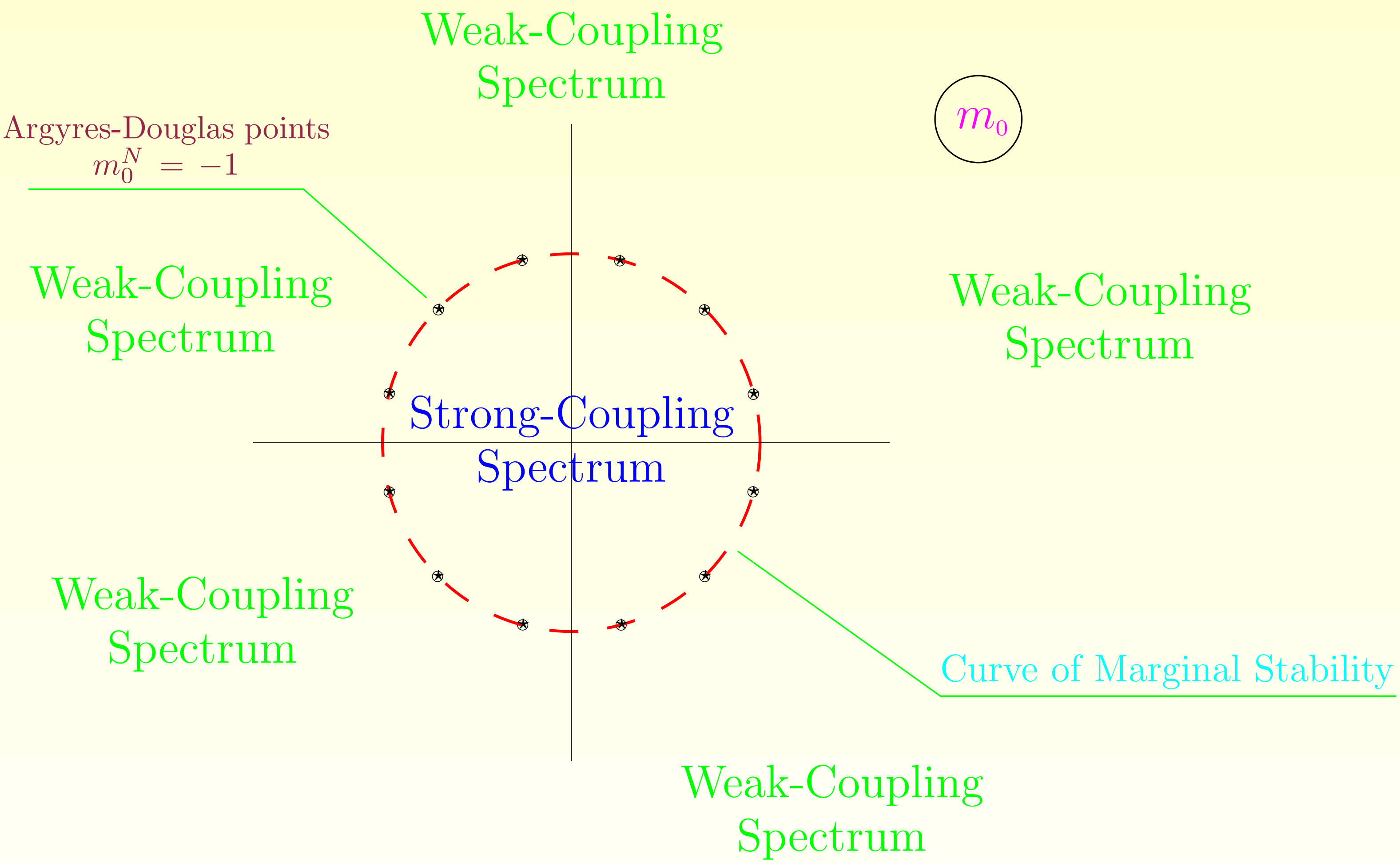
$$\prod_j X_j = 1\,.$$

This theory at small m_0 predicts the existence of N kinks

$$m_{\text{BPS}} \approx \left|\frac{N}{2\pi}\left(e^{2\pi i/N} - 1\right)\Lambda - i\,m_j\right|, \qquad j = 0, \ldots, N-1\,.$$

[Shifman, Yung '10]

These are the states that become massless at the corresponding Argyres-Douglas points $\sigma_p = \sqrt[N]{1 + m_0^N} \cdot e^{2\pi i p/N} = 0$

