Fluctuation Properties of Ideal 1/f Noise:

from Statistical Mechanics to Random Matrices,

Riemann Zeta-Function and Burgers Turbulence

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References:

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Definition of ideal 1/f noises:

Random signals such that **spectral power** associated with a given Fourier harmonic is **inversely proportional** to the frequency. Believed to be **frequently encountered** in Nature: voltage fluctuations, non-equilibrium phase transitions, spontaneous brain activity, etc.

- **Periodic version:** random **Fourier series** of the form

$$V(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[v_n e^{int} + \overline{v}_n e^{-int} \right] \,, \quad t \in [0, 2\pi)$$

where $v_n, \overline{v}_n$ are **complex Gaussian i.i.d.** with zero mean and the variance $\langle v_n \overline{v}_n \rangle = 1$. It implies

$$\langle V(t_1) V(t_2) \rangle_V = 2 \sum_{n=1}^{\infty} \frac{1}{n} \cos n(t_1 - t_2) \equiv -2 \ln |2 \sin \frac{t_1 - t_2}{2}|, \quad t \neq t'$$

- **Aperiodic version:** similarly, random Gaussian **Fourier integral** defines a Gaussian process on the whole line $-\infty < t < \infty$ by

$$V(t) = \int_0^{\infty} \frac{d\omega}{\sqrt{\omega}} \left[e^{i\omega t} v(\omega) + e^{-i\omega t} \overline{v}(\omega) \right] \,, \quad \langle V(t_1) V(t_2) \rangle_V = -2 \ln |t_1 - t_2|$$

with $\delta-$correlated complex Gaussian $v(\omega)$. **The corresponding definitions are formal, as sums/integrals do not converge pointwise, and should be understood as random generalized functions (e.g. 1D "projections" of the Gaussian Free Field) or after a proper regularization.**

A regularization:

Subdivide the interval $t \in [0, 2\pi)$ by **finite number** of points $t_k = \frac{2\pi}{M}k$ where $k = 1, \ldots, M < \infty$ and associate with each k Gaussian-distributed real variables V_k with **covariances**

$$\langle V_k V_m \rangle = -2 \ln |2 \sin \frac{t_k - t_m}{2}|, \quad \text{for} \quad k \neq m$$

For the problem to be well-defined we have to choose the **variance** accordingly:

$$\langle V_k^2 \rangle = 2 \ln M + W, \quad \text{with any} \quad W > 0$$

In the limit $M \to \infty$ this is expected to approximate the $2\pi-$ periodic **1/f noise** with the covariance $\langle V(t_1) V(t_2) \rangle_V = -2 \ln |2 \sin \frac{t_1 - t_2}{2}|, \quad t \neq t' \in [0, 2\pi)$.

- Our aim is to understand the statistics of **high/low** and **extreme** values of this **strongly correlated** sequence. **The problem turns out to be intimately connected to the mechanism of freezing transitions in disordered systems theory (Random Energy Models, Dirac fermions in random magnetic field). It has also interesting relations to Liouville Quantum Gravity, random conformal weldings, to multifractal random measures in turbulence and mathematical finance, as well as to various aspects of the Random Matrix Theory and value distribution of the Riemann zeta-function along the critical line.**

Part I: mapping to Statistical Mechanics:

We interpret the sequence V_k for $k = 1, \ldots, M < \infty$ as a set of **random energies** and consider the associated equilibrium Statistical Mechanics by introducing the **temperature** $T = \beta^{-1}$ and defining the **partition function** $Z(\beta) = \sum_{i=1}^{M} e^{-\beta V_i}$.

