Spin and charge correlations in quantum dots: The effect of level statistics

Igor Burmistrov

Landau Institute

In collaboration with

Yuval Gefen (Weizmann Institute, Rehovot, Israel)
Mikhail Kiselev (ICTP, Triest, Italy)

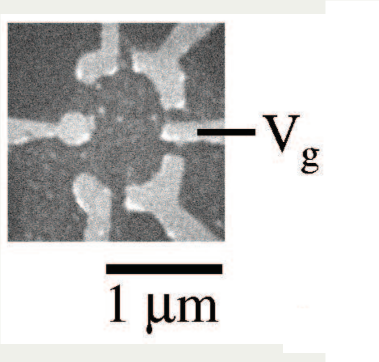
Acknowledgement to
V. Kravtsov, Yu. Makhlin, M. Skvortsov

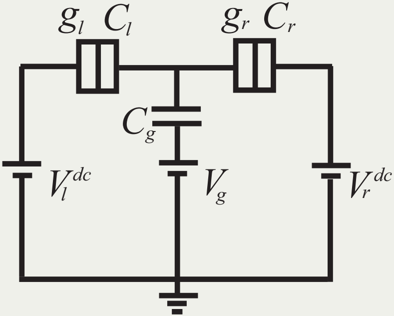
JETP Letters 92, 766 (2010)

Symposium on Theoretical and Mathematical Physics, St. Peterburg, Russia 2011

Motivation

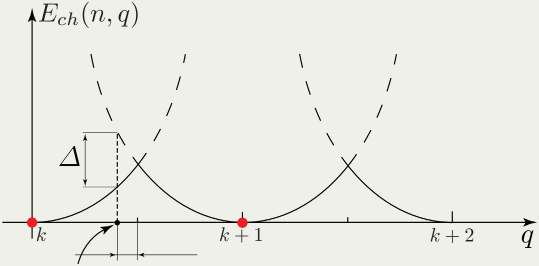
- Transport through quantum dots (QDs)
  - metalic grains, e.g. Al, Au, Pd, with diameter $1 - 10\,nm$
  - large molecules
  - regions of confinement for electrons in 2DEG with typical size $1\,\mu m$





Charging energy $E_c = e^2/L \approx 10K$ for $L = 1\mu m$

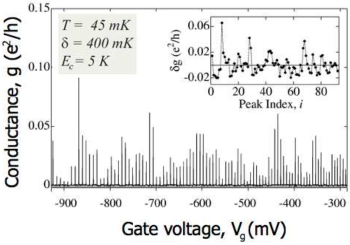
$$\frac{(Q-q)^2}{2C} \Longrightarrow \qquad \hat{H} = E_c(\hat{n}-q)^2, \qquad q \equiv N_0 = C_g V_g/e$$



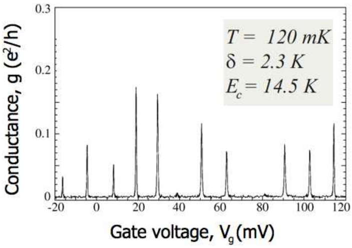
Periodic dependence on V_g

Coulomb blockade at $N_0 \neq k + 1/2$

from Patel et al. (1998)



from Luscher et al. (2001)



No periodicity in gate voltage

Different spacings between conductance peaks

Different heights of conductance peaks

Possible explanation: the effect of single-particle level fluctuations

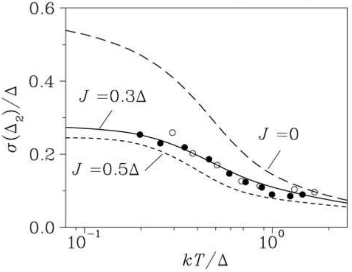
Parameters of QDs

$E_{\rm Th}$	-	Thouless energy
E_c	-	charging energy
T	-	temperature
δ	-	mean level spacing
J	-	exchange energy
$g_{l,r}$	-	left/right tunnel conductances
N_0	-	external charge

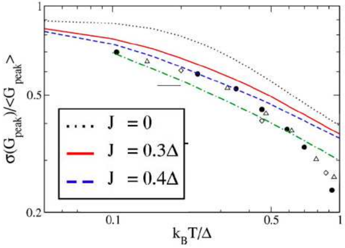
Thouless conductance $g_{\rm Th} = E_{\rm Th}/\delta \gg 1$

Low temperatures: $T \ll E_{\rm Th}, E_c$

from Alhassid&Rupp (2003)