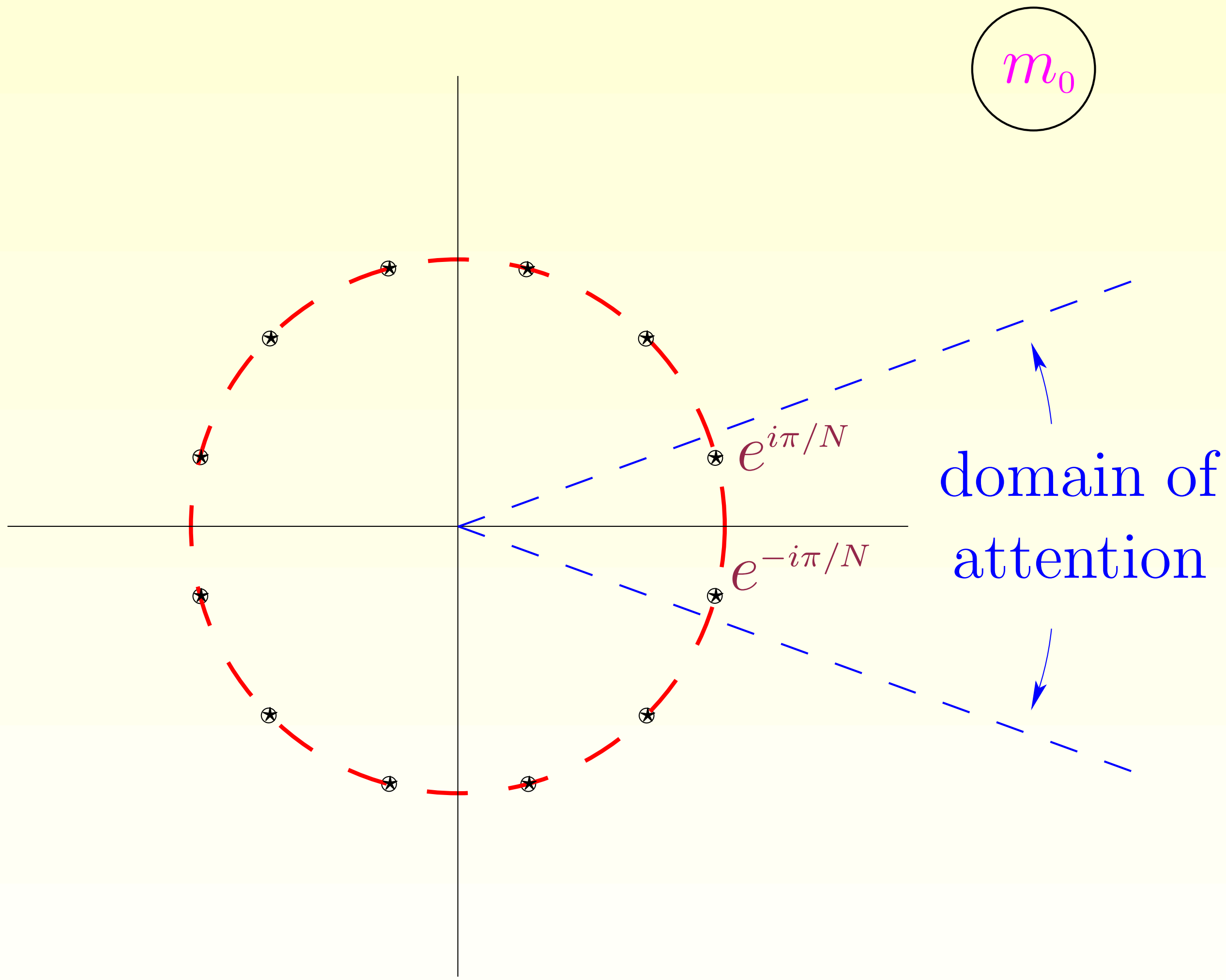


Qualitatively this is seen from the formula

$$\mathcal{Z} = \text{“}\mathcal{W}_{\text{eff}}(\sigma_1) - \mathcal{W}_{\text{eff}}(\sigma_0)\text{”}$$

We prefer to work with our function $U_0(m_0)$

$$U_0(m_0) \;=\; -\,\frac{1}{2\pi}\left(e^{2\pi i/N}\,-\,1\right)\left\{N\,\sqrt[N]{1\,+\,m_0^N}\;+\;\sum_j m_j\,\ln\left(\sqrt[N]{1\,+\,m_0^N}\,-\,m_j\right)\right\}.$$



$$m_{\rm BPS} \;=\; U_0(m_0) \;+\; i\,\vec{N}\cdot\vec{m}\,,$$

General Criteria

We can now formulate the requirements for the spectrum of BPS states in the overall region of the complex mass parameter m_0 :

- Quasi-classical limit — the spectrum at large m_0 and large excitation number n must reproduce the semiclassical result:

$$m_{\rm BPS} \simeq \frac{N}{2\pi}(m_1 - m_0)\cdot\ln|m_0| + i\,n\cdot(m_1 - m_0)\,;$$

- Argyres–Douglas point — the only states that survive when crossing from weak coupling into the strong coupling region are those N states which become massless at the AD points;

- Mirror spectrum — the latter N kinks must reflect the spectrum given by mirror formula in the small m_0 limit:

$$m^{(j)}_{\rm BPS} \approx \left| -\frac{N}{2\pi}\left(e^{2\pi i/N} - 1\right) + i\,m_j\right|.$$

Dorey's formula for the spectrum

$$\mathcal{Z} \;=\; \vec{T}\cdot\vec{m}_D \;+\; i\,\vec{N}\cdot\vec{m}$$

with $\vec{N} \;=\; n\,\vec{T}$ cannot describe the set of the N states in the strong coupling region.

The right answer is obtained by comparing our formula

$$m_{\rm BPS} \;=\; U_0(m_0) \;+\; i\,\vec{N}\cdot\vec{m}\,,$$

$$U_0(m_0) \;=\; -\frac{1}{2\pi}\left(e^{2\pi i/N}-1\right)\left\{N\,\sqrt[N]{1+m_0^N} \;+\; \sum_j m_j\,\ln\left(\sqrt[N]{1+m_0^N}-m_j\right)\right\}$$

with the mirror result

$$m_{\rm BPS}^{(j)} \;\approx\; \left|-\frac{N}{2\pi}\left(e^{2\pi i/N}-1\right) \;+\; i\,m_j\right|$$

in the small mass limit.

Near the origin we obtain

$$m_{\text{BPS}} \approx -\frac{N}{2\pi}\left(e^{2\pi i/N} - 1\right) + i\,\vec{N}\cdot\vec{m}\,,$$

The spectrum for the strong coupling area then is found to be

$$\vec{N} = \begin{array}{l} (1, \quad 0, \quad 0, \quad ..., \quad 0)\,, \\ (0, \quad 1, \quad 0, \quad ..., \quad 0)\,, \\ (0, \quad 0, \quad 1, \quad ..., \quad 0)\,, \\ \quad . \quad . \quad . \quad . \\ (0, \quad 0, \quad 0, \quad ..., \quad 1)\,. \end{array}$$

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Precisely one of these states becomes massless at each corresponding AD point.