In this way we arrive to **1D generalization** of the **Derrida**'s **Random Energy** Model to be studied in the thermodynamic limit $M \to \infty$. In particular, **minimal energy** can be extracted from the zero-temperature limit of the free energy as

$$V_{min} = \min\left(V_1, \ldots, V_M\right) = \lim_{\beta\to\infty} f(\beta), \quad f(\beta) = -\beta^{-1} \log Z(\beta)$$

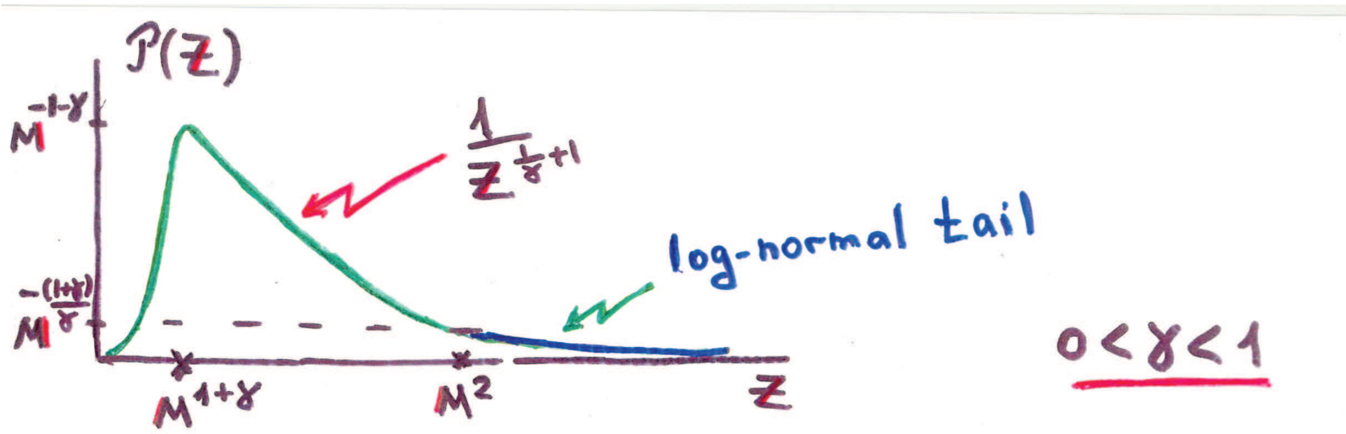
Conclusion: We need to know the statistics of the **free energy** to extract the **extreme value statistics** of the energy sequence.

Observation: The positive integer moments $\langle Z^n(\beta)\rangle$, $n = 1, 2, \ldots$ of the partition function $Z(\beta) = \sum_{i=1}^{M} e^{-\beta V_i}$ for the (regularized) periodic 1/f noise sequence V_k in the high-temperature phase $0 < \beta < 1$ turn out to be given in the thermodynamic limit $M \gg 1$ by the Dyson-Morris-Selberg integral:

$$\langle Z^n(\beta)\rangle = \begin{cases} M^{1+\beta^2 n^2}\, O(1) & \text{for } n > 1/\beta^2 \\ M^{n(1+\beta^2)} \frac{\Gamma(1-n\beta^2)}{\Gamma^n(1-\beta^2)} & \text{for } 1 < n < 1/\beta^2 \end{cases}$$

We reconstruct the probability density $\mathcal{P}(Z)$ from its moments.

Outcome of the analysis: The probability density $\mathcal{P}(Z)$ of the partition function $Z(\beta)$ in the high-temperature phase $\gamma = \beta^2 < 1$ consists of two pieces:



The **"body"** of the distribution has a pronounced maximum at $Z \sim Z_e = \frac{M^{1+\beta^2}}{\Gamma(1-\beta^2)} \ll M^2$, and the powerlaw decay at $Z_e \ll Z \ll M^2$:

$$\mathcal{P}(Z) = \frac{1}{\beta^2}\frac{1}{Z}\left(\frac{Z_e}{Z}\right)^{\frac{1}{\beta^2}} e^{-\left(\frac{Z_e}{Z}\right)^{\frac{1}{\beta^2}}}, \quad Z \ll M^2$$