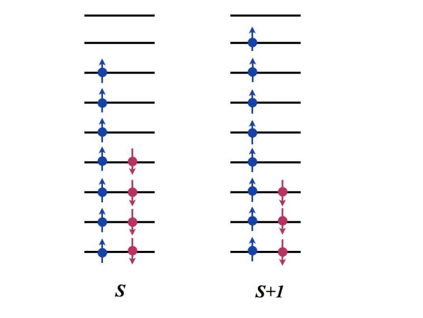
from Usaj&Baranger (2003)



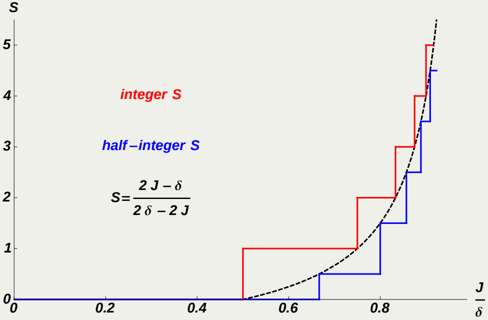
Data points from Patel et al.

Theory is applicable at relatively low temperatures:

Exchange J>0 is important



Kurland, Aleiner, Altshuler, PRB (2000)



$$H_J = -J\hat{\boldsymbol{S}}^2 \qquad E_{S+1} - E_S \equiv (2S+1)\delta = J(2S+2)$$

- Nearly ferromagnetic materials
  - Pd $J/\delta = 0.9$
  - YFe_2Zn_{20} $J/\delta = 0.94$ Jia, Bud'ko, Samolyuk, Canfield, Nat Phys (2007)

- Can we find signatures of exchange J in physical observables, e.g. conductance, tunneling DOS, spin susceptibility, at $T \gg \delta, J$?

- How the single-particle level statistics affects results, in particular, the mesoscopic Stoner instability, at $T \gg \delta, J$?

Universal Hamiltonian

- Hamitonian

Kurland, Aleiner, Altshuler PRB (2000); Aleiner, Brouwer, Glazman, Phys Rep (2002)

$$\mathcal{H} = \sum_{\alpha,\sigma} \epsilon_{\alpha,\sigma} a^\dagger_{\alpha,\sigma} a_{\alpha,\sigma} + E_c \left(\hat{N} - N_0\right)^2 - J \hat{\boldsymbol{S}}^2$$

$\epsilon_{\alpha,\sigma} = \epsilon_\alpha + b\sigma/2$ – single particle spectrum in the presence of Zeeman splitting $b = g\mu_B B$

$\hat{N} = \sum_{\alpha,\sigma} a^\dagger_{\alpha,\sigma} a_{\alpha,\sigma}$ – particle number operator

$\hat{\boldsymbol{S}}_{\sigma\sigma'} = \frac{1}{2}\sum_{\alpha,\sigma} a^\dagger_{\alpha,\sigma} \boldsymbol{\sigma}_{\sigma\sigma'} a_{\alpha,\sigma'}$ – spin operator

- Spin susceptibility

$$\chi = T \frac{\partial^2}{\partial b^2} \ln \operatorname{Tr} \exp\left(-\beta\mathcal{H}\right)$$

- Tunneling DOS

$$\nu(\varepsilon) = -\frac{1}{\pi} \operatorname{Im} \sum_{\alpha,\sigma} \mathcal{G}^R_{\alpha\sigma;\alpha\sigma}(\varepsilon), \quad \mathcal{G}^R_{\alpha_1\sigma_1;\alpha_2\sigma_2}(t_1, t_2) = -i\theta(t_1 - t_2) \left\langle \left\{ a_{\alpha_1\sigma_1}(t_1), a^\dagger_{\alpha_2\sigma_2}(t_2) \right\} \right\rangle$$

Standard approach

- Non-Abelian action in imaginary time after Hubbard-Stratonovich transformation

$$\mathcal{S} = \int_0^\beta d\tau \left[\sum_{\alpha,\sigma,\sigma'} \overline{a}_{\alpha,\sigma} \left[\partial_\tau - \epsilon_{\alpha,\sigma} + \mu + i\phi + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\theta}}{2} \right]_{\sigma\sigma'} a_{\alpha,\sigma'} + \frac{\boldsymbol{\theta}^2}{4J} + \frac{\phi^2}{4E_c} - iN_0\phi \right]$$

- ϕ-field can be removed by gauge $U(1)$ transformation

Ambegaokar, Eckern, Schoen (1982), ..., Kamenev, Gefen (1996), Efetov, Tschersich (2003), Sedlmayr, Yurkevich, Lerner (2006)

- $\boldsymbol{\theta}$-field can be removed only in the Ising case by $U(1)$ transformation

Kiselev, Gefen (2006)

- In general $SU(2)$ case, $\boldsymbol{\theta}$-field cannot be removed by gauge transformation.