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Precisely one of these states becomes massless at each corresponding AD point.

Neither can this spectrum be described by the Dorey's formula, nor can it be fit into one tower of states

In order to describe the quasi-classical asymptotics

$$m_{\text{BPS}} \simeq \frac{N}{2\pi}(m_1 - m_0)\cdot \ln|m_0| + i\,n\cdot(m_1 - m_0)\,;$$

one has to introduce $N-1$ towers

$$\begin{aligned}
\vec{N}_{(1)} &= (\,-n_{(1)}+1, & n_{(1)}, && 0, && 0, && ..., && 0\,)\,,\\
\vec{N}_{(2)} &= (\,\quad -n_{(2)}, & n_{(2)}, && 1, && 0, && ..., && 0\,)\,,\\
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&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
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This is the spectrum which satisfies all three criteria

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This is the spectrum which satisfies all three criteria

$$m_{\text{BPS}} \;=\; U_0(m_0) \;+\; i\,n_{(k)}\cdot(m_1 - m_0) \;+\; i\,m_k\,, \qquad k = 1, ..., N-1\,.$$

Curves of Marginal Stability

For each tower,

$$m_{\text{BPS}} \;=\; U_0(m_0) \;+\; i\,n_{(k)}\cdot(m_1 - m_0) \;+\; i\,m_k\,, \qquad k = 1,...,N-1\,.$$

there will be its own decay curve. The CMS condition is

$$\text{Re}\;\frac{U_0(m_0) \;+\; i\,m_k}{m_1 \;-\; m_0} \;=\; 0\,.$$

We will be drawing the curves in the m_0^N plane, otherwise a curve would repeat itself N times.

The first tower which we call primary is special.

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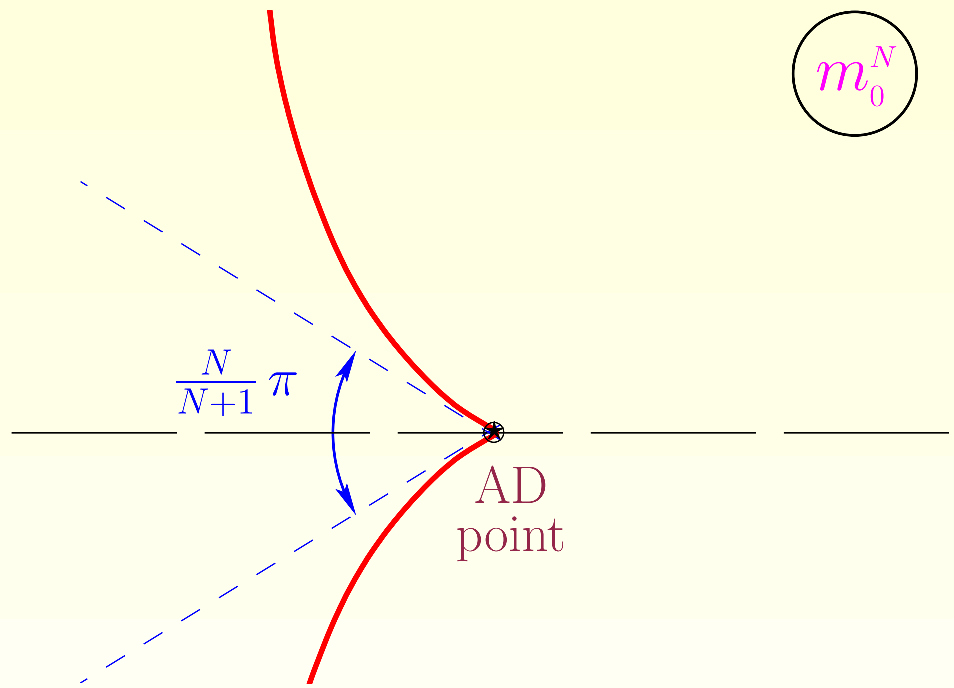