Limit $\beta \to 0$ can be used to reproduce the Gumbel distribution of "roughness" by **Antal et al. '01**
At $Z \gg M^2$ the above expression is replaced by a **lognormal tail**. **Now we define $z = Z/Z_e$ and consider the generating function**

$$g_\beta(x) = \left\langle \exp(-e^{\beta x} z) \right\rangle_{M \gg 1}, \quad \beta = 1/T$$

High-temperature "duality" and freezing scenario:

In the high-temperature phase the generating function $g_\beta(x)$ can be found explicitly and turned out to satisfy a remarkable duality relation:

$$g_\beta(x) = \int_0^\infty dt \exp\left\{-t - e^{\beta x} t^{-\beta^2}\right\} = g_{\frac{1}{\beta}}(x)\,, \quad \beta < \beta_c = 1$$

This however does not allow to continue to $\beta > \beta_c$ regime. Instead, the phase transition at $\beta = \beta_c$ is conjectured to be described by the following freezing scenario: $g_\beta(x)$ freezes to the temperature independent profile $g_{\beta=\beta_c}(x)$ in the "glassy" phase $T \le T_c$. The scenario is supported by

(i) a (one-loop) renormalization group arguments for the logarithmic models (Carpentier, Le Doussal '01) revealing an analogy to the travelling wave analysis of polymers on disordered trees (Derrida, Spohn 1989). In particular, the latter model shares with our model the REM-like mean free energy which in the limit $M \gg 1$ behaves in the high-temperature phase $\beta < 1$ as $\langle f(\beta)\rangle \approx -\left(\beta + \frac{1}{\beta}\right) \log M$ and "freezes" to the critical value $\langle f(\beta_c)\rangle \approx -2 \log M \equiv V_{min}$ for all temperatures below the transition.

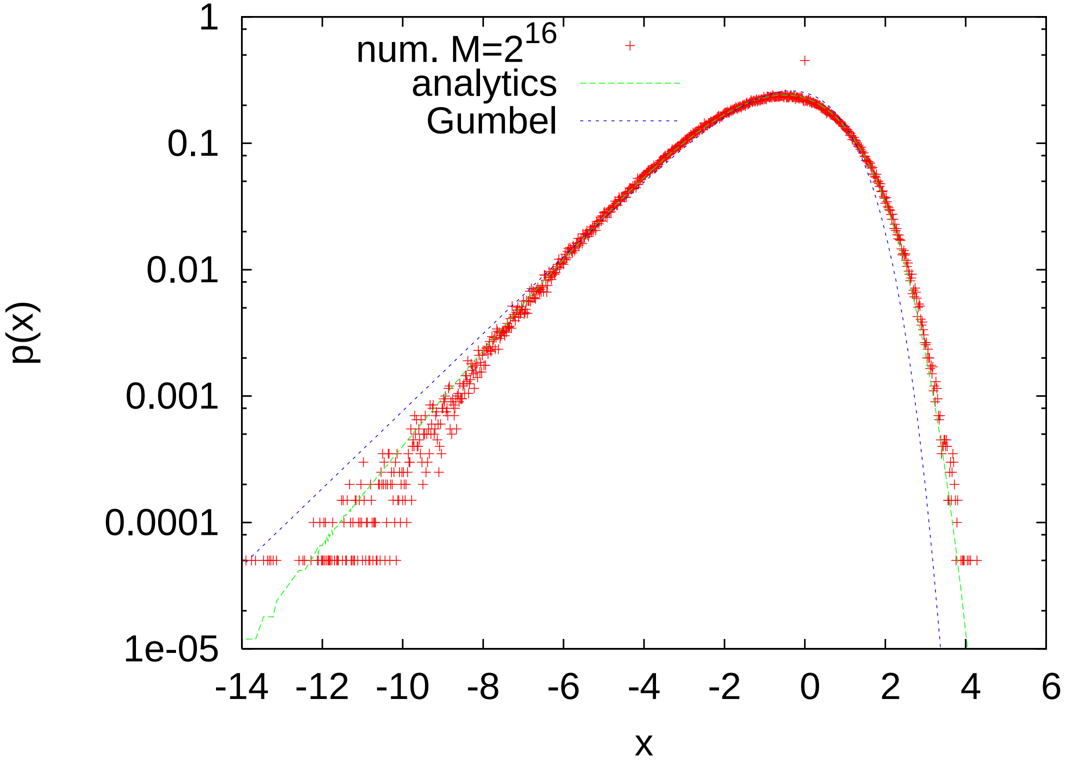
(ii) compatibility with duality which implies $\partial_\beta g_\beta(x)|_{\beta=\beta_c^-} = 0$, for all x, so that the "temperature flow" of this function vanishes at the critical point $\beta = \beta_c = 1$.