The standard trick does not apply for $SU(2)$ case!

Spin problem

- The Hubbard-Stratonovich transformation of evolution operator

$$e^{\mp itJ\hat{\boldsymbol{s}}^2} = \prod_\alpha \int \mathcal{D}[\boldsymbol{\theta}]\, e^{\pm \frac{i}{4J}\int_0^t dt'\, \boldsymbol{\theta}^2} \mathcal{T} e^{i\int_0^t dt'\, \boldsymbol{\theta}\hat{\boldsymbol{s}}_\alpha}$$

Time-ordering $\mathcal{T}$ due to noncommutativity of the spin operators!

- Wei-Norman-Kolokolov transformation

Wei, Norman, J. Math. Phys. (1963), Kolokolov, Ann. Phys. (1990)

$$\mathcal{T} e^{i\int_0^t dt'\, \boldsymbol{\theta}\hat{\boldsymbol{s}}} = e^{\hat{s}_- \psi_+(t)} e^{i\hat{s}_z \int_0^t dt' \rho(t')} \exp\left[i\hat{s}_+ \int_0^t dt' \psi_-(t') e^{-i\int_0^{t'} d\tau \rho(\tau)} dt'\right]$$

$$\theta_z = \rho - 2\psi_+\psi_-, \qquad \frac{\theta_x - i\theta_y}{2} = \psi_-, \qquad \frac{\theta_x + i\theta_y}{2} = -i\dot{\psi}_+ + \rho\psi_+ - \psi_-\psi_+^2$$

$$Z = \sum_{n_{\uparrow,\downarrow}\in\mathbb{Z}} Z_{n_\uparrow} Z_{n_\downarrow} \frac{\sinh(\beta b(2m+1)/2)}{\sinh(\beta b/2)} \exp\left[-\beta E_c (n-N_0)^2 + \beta J m(m+1)\right]$$

where $n = n_\uparrow + n_\downarrow$, $m = (n_\uparrow - n_\downarrow)/2$ and the partition function for n electrons

$$Z_n = \oint\limits_{|z|=1} \frac{dz}{2\pi i}\, z^{-n-1} \prod_\gamma \left(1 + z\, e^{-\beta\epsilon_\gamma}\right)$$

N.B.: At $b = 0$ it coincides with Alhassid&Rupp (2003)

Exact results/ TDOS

$$\nu_\sigma(\varepsilon) = \frac{1+e^{-\beta\varepsilon}}{2Z} \sum_{\alpha, n_{\uparrow,\downarrow}\in\mathbb{Z}} e^{-\beta[E_c(n-N_0)^2 - Jm(m+1)]}$$

$$\times\, \delta\Big(\varepsilon - \epsilon_{\alpha\sigma} - E_c(2n-2N_0+1) - J(m+1/4)\Big)$$

$$\times \Big\{ e^{\beta b\sigma/2} Y(\beta b\sigma/2, 2m+1)\Big[Z_{n_\uparrow}(\epsilon_\alpha)Z_{n_\downarrow} - Z_{n_\uparrow+1}Z_{n_\downarrow-1}(\epsilon_\alpha)\Big]$$

$$- Y(-\beta b\sigma/2, -2m)\Big[Z_{n_\uparrow}Z_{n_\downarrow}(\epsilon_\alpha) - Z_{n_\uparrow}(\epsilon_\alpha)Z_{n_\downarrow}\Big]\Big\}$$

where

$$Y(z,x) = \frac{e^{(x-1)z}}{\sinh z} - \frac{\sinh(xz)}{x\sinh^2 z}, \qquad Z_n(\epsilon_\alpha) = \oint_{|z|=1} \frac{dz}{2\pi i} z^{-n-1} \prod_{\gamma\neq\alpha} \big(1 + z\, e^{-\beta\epsilon_\gamma}\big)$$

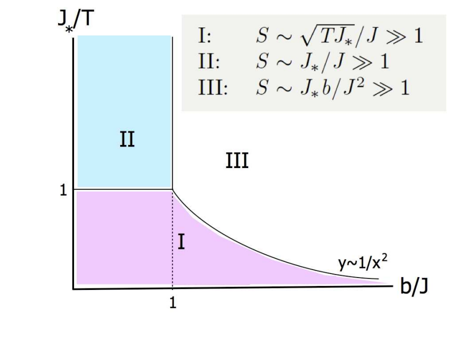
N.B.: TDOS coincides with result of Sedlmayr, Yurkevich, Lerner (2006) at $J = b = 0$