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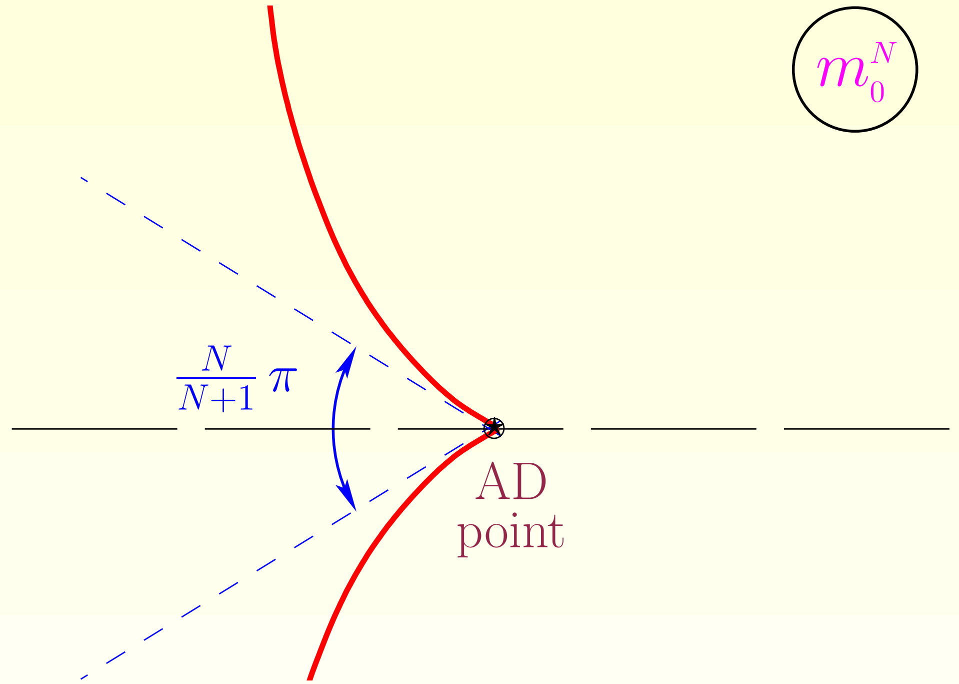
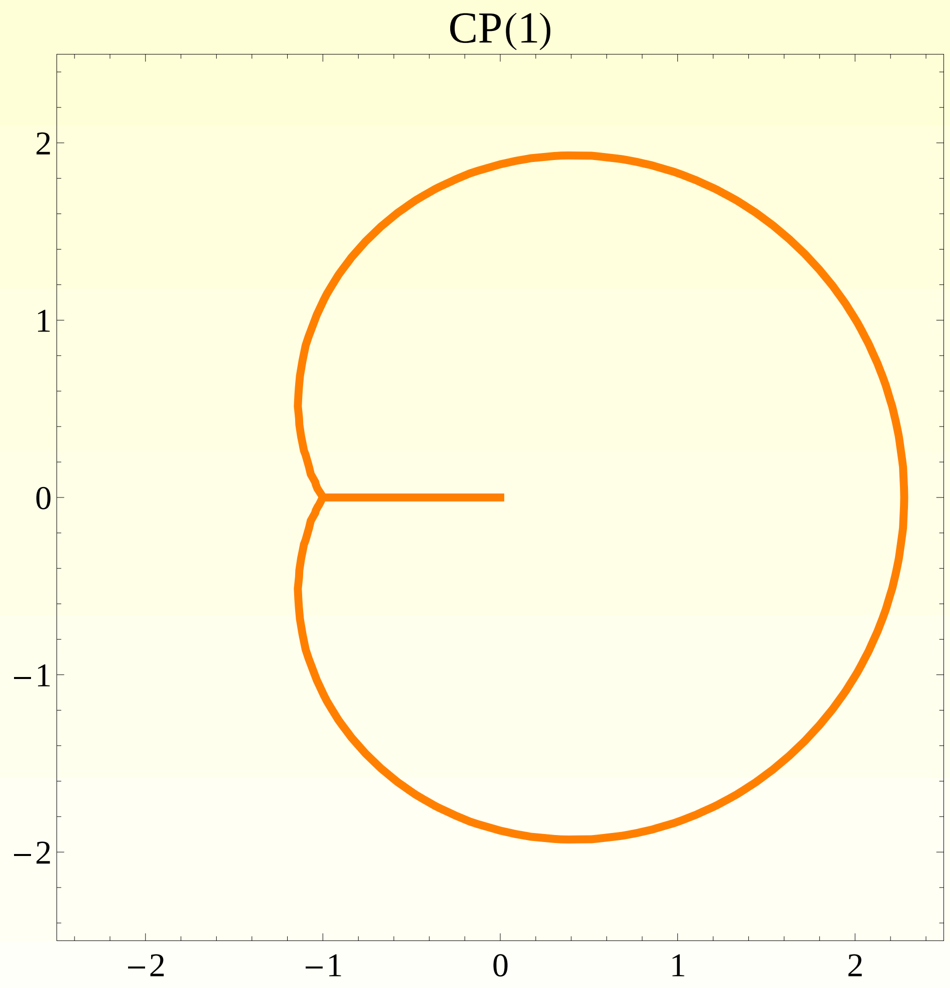
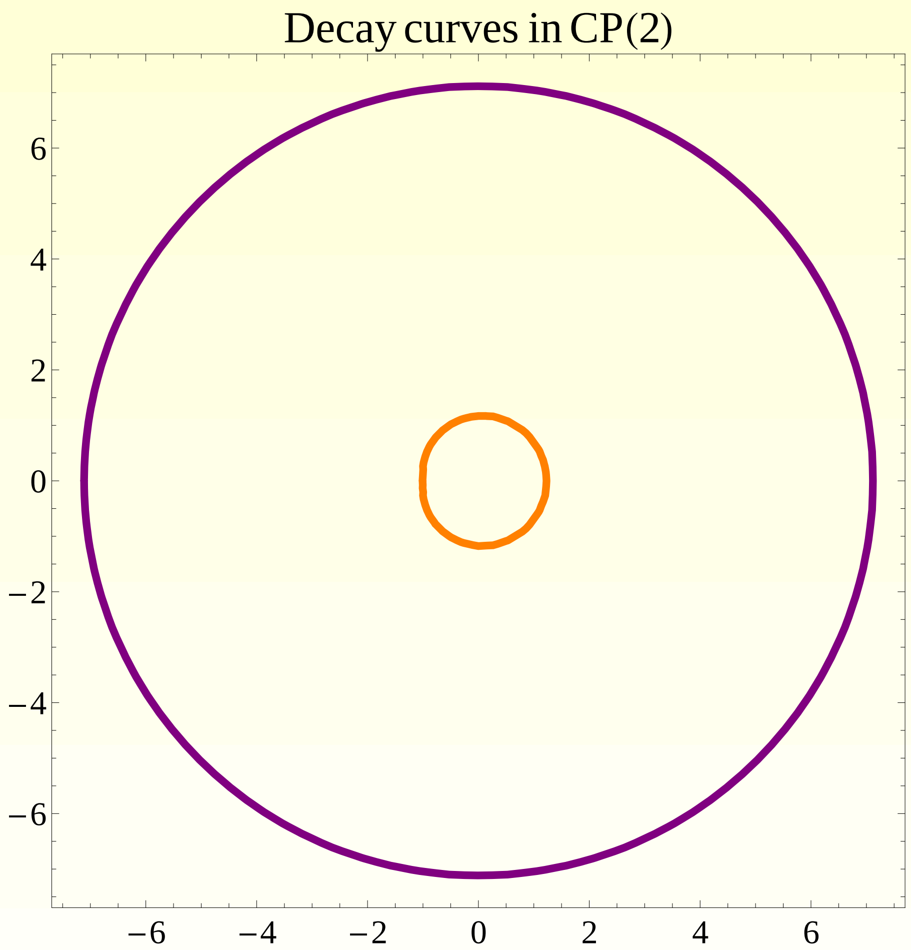


At infinite N the cusp flattens out, as the curves tend to circles.