(iii) our numerics, also confirming the relation between the freezing and the one-step replica symmetry breaking mechanism operative in the "glassy" phase.

Assuming the freezing scenario, the absolute minimum of the random sequence is simply given by $V_{min} = -\lim_{T\to 0} f = -2\log M + c\log\log M + \mathrm{x}$, with unknown c (conjectured to be $c = 3/2$) and the probability density of x related to the frozen profile $g_{\beta_c}(x)$ by

$$p(x) = -g'_{\beta_c}(x) = -\frac{d}{dx}\left[2e^{x/2}K_1(2e^{x/2})\right], \quad (1)$$

different from the Gumbel law $p_{Gum}(x) = -\frac{d}{dx}\exp\left(-e^{x}\right)$. Note the tail: $p|_{x\to-\infty} \approx -xe^{x}$.



Distribution of extremes: we compare three distributions: (i) the histrogram for ensemble of 10^6 realizations of the periodic 1/f model sampled at $M = 2^{16}$ equispaced points, (ii) the analytical prediction (1), and (iii) the Gumbel distribution for the mean & variance given by (1)

Multifractal structure of low/high values of the ideal 1/f noises:

Remembering that asymptotically $V_{min} = -2\log M$, let us call the value V_i "x−low" if $V_i < 2x\log M$, for some $x \in (-1,0)$. Our next goal is to count the total number $\mathcal{N}_M(x)$ of x−low points in the 1/f sequence. A direct calculation for periodic case shows that the number of x−low points is multifractal:

$$\mathcal{N}_{M\gg 1}(x) \sim \mathcal{N}_e = \frac{M^{1-x^2}}{2|x|\sqrt{\pi\log M}}\frac{1}{\Gamma(1-x^2)}, \quad 0<|x|<1.$$

The ratio $n = \frac{\mathcal{N}_M(x)}{\mathcal{N}_e}$ is randomly fluctuating from one realization of the process to the other according to the probability density (YF, Le Doussal, & Rosso, in progress):

$$\mathcal{P}(n) \approx \frac{1}{x^2}\, n^{-\frac{1}{x^2}-1}\exp\left(-n^{-\frac{1}{x^2}}\right), \quad n \ll n_{max} = \frac{M}{\mathcal{N}_e} \sim M^{x^2}, \quad 0<|x|<1$$

and vanishing very fast when $n \sim n_{max}$.

Note: $M \gg 1$ limit of the discrete regularization of the aperiodic version of the 1/f noise can be considered by similar techniques, see Y F, Le Doussal, Rosso, 2009.

Qualitatively, the two cases share all the major features of the extreme and high value statistics (e.g. multifractality and the same power-law tail for the distribution of the number of points above given high level). Details of the distributions are however different. In particular, the high-temperature duality becomes rather nontrivial to verify and was found to follow from the properties of one of the double Barnes functions featuring in the talk by L.D. Faddeev.