Spin susceptibility / No fluctuations of single-particle levels

$$\chi(T, b) = \frac{1}{2(\delta - J)} - \frac{\delta^2}{4T(\delta - J)^2} \sinh^{-2}\left(\frac{\delta b}{2T(\delta - J)}\right) + \frac{1}{4T} \sinh^{-2}\left(\frac{b}{2T}\right)$$

- Emergence of new energy scale: renormalized exchange energy

$$J_\star = \frac{J}{1 - J/\delta} \gg \delta, J \qquad \text{at} \qquad \delta - J \ll \delta$$



Fluctuations of single-particle levels

- Fluctuations of single-particle thermodynamic density of states ($dn/d\mu$)

Dyson (1962), Mehta, Dyson (1963), Efetov (1982)

$$\langle \Delta\nu_0(E)\Delta\nu_0(E+\omega)\rangle = \frac{1}{\delta^2}\left[\delta(\omega/\delta) - R_{U/O}(\pi\omega/\delta)\right]$$

Unitary ensemble : $$R_U(x) = \frac{\sin^2 x}{x^2}$$

Orthogonal ensemble : $$R_O(x) = \frac{\sin^2 x}{x^2} + \left(\frac{d}{dx}\frac{\sin x}{x}\right)\int_x^\infty \frac{\sin t}{t}\,dt$$

Fluctuations of level spacing at $T \gg \delta$:

$$\frac{\overline{(\Delta-\delta)^2}}{\delta^2} = \frac{c\delta^2}{\beta_{U/O}T^2} \ll 1, \qquad c \approx 0.02, \qquad \beta_U = 2\beta_O = 2$$

Common wisdom: Fluctuations of single-particle levels are not important at $T \gg \delta$

Fluctuations of single-particle levels near Stoner instability

Renormalized exchange energy $\mathcal{J}$:

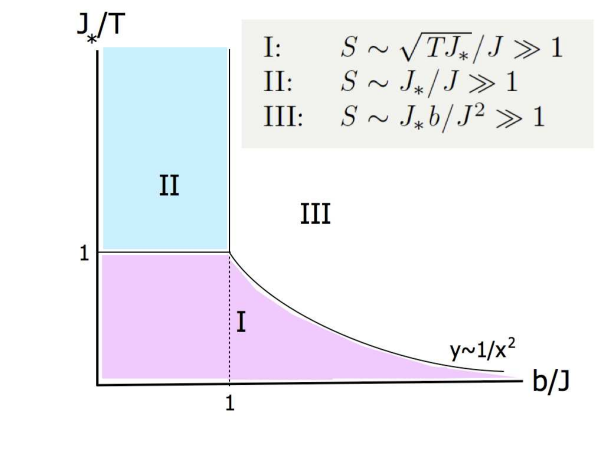
$$\frac{1}{\mathcal{J}} = \frac{1}{J} - \frac{1}{\Delta}$$

Fluctuations of Δ can lead to $\mathcal{J} < 0$ (Stoner instability)

$$\frac{\overline{(\Delta-\delta)^2}}{\delta^4} \ll \frac{1}{J_*^2} \equiv \left(\frac{1}{J} - \frac{1}{\delta}\right)^2, \qquad \Longrightarrow \qquad J_* \ll T$$

Near Stoner instability level fluctuations are not important at $T \gg J_* \gg \delta, J$

Spin susceptibility in region I: $J_* \max\{1, b^2/J^2\} \ll T$



Spin susceptibility in region I: $J_* \max\{1, b^2/J^2\} \ll T$

Spin susceptibility for a given realization of levels:

$$\chi(T,b) = \frac{\mathcal{J}}{2J\delta}\left[1 + \frac{\mathcal{J}}{6\,T} - \frac{\mathcal{J}^3 b^2}{120\,T^3 J^2}\right]$$

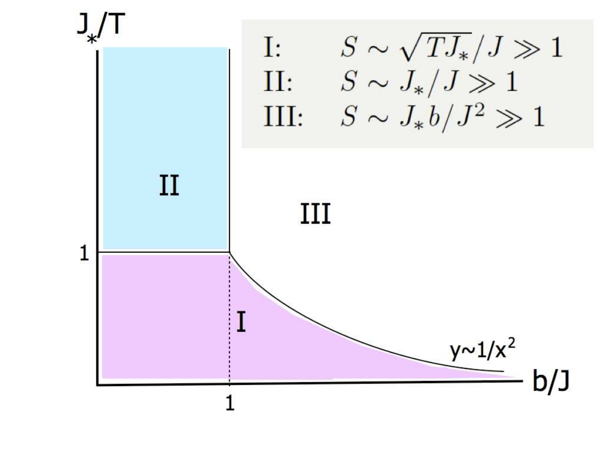
Averaged spin susceptibility:

$$\overline{\chi(T,b)} = \frac{J_\star}{2J\delta}\left\{1 + \frac{J_\star}{6\,T} + \frac{c}{\beta_A}\frac{J_\star^2}{T^2}\left[1 + \frac{J_\star}{2\,T}\right] - \frac{J_\star^3 b^2}{120\,T^3 J^2}\left[1 + \frac{10\,c}{\beta_A}\frac{J_\star^2}{T^2}\right]\right\}$$

where $c \approx 0.02$, $\beta_A = 2\,(1)$ for unitary (orthogonal) ensemble.