The Curve for CP(1)

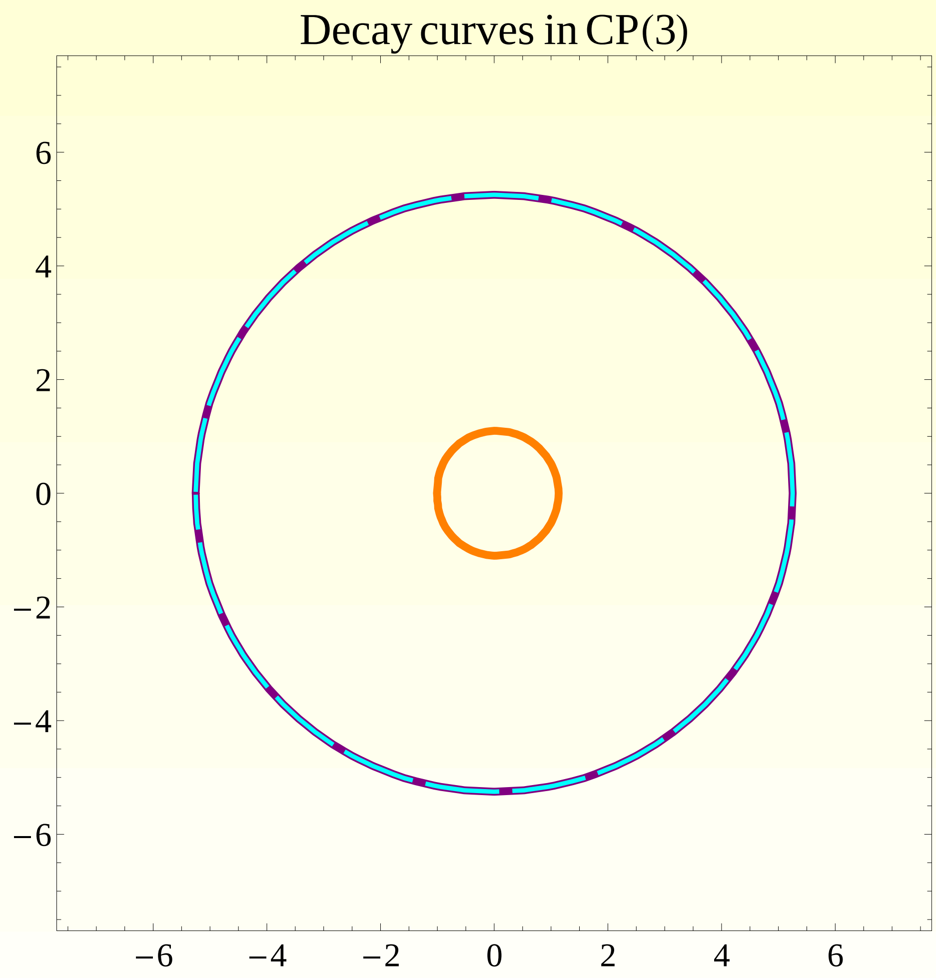


The Curves for CP(2)



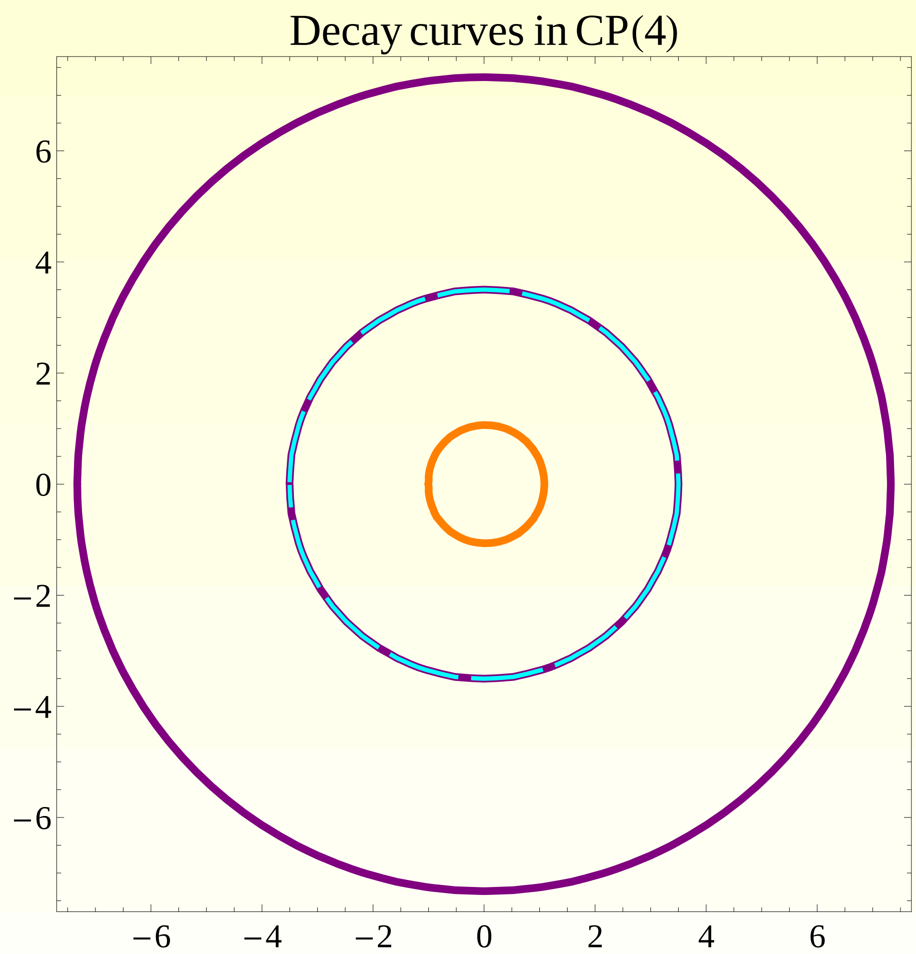
the plane has been rescaled in order to fit two curves