Part II: From1/f noises to Random Matrices and Riemann zeta-function:

let U_N be a $N \times N$ **unitary matrix**, chosen at random from the unitary group $\mathcal{U}(N)$. Introduce its **characteristic polynomial** $p_N(\theta) = \det\left(1 - U_N\, e^{-i\theta}\right)$ and define the following objects which are formal analogues of the **partition function**

$$\mathcal{Z}_N(\beta; L) = \frac{N}{2\pi}\int_0^{2\pi} |p_N(\theta)|^{2\beta} d\theta \equiv \frac{N}{2\pi}\int_0^{2\pi} e^{-\beta V_N(\theta)}\, d\theta, \quad \beta > 0$$

where $V_N(\theta) = -2\log|p_N(\theta)|$. We observe that it is possible to evaluate the $N \gg 1$ limit of the positive integer moments

$$\mathbb{E}\left\{\mathcal{Z}_N^n(\beta; L)\right\} = N^n \int_0^{2\pi} \ldots \int_0^{2\pi} \mathbb{E}\left\{|p_N(\theta_1)|^{2\beta} \ldots |p_N(\theta_n)|^{2\beta}\right\} \prod_{j=1}^{n} \frac{d\theta_j}{2\pi},$$

where the expectation $\mathbb{E}\{\ldots\}$ stand for the $\mathcal{U}(N)$ group average. Indeed, the expectation value in the integrand is a **Toeplitz determinant** whose asymptotic $N \gg 1$ behaviour is known due to **Fisher-Hartwig** and **Widom**:

$$\mathbb{E}\left\{|p_N(\theta_1)|^{2\beta} \ldots |p_N(\theta_n)|^{2\beta}\right\} \sim \left[N^{\beta^2} \frac{G^2(1+\beta)}{G(1+2\beta)}\right]^n \prod_{r<s}^{n} |e^{i\theta_r} - e^{i\theta_s}|^{-2\beta^2}$$

We then see that the $\theta-$integral is again of the same **Dyson-Morris-Selberg** type that we have studied before.

Conclusion: Log-Mod of the characteristic polynomial of CUE matrices is just a different regularization of the same periodic 1/f noise!

In the last decade, following Keating $\&$ Snaith 2001, it became a well-accepted paradigm that many properties of the Riemann zeta-function $\zeta(s)$ along the critical line $s = \frac{1}{2} + it,\ t \in \mathbb{R}$ can be successfully understood by comparing them to analogous properties of characteristic polynomials of random matrices of large size $N \sim \log t$. The idea is inspired by random-matrix properties of zeta-function zeroes (Montgomery (1973); Odlyzko, ...). To this end, define for a fixed real t the function

$$V_t^{(\zeta)}(x) = \log\left|\zeta\left(\tfrac{1}{2} + i(t+x)\right)\right| = \text{Re}\log\zeta\left(\tfrac{1}{2} + i(t+x)\right)$$

For large $t \to \infty$ the function $V_t^{(\zeta)}(x)$ actually mimics a Gaussian random function of variable x of mean zero and variance $\frac{1}{2}\log\log t$ (Selberg, see also Hughes, Keating, O'Connel). Moreover, a simple consideration which uses the Euler product formula for Riemann zeta and the probabilistic properties of primes given by the Prime Number Theorem allows one to show that the small-x behaviour of the covariance high up the critical line:

$$\left\langle V_t^{(\zeta)}(x_1) V_t^{(\zeta)}(x_2)\right\rangle \approx \begin{cases} -\frac{1}{2}\log|x_1 - x_2|, & \text{for } \frac{1}{\log t} \ll |x_1 - x_2| \ll 1 \\ \frac{1}{2}\log\log t, & \text{for } |x_1 - x_2| \ll \frac{1}{\log t} \end{cases}$$

with the averaging going over an interval $[t - h/2, t + h/2]$ such that $\frac{1}{\log t} \ll h \ll t$.

Message: locally the log-mod of the Riemann zeta-function resembles an aperiodic 1/f noise. One can exploit this fact to cast new light on statistics of moments and high values of the Riemann zeta along the critical line using the idea of freezing (Y F $\&$ J P Keating, in progress.)