Averaged spin susceptibility is Fermi-liquid like: $\overline{\chi(T,b)} = [2(\delta - J)]^{-1} + \dots$

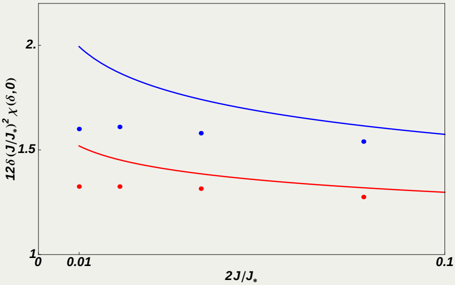
Spin-susceptibility in region II: $T \ll J_*$ and $b \ll J$



Spin-susceptibility in region II: $T \ll J_*$ and $b \ll J$

Averaged spin susceptibility:

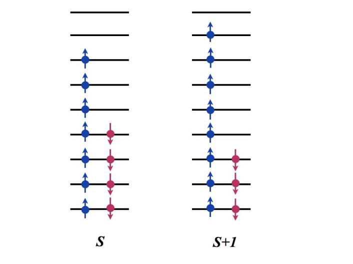
$$\overline{\chi(T, b=0)} = \frac{1}{3T}\left(\frac{J_*}{2J}\right)^2 \left[1 + \frac{2}{\beta_A \pi^2}\left(\ln \frac{J_*}{2T} + c_2\right)\right], \qquad c_2 \approx 1.43$$



Points are numerical simulations from Kurland, Aleiner, Altshuler (2000)

At $T \ll J_*$, $b \ll J$ level fluctuations are important but numerically small

Mesoscopic Stoner instability ($T = 0$): / Effect of level statistics

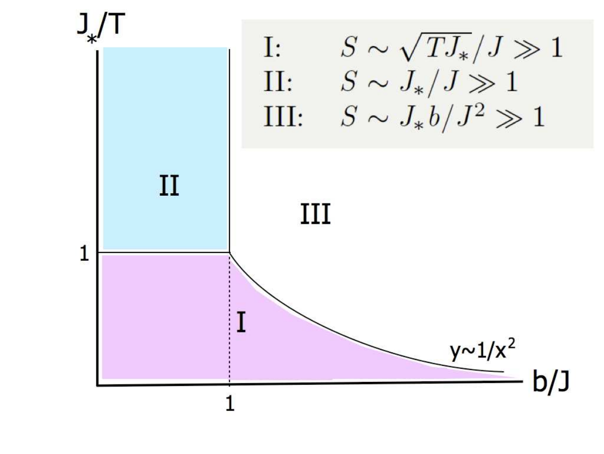


$$E_{S+1} - E_S \equiv (2S+1)\delta + \Delta E = J(2S+2), \qquad \Delta E = \Delta n_{2S}\delta$$

$$S \equiv \overline{S} + \Delta S = \frac{J_*}{2J}\Big[1 - \Delta n_{2S}\Big] \quad \Longrightarrow \quad \overline{\boldsymbol{S}^2} = \left(\frac{J_*}{2J}\right)^2 \Big[1 + \overline{(\Delta n_{2S})^2}\Big]$$

$$\overline{\chi} = \frac{\overline{\boldsymbol{S}^2}}{3T}, \qquad \overline{(\Delta n_{2S})^2} = \frac{2}{\beta_A \pi^2}\left(\ln \frac{J_*}{J} + \text{const}\right)$$

Spin-susceptibility in region III: $J \max\{1, \sqrt{T/J_*}\} \ll b$



Spin-susceptibility in region III: $J\max\{1,\sqrt{T/J_*}\} \ll b$

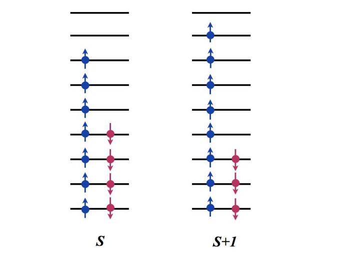
Averaged spin susceptibility:

$$\overline{\chi(T,b)} = \frac{J_*}{2J^2}\left\{1 + \frac{2\,TJ^2}{J_* b^2}\left[1 - \frac{2}{\beta_A \pi^2}\frac{J_*}{T}\right]\right\}$$