The Curves for CP(3)



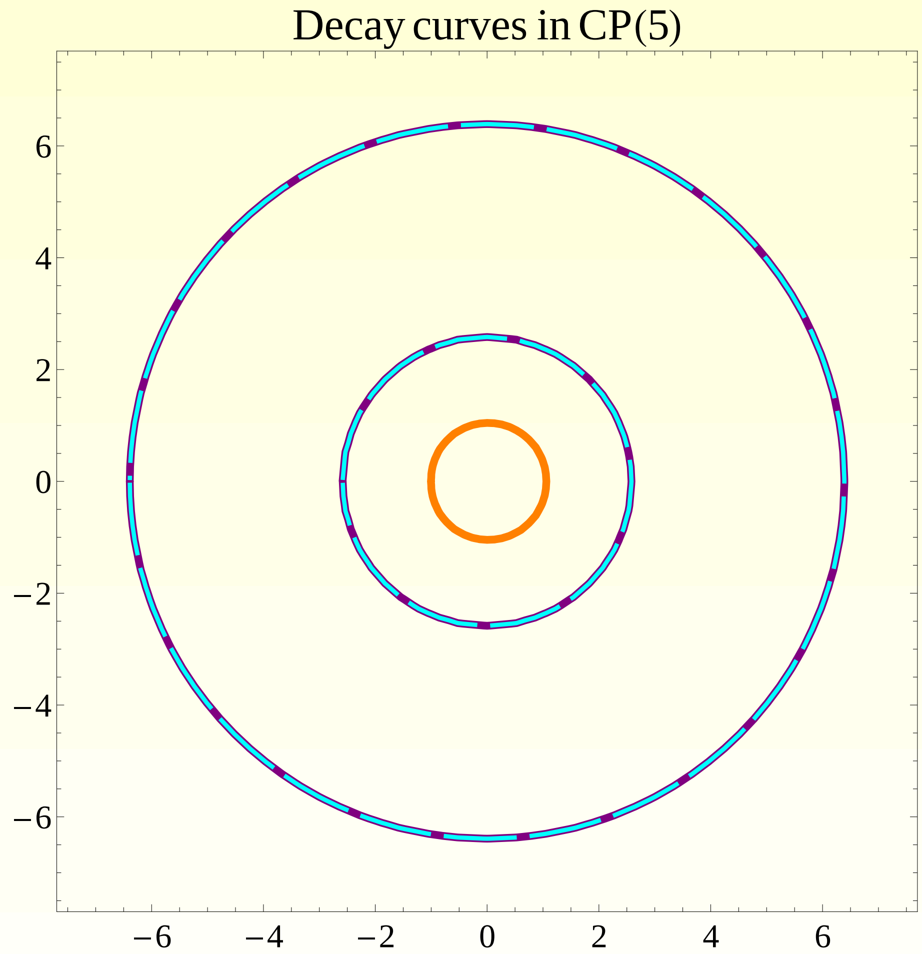
the plane has been rescaled in order to fit the curves

The Curves for CP(4)



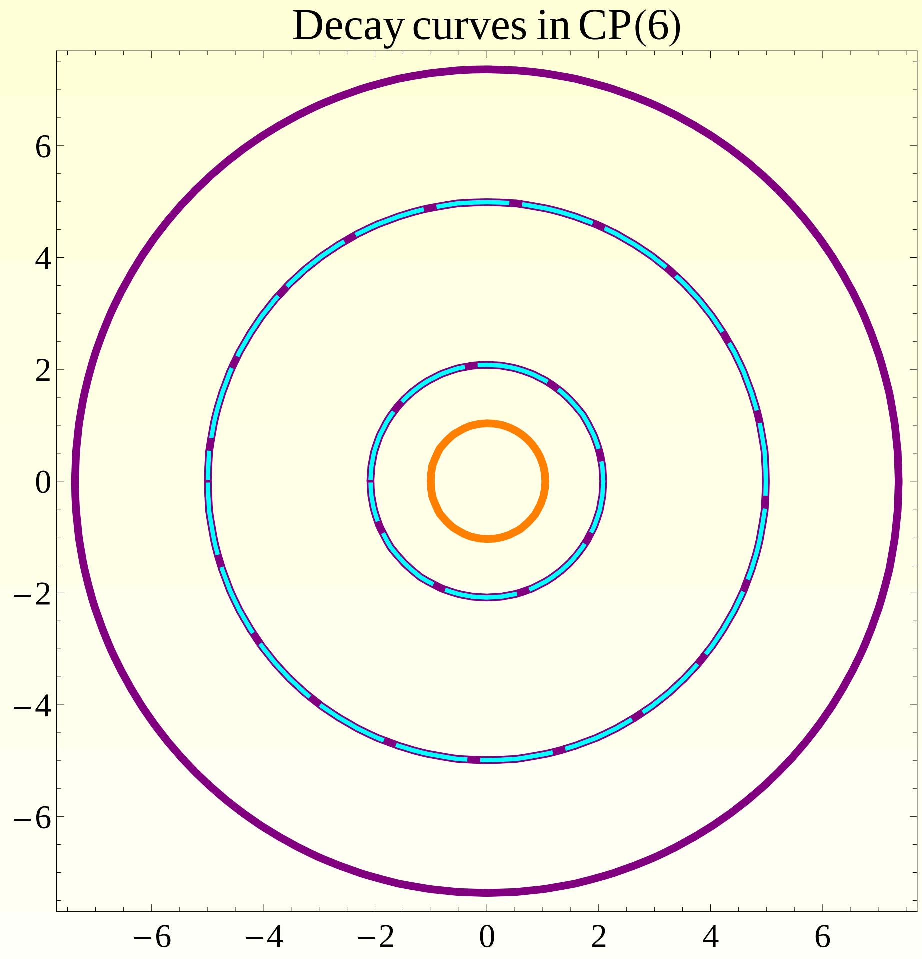
the plane has been rescaled in order to fit the curves