Part III: Decaying Burgers Turbulence:

• The problem of analysis of solutions of the (unforced) Burgers equation

$$\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} = \nu\nabla^2\mathbf{v}, \quad \mathbf{v}(\mathbf{x}, t=0) = -\nabla\Psi_0(\mathbf{x}),\ \nu > \mathbf{0}$$

with random initial condition, usually assumed to be Gaussian and specified in terms of the two-point correlation function $\overline{\mathbf{v}(\mathbf{x},0)\mathbf{v}(\mathbf{x}',0)}$, or alternatively $\overline{\Psi_0(\mathbf{x})\Psi_0(\mathbf{x}')}$. General reference: **Bec** $\&$ **Khanin** Physics Reports **447** (2007), 1

• The problem appears as an important reference model not only in fluid dynamics, but also in such diverse physical contexts as statistical mechanics of systems with quenched disorder (**Balents, Bouchaud, Mezard '95; Le Doussal '08**), and formation of large scale structures in cosmology (**Gurbatov, Saichev, Shandarin '89; Vergassola, Dubrulle, Frish, Noullez '94)** . In particular, the cosmological applications stimulated interest in **dBT** for **vanishing viscosity** $\nu \to 0$ and **scale-free** power-law random initial conditions:

$$\overline{\mathbf{v}(\mathbf{x},0)\mathbf{v}(\mathbf{x}',0)} \sim |\mathbf{x}-\mathbf{x}'|^{-n-1} \quad \text{at large distances}$$

Cole-Hopf solution, mapping to Statistical Mechanics:

Solution to decaying Burgers turbulence for a given initial potential $\Psi_0(\mathbf{x})$ can be written explicitly as

$$\mathbf{v}(\mathbf{x},t) = -2\nu\nabla_x \ln Z, \quad Z(\mathbf{x},t) = \int e^{-\frac{1}{2\nu}\mathcal{H}_x(\mathbf{y})}\frac{d\mathbf{y}}{(4\pi\nu t)^{d/2}}$$

in terms of the **"effective potential"**

$$\mathcal{H}_x(\mathbf{y}) = \Psi_0(\mathbf{y}) + \frac{1}{2t}(\mathbf{x}-\mathbf{y})^{\mathbf{2}}$$

Such $Z(\mathbf{x},t)$ can be interpreted as the **partition function** of a particle equilibrated in the energy potential $\mathcal{H}_x(\mathbf{y})$ at an **effective temperature** $T = 2\nu$, so that **inviscid** limit = **zero temperature**).

Observation: for $\mathbf{n} = 1$ the initial velocity decays as $\overline{v(x,0)v(x',0)} \sim |x-x'|^{-2}$ which implies the potential $\Psi_0(\mathbf{x})$ is a version of 1/f noise:

$$\overline{\Psi_0(x)\Psi_0(x')} = -2\ln\left[|x-x'|/L\right], \quad \epsilon < |x-x'| < L$$

where $L \gg 1$ and $\epsilon \ll 1$ are the infrared and ultraviolet cutoff scales. We therefore may anticipate a kind of freezing transition inducing changes in the shape of the p.d.f. for velocity $v(x,t) = -2\nu\partial_x \ln Z(x,t)$ at finite critical viscosity $\nu_c(\equiv T_c/2) > 0$.

Note: it is commonly accepted in the literature that that the velocity $v(x,t)$ remains always Gaussian-distributed in the inviscid limit $\nu = 0$.

Statistical mechanics in random potential and Burgers velocity:

In the language of statistical mechanics with $T = 2\nu$ the velocity is given by

$$v = -\frac{1}{t} \prec y \succ_T \quad \text{where} \quad \prec \mathcal{O} \succ_T = \frac{1}{Z(x,t)} \int \frac{dy}{\sqrt{2\pi T t}} \mathcal{O}(y) e^{-\mathcal{H}_0(y)/T}$$

where $\mathcal{H}_0(y) = \Psi_0(y) + \frac{y^2}{2t}$ is the random energy function.