At $J\max\{1,\sqrt{T/J_*}\} \ll b$ level fluctuations are suppresed by magnetic field

Averaged spin susceptibility is Fermi-liquid like: $\overline{\chi(T,b)} = [2(\delta - J)]^{-1} + \dots$

Mesoscopic Stoner instability ($T = 0$): / Effect of magnetic field



$$E_{S+1} - E_S \equiv (2S+1)\delta - b + \Delta E = J(2S+2), \qquad \Delta E = \Delta n_{2S}\delta$$

$$S \equiv \overline{S} + \Delta S = \frac{b - \delta \Delta n_{2S}}{2(\delta - J)} \quad \Longrightarrow \quad \overline{\Delta S^2} = \frac{\delta^2}{b^2}\left(\overline{S}\right)^2 \overline{(\Delta n_{2\overline{S}})^2}$$

$$\overline{\chi} = \frac{\partial \overline{S}}{\partial b} = \frac{1}{2(\delta - J)}, \qquad \overline{(\Delta n_{2S})^2} = \frac{2}{\beta_A \pi^2}\left(\ln \frac{J_*}{J} + \text{const}\right)$$

Charge and spin separation

$$\mathcal{H} = \sum_{\alpha,\sigma} \varepsilon_\alpha a^\dagger_{\alpha,\sigma} a_{\alpha,\sigma} + E_c \left(\hat{N} - N_0\right)^2 - J\hat{\boldsymbol{S}}^2,$$

$$\hat{N} = \sum_{\alpha,\sigma} a^\dagger_{\alpha,\sigma} a_{\alpha,\sigma}, \quad \hat{\boldsymbol{S}}_{\sigma\sigma'} = \frac{1}{2}\sum_{\alpha,\sigma} a^\dagger_{\alpha,\sigma} \vec{\boldsymbol{\sigma}}_{\sigma\sigma'} a_{\alpha,\sigma'}$$

- Functional integral in imaginary time
- Decoupling Coulomb interaction $E_c(\hat{N} - N_0)^2$ by the Hubbard-Stratonovich field ϕ

$$Z = \int_{-\pi T}^{\pi T} d\phi_0\, \mathcal{D}(\tau_1, \tau_1|\phi_0) \tilde{Z}(\phi_0)$$

Charge problem:

$$\mathcal{D}(\tau_1, \tau_2|\phi_0) = \sum_{m\in\mathbb{Z}} \int \mathcal{D}[\tilde{\phi}] e^{-\int_0^\beta d\tau \frac{\tilde{\phi}^2(\tau)}{4E_c} - i\int_{\tau_1}^{\tau_2} d\tau \tilde{\phi}(\tau) - \frac{\pi^2 T}{E_c}(m+\frac{\beta\phi_0}{2\pi})^2 + 2\pi i N_0 (m+\frac{\beta\phi_0}{2\pi}) - i2\pi m T(\tau_1-\tau_2)}$$

Spin problem:

$$\tilde{Z}(\phi_0) = \operatorname{Tr} e^{-\beta\mathcal{H}_J}, \qquad \mathcal{H}_J = \sum_{\alpha,\sigma} \tilde{\varepsilon}_\alpha a^\dagger_{\alpha,\sigma} a_{\alpha,\sigma} - J\hat{\boldsymbol{S}}^2 \qquad \tilde{\varepsilon}_\alpha = \varepsilon_\alpha - i\phi_0$$

N.B.: Charge and spin are entangled due to integration over ϕ_0!

Spin problem

- The Hubbard-Stratonovich transformation of exchange interaction:

$$e^{\mp itJ\hat{\mathbf{s}}^2} = \lim_{N\to\infty} \prod_\alpha \prod_{n=1}^N \int d\boldsymbol{\theta}_n \, e^{\pm \frac{i}{4J} t\boldsymbol{\theta}_n^2/N} e^{it\boldsymbol{\theta}_n \hat{\mathbf{s}}_\alpha/N} = \prod_\alpha \int \mathcal{D}[\boldsymbol{\theta}] \, e^{\pm\frac{i}{4J}\int_0^t dt' \, \boldsymbol{\theta}^2} \mathcal{T} e^{i\int_0^t dt' \, \boldsymbol{\theta}\hat{\mathbf{s}}_\alpha}$$

Time-ordering $\mathcal{T}$ due to noncommutativity of the spin operators!