The Curves for CP(5)



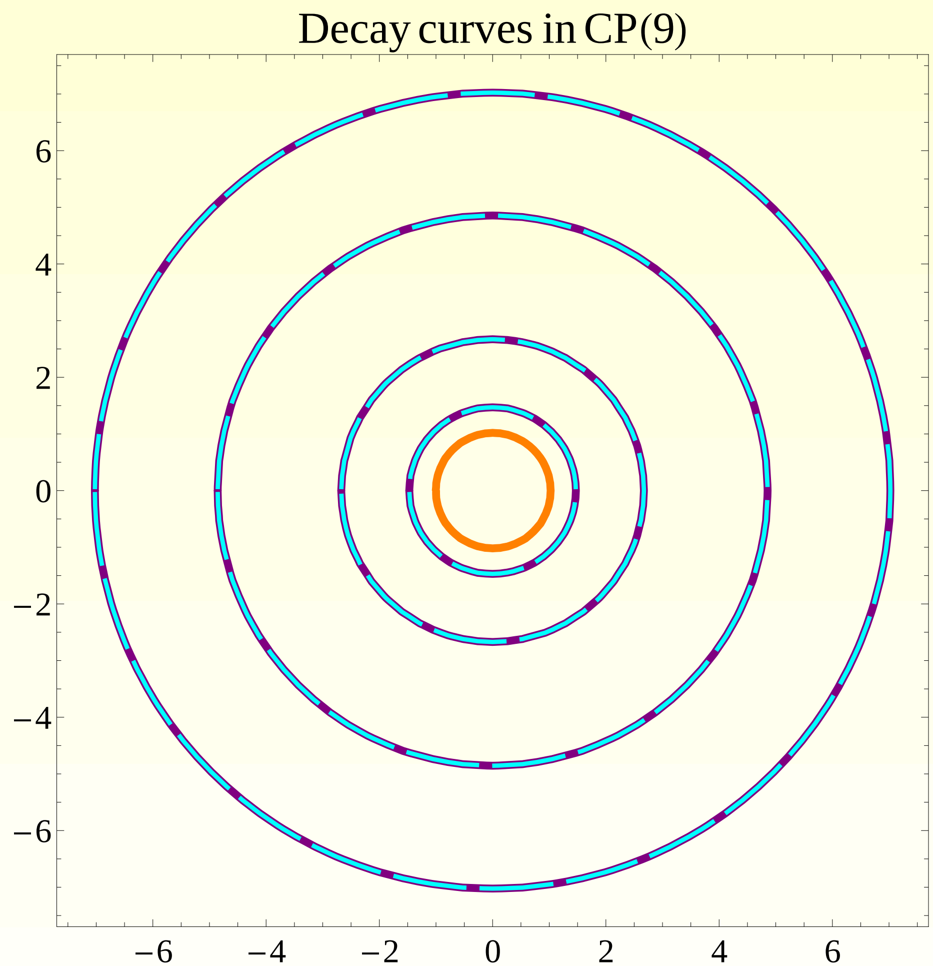
the plane has been rescaled in order to fit the curves

The Curves for CP(6)



the plane has been rescaled in order to fit the curves

The Curves for CP(9)



the plane has been rescaled in order to fit the curves

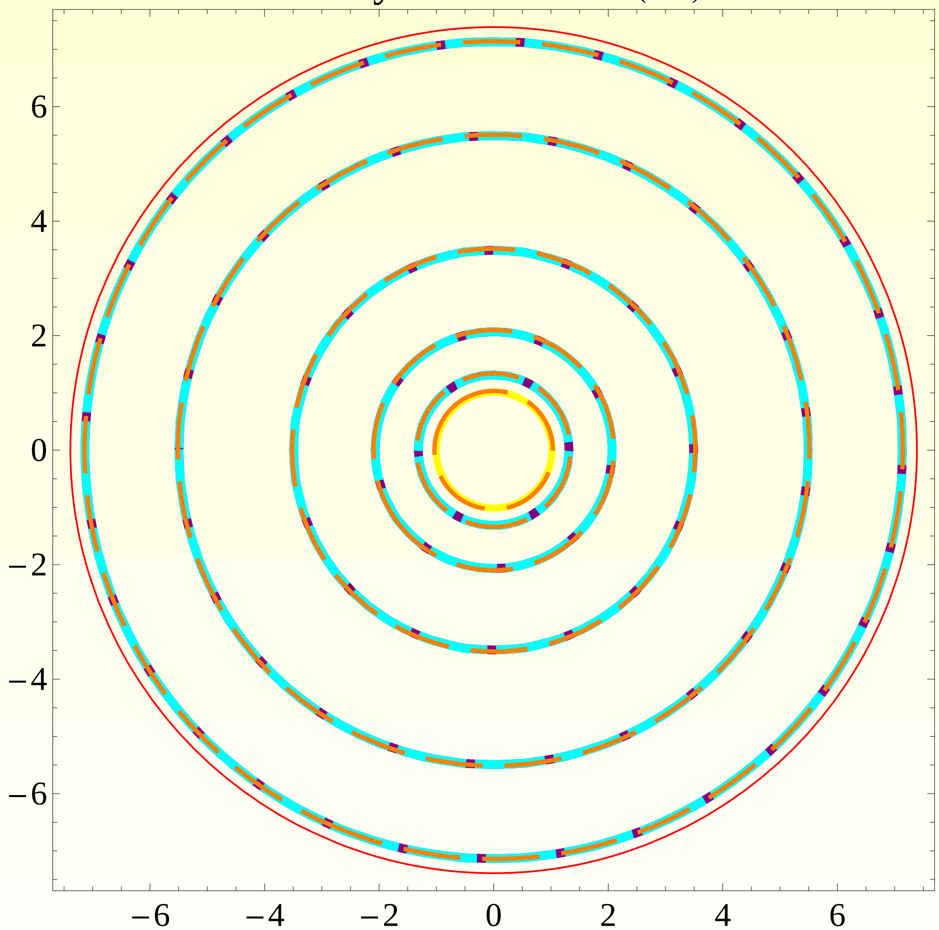
At larger N all the curves become circles with radii