To understand better **thermodynamics** of our system and the nature of the anticipated **freezing transition** it turns out to be instructive to consider also a different object:

$$\mathcal{P}_Y(Y) = \overline{\prec \delta(Y - y) \succ_T} = \overline{\frac{1}{Z} e^{-\mathcal{H}_0(Y)/T}}$$

interpreted as the averaged **Boltzmann-Gibbs** probability measure of the coordinate of a particle **equilibrated** at a given temperature T in the random energy landscape $\mathcal{H}_0(Y)$. At $T \to 0$ the thermal average is obviously dominated by the **deepest minimum** of the landscape whose position $\mathbf{Y}_{\min}$ fluctuates from one realization of disorder to the other. This mechanism immediately implies for velocity p.d.f. in **zero viscosity** (= $T \to 0$) limit the relation

$$\mathcal{P}(v)|_{T=0} = t\mathcal{P}_Y(vt)|_{T=0}$$

Statistical mechanics in random potential and λ- Hermite ensemble:

The disorder averaging procedure for $\mathcal{P}_Y(Y)$ can be performed via the standard **replica trick** after representing $Z^{-1} = Z^{n-1}|_{n\to 0}$ and using the Gaussian nature of the random potential $\Psi_0(y)$ by employing $\overline{\Psi_0(y)\Psi_0(y')} = -2\ln\left[|y-y'|/L\right]$. One finds

$$\mathcal{P}_Y(Y) = \lim_{n\to 0}\left\langle \frac{1}{n}\sum_{j=1}^{n}\delta\left(Y - z_j\sqrt{Tt}\right)\right\rangle_{n,-\gamma}$$

where $\gamma = 1/T^2 > 0$ and we have defined for $1 \le n < 1/\gamma$

$$\langle\ldots\rangle_{n,\lambda} = \frac{1}{S_n(\lambda)}\int_{-\infty}^{\infty}(\ldots)\prod_{i<j}^{n}|z_i - z_j|^{2\lambda}\prod_{j=1}^{n}\frac{dz_j}{\sqrt{2\pi}}e^{-\frac{z_j^2}{2}},$$

with $S_n(\lambda) = \prod_{j=1}^{j=n}\left[\Gamma(1+j\lambda)/\Gamma(1+\lambda)\right]$ being the famous **Selberg** integral. For finite integer $n \ge 1$ and $\lambda > 0$ the above expression is nothing else but the mean density of the so-called λ-**Hermite** ensemble of $n\times n$ random matrices introduced by **Dumitriu** $\&$ **Edelman** '02 .
Note that the corresponding random matrix-like integrals are still **convergent** for $\lambda = -\gamma$ as long as $0 < \gamma < 1$. The **replica limit** implies $n \to 0$.

Statistical mechanics in random potential and λ- Hermite ensemble:

Although a closed-form expression for the eigenvalue density for **λ-Hermite** ensemble does not seem to be available, one can use the **Jack polynomials** expansion developed by **Dumitriu & Edelman** '02,'06 and find **a few lower moments** of that density explicitly. Performing the analytical continuation $n \to 0$ and $\lambda \to -\gamma$ we obtained the lower nonvanishing moments $M_{2q} = \int \mathcal{P}_Y(Y) Y^{2q} dY$ up to $2q = 16$. We present below the corresponding **cumulants** C_{2q}:

$$C_2 = t\left(T + T^{-1}\right), \quad C_4 = -t^2, \quad C_6 = 2t^3\left(T + T^{-1}\right)$$

$$C_8 = -t^4\left[26 + 6\left(T^2 + T^{-2}\right)\right], \quad C_{10} = t^5\left[300\left(T + T^{-1}\right) + 24\left(T^3 + T^{-3}\right)\right]$$

and similar but longer expressions for $C_{2q},\ q = 6, 7, 8$.