- Wei-Norman-Kolokolov transformation

$$\mathcal{T} e^{i\int_0^t dt' \, \boldsymbol{\theta}\hat{\mathbf{s}}} = e^{\pm \hat{S}_\mp \psi_\pm(t)} e^{i\hat{S}_z \int_0^t dt' \rho(t')} \exp\left[i\hat{S}_\pm \int_0^t dt' \psi_\mp(t') e^{\mp i\int_0^{t'} d\tau \rho(\tau)} dt' \right] e^{\mp \hat{S}_\mp \psi_\pm(0)}$$

$$\theta_z = \rho - 2\psi_+\psi_-, \quad \frac{\theta_x \mp i\theta_y}{2} = \psi_\mp, \quad \frac{\theta_x \pm i\theta_y}{2} = \mp i\dot{\psi}_\pm + \rho\psi_\pm - \psi_\mp\psi_\pm^2, \quad \boldsymbol{\theta}^2 = \rho^2 \mp 4i\psi_\mp\dot{\psi}_\pm$$

N.B.: The Jacobian of transformation from $\boldsymbol{\theta}$ to $\rho, \psi_\pm$ is $\mathcal{J} = \exp\left[\frac{i}{2}\int_0^t dt' \rho(t')\right]$
Initially, $\theta_{x,y,z}$ are real variables, but now $(\theta_x - i\theta_y)^* \neq \theta_x + i\theta_y$
We impose constraints $\psi_+ = \psi_-^*$ and $\rho = -\rho^*$

From spin problem to quantum mechanics

- Two sets of variables: $\boldsymbol{\theta_{1,2}}$ $\Longrightarrow$ two sets of new variables $\rho_{1,2}$ and $\psi^{\pm}_{1,2}$
- We choose initial condition $\psi_1^+(0) = \psi_2^-(0) = 0$
- Exact integration over $\psi^{\pm}_{1,2}$
- New variables: $\rho_{1,2}(t) = \mp i\dot{\xi}_{1,2}$, $\xi_1(0) = \xi_2(0)$, $\xi_1(t_1) + \xi_2(t_2) = 0$,

$$\tilde{Z}[\phi_0] = \prod_\gamma \left(- \oint\limits_{|z|=1} \frac{dz_\gamma}{2\pi i z_\gamma^2} e^{-z_\gamma \left[1+e^{-2\beta\tilde{\varepsilon}_\gamma}\right]} \right) \int\limits_0^\infty \frac{dy}{4yd} e^{-y} \int\limits_{-\infty}^\infty d\xi_1\, d\xi_2 \prod_\gamma e^{-2z_\gamma e^{-\beta\tilde{\varepsilon}_\gamma} \cosh\frac{\xi_1-\xi_2}{2}}$$

$$\times \delta\left(\xi_1 + \xi_2 + 2\ln\left[4y\sum_\gamma z_\gamma e^{-\beta\tilde{\varepsilon}_\gamma}\right]\right) \langle \xi_1| e^{-i\mathcal{H}_0 t_1} e^{-\xi} e^{i\mathcal{H}_0(t_1+i\beta)}|\xi_2\rangle$$

$$\mathcal{H}_0 = -J\frac{\partial^2}{\partial\xi^2} + \frac{J}{4}e^{-\xi}, \quad E_\nu = J\nu^2, \quad \Psi_\nu(\xi) = \frac{2}{\pi}\sqrt{\nu\sinh 2\pi\nu}\,K_{2i\nu}(e^{-\xi/2})$$

The final step

$$\tilde{Z}[\phi_0] = \int_{-\infty}^{\infty} dh\, h \sinh(\beta h) e^{-\beta h^2/J} \prod_{\gamma} \left(1 + e^{-\beta(\tilde{\varepsilon}_\gamma + h)}\right) \left(1 + e^{-\beta(\tilde{\varepsilon}_\gamma - h)}\right)$$

- The grand partition function

$$Z = \frac{1}{\sqrt{\pi\beta J}} e^{-\beta J/4} e^{-\beta b^2/4J} \sum_{n\in\mathbb{Z}} e^{-\beta E_c (n-N_0)^2} \int_{-\pi T}^{\pi T} \frac{d\phi_0}{2\pi T} e^{i\beta\phi_0 n}$$
$$\times \int_{-\infty}^{\infty} dh\, \sinh(h) \frac{\sinh(bh/J)}{\sinh(\beta b/2)} e^{-Th^2/J} \prod_{\sigma=\pm} e^{-\beta\Omega_0(\mu - i\phi_0 + Th\sigma)}$$
$$e^{-\beta\Omega_0(\mu)} = \prod_{\alpha} \left(1 + e^{-\beta(\varepsilon_\alpha - \mu)}\right)$$