$$\left|m_0\right| \quad = \quad e^{1\,-\,\cos\,\frac{2\,k\,-\,1}{N}\,\pi}\,, \qquad\qquad k \;=\; 1, ..., N-1\,.$$

At larger N all the curves become circles with radii

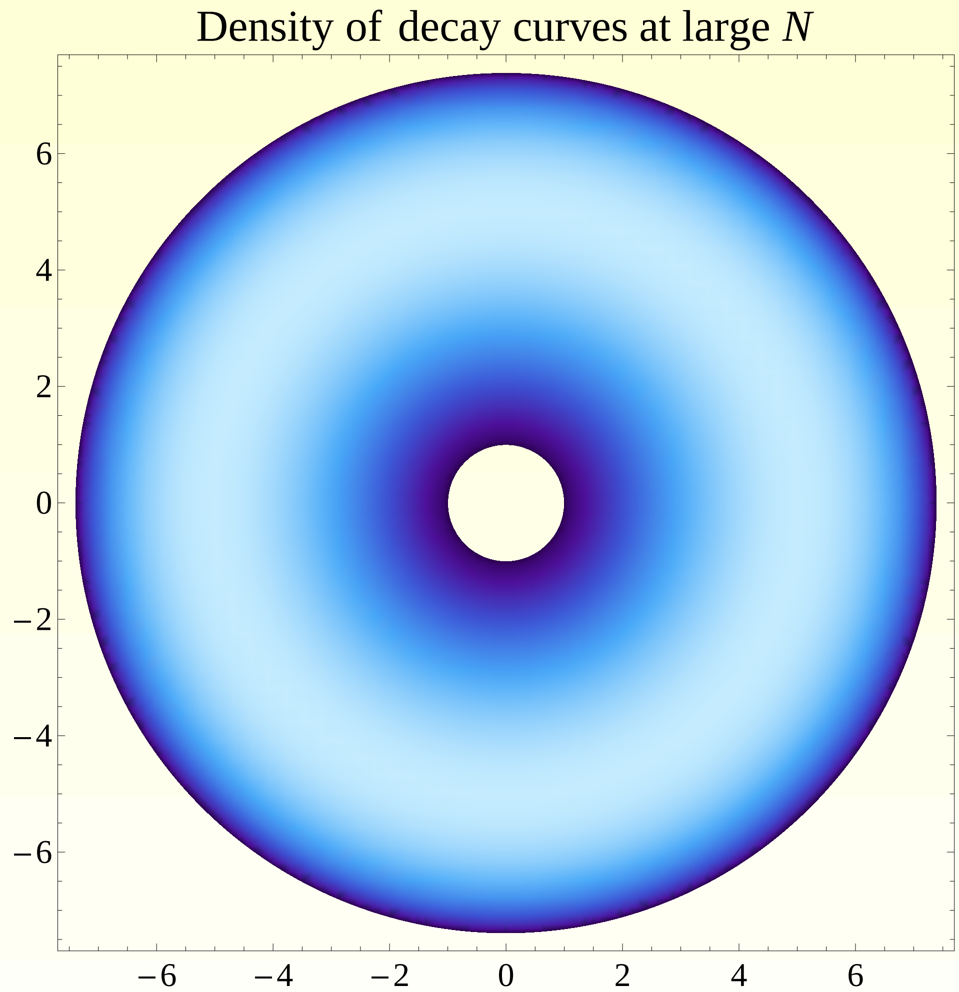
$$|m_0| = e^{1-\cos\frac{2k-1}{N}\pi}, \qquad k = 1, ..., N-1\,.$$

Decay curves in CP(11)



the plane has been rescaled in order to fit the curves

At larger N the curves fill the whole interval $1 \;\; \ldots \;\; e^2$



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We formulated three criteria which must be met

Based on the strong coupling spectrum, and AD points, we conclude that the weak-coupling spectrum must contain $N-1$ towers:

$$\begin{array}{lcl}
\vec{N}_{(1)} & = & (-n_{(1)}+1, \quad n_{(1)}, \quad 0, \quad 0, \quad \ldots, \quad 0)\,, \\
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. \quad . & . & . \quad . \quad . \quad . \quad . \quad . \\
\vec{N}_{(N-1)} & = & (\quad -n_{(N-1)}, \; n_{(N-1)}, \quad 0, \quad 0, \quad \ldots, \quad 1)\,.
\end{array}$$

In terms of masses

$$m_{\text{BPS}} \;=\; U_0(m_0) \;+\; i\, n_{(k)} \cdot (m_1 - m_0) \;+\; i\, m_k\,,$$

for the towers $k = 1, \ldots, N-1$.

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Implications for SQCD are to be determined

THANK YOU!