The main feature apparent from the above (and proved in full generality) is that **all the cumulants** (and hence the whole function $\mathcal{P}_Y(Y)$) are **invariant** with respect to the **duality transformation** $T \to 1/T$. Employing the **freezing conjecture** we thus predict that **the whole Gibbs-Boltzmann probability density** $\mathcal{P}_Y(Y)$ **freezes** at the critical point $T = T_c = 1$ providing a vivid picture of what freezing entails.

Freezing scenario vs. numerics for zero viscosity velocity moments:

If this scenario were correct, the values of the above cumulants evaluated at $T = 1$ should immediately provide, in view of the discussed zero-temperature correspondence, the **cumulants of the velocity p.d.f.** in **zero viscosity** limit:

$$\overline{v^2}|_{\nu=0} = \frac{2}{t}, \quad \overline{v^4}^c = \left[\overline{v^4} - 3\overline{v^2}^2\right]|_{\nu=0} = -\frac{1}{t^2}, \ etc.$$

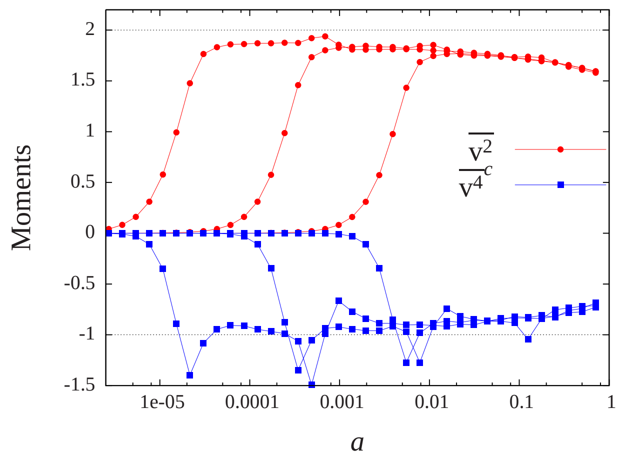


Figure 1: **Numerical evaluation of $\overline{v^2}$ and $\overline{v^4}^c$ in the inviscid limit $\nu = 0$ for discretized Burgers equation (number of points $M = 2^{10}, 2^{14}, 2^{18}$) with periodic version of the logarithmically correlated potential (averaged over 10^6 samples) against the theoretical prediction at $t = 1$.**

In fact, we can show that the velocity is **non-Gaussian** everywhere in the **low-viscosity** phase $\nu < \nu_c = 1/2$. Above the critical viscosity Gaussianity is restored.

Summary:

I. Using the methods of statistical mechanics of disordered systems we studied the statistics of minima/maxima of the Gaussian 1/f noise, both periodic and aperiodic. The distributions are manifestly non-Gumbel and show universal backward tail $p|_{x\to-\infty} \approx -xe^{x}$. This is heavily based on the conjectured freezing scenario, supported by numerics, high-temperature duality, REM-like replica-symmetry breaking, and renormalization-group arguments, but still lacking rigorous justification. We have also predicted a strongly-fluctuating multifractal pattern in the powerlaw-distributed number of high/low points in the 1/f signals.

II. We reveal strong links between the 1/f noise and Log-mod of characteristic polynomials of random matrices, and of the Riemann zeta-function along the critical line. This allows to put forward new conjectures about the statistics of high and extreme values of the latter objects.

III. Combining the methods of statistical mechanics with insights from the random matrix theory we reveal a phase transition with decreasing viscosity ν at finite $\nu = \nu_c > 0$ in one-dimensional decaying Burgers turbulence with a power-law correlated random profile of initial velocities $\overline{v(x,0)v(x',0)} \sim |x-x'|^{-2}$. The low-viscosity phase exhibits non-Gaussian one-point probability density of velocities, reflecting a spontaneous one step replica symmetry breaking (RSB) in the associated statistical mechanics problem. We obtain the low orders cumulants analytically which favourably agree with numerical simulations.