Region II: $\delta \ll T \ll J_*$ and $b \ll J$ / The effect of level fluctuations

- Partition function $Z = Z_C Z_S$
- Charging sector

$$Z_C = \sqrt{\frac{\Delta}{4\pi T}} \sum_{n\in\mathbb{Z}} e^{-\beta[E_c(n-N_0)^2-(\tilde{\mu}-\mu)n+2\Omega_0(\tilde{\mu})]}$$

- Spin sector $(b=0)$

$$Z_S = \frac{2}{\sqrt{\pi}(\beta J)^{3/2}} e^{-\beta J/4} \int_{-\infty}^{\infty} dh\, h \sinh(h)\, e^{-Th^2/J_* - F(h)}$$

where the Gaussian random function

$$F(h) = -\int_{-\infty}^{\infty} dE\, \Delta\nu_0(E)\, \ln\left[1 + \frac{\cosh h - 1}{2\cosh^2(E/2T)}\right]$$

- Averaged spin susceptibility:

$$\overline{\chi(T, b=0)} = \frac{1}{3}\frac{\partial \overline{\ln Z_S}}{\partial J}$$

In the absence of $F(h)$ the typical value of h is of the order of $\gamma = J_*/T \gg 1$

- Averaged free energy:

$$\overline{\ln Z_S} = \overline{\ln \int_0^\infty dh\, h\, \exp[\gamma f(h)]} \quad , \qquad f(h) = h - h^2 - \frac{1}{\gamma} F(\gamma h)$$

- Saddle-point approximation is justified by $\gamma \gg 1$:

$$\overline{\ln Z_S} = \gamma \overline{f(h_*)}, \qquad f'(h_*) = 1 - 2h_* - F'(\gamma h_*) = 0$$

- Approximate solution (small shift of h_* from $1/2$)

$$\overline{f(h_*)} = \frac{1}{4}\left\{1 + \overline{F'^2(\gamma/2)} + \frac{\gamma^2}{12}\frac{d}{dz}\overline{F''(z)F'^3(z)}\Big|_{z=\gamma/2} + \dots\right\}$$

- Two-point correlation function

$$C_{ab}(h_1, h_2) = \frac{d^{a+b}}{dh_1^a\, dh_2^b} \overline{F(h_1) F(h_2)}$$

- For $T \gg \delta$ and $|h| \gg 1$:

$$C_{00}(h,h) = \frac{4 \ln 2}{\beta_A \pi^2} h^2$$

$$C_{11}(h,h) = \frac{2}{\beta_A \pi^2}\Big[\ln|h| + c_1\Big], \quad c_1 = -\int_0^1 \frac{d\omega}{\omega^2}[1 - \omega \coth \omega] + \int_1^\infty \frac{d\omega\, \ln \omega}{\sinh^2 \omega} \approx 0.43$$

$$C_{01}(h,h) = \frac{4 \ln 2}{\beta_A \pi^2} |h|$$

$$C_{12}(h,h) = \frac{1}{\beta_A \pi^2 |h|}$$

$$C_{02}(h,h) = \frac{4 \ln 2}{\beta_A \pi^2} - C_{11}(h,h)$$

Region II: $\delta \ll T \ll J_*$ and $b \ll J$ / The effect of level fluctuations

$$\overline{f(h_*)} = \frac{1}{4}\left\{1 + C_{11}(\gamma/2) + \frac{\gamma^2}{4}\frac{d}{dz}C_{12}(z)C_{11}(z)\Big|_{z=\gamma/2} + \dots\right\}$$

$$\overline{f(h_*)} = \frac{1}{4}\left\{1 + \frac{2}{\beta_A \pi^2}\left[a_1 \ln\frac{\gamma}{2} + a_2\right]\right\},$$

$$a_1 = 1 - \frac{1}{\beta_A \pi^2} + \dots$$

$$a_2 = c_1 - \frac{c_1 - 1}{\beta_A \pi^2}\dots$$

Conclusions

- Exact analytical results for the spin susceptibility and TDOS in a QD with direct Coulomb and exchange interactions in the presence of Zeeman splitting
- Level fluctuations ehnance the spin susceptibility at $T \ll J_*$
- Magnetic field $b \gg J$ suppresses the effect of level fluctuations

- Future work:
  - Anisotropic exchange $H_J = -J_\perp(\hat{S}_x^2 + \hat{S}_y^2) - J_z \hat{S}_z^2$
  - Interaction in the Cooper channel
  - Dynamic spin susceptibility
  - $I(V)$ curve in the co-tunneling approximation
  - Analysis at low temperatures $T \lesssim \delta$