New approaches to QCD at finite matter density

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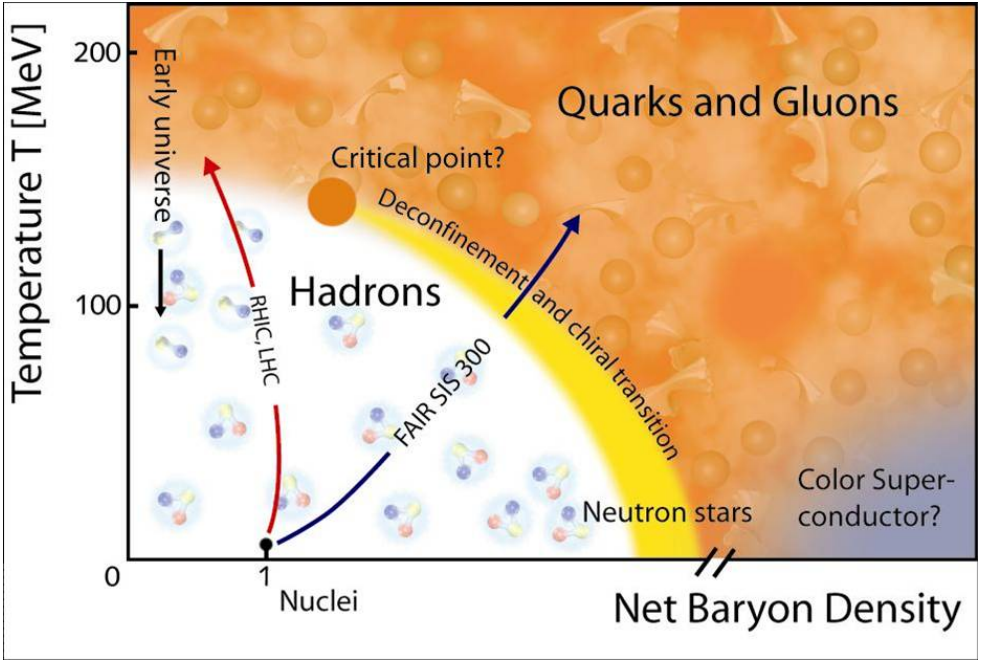
Motivation and basic concepts of lattice QCD

Something we would like to understand in detail ...



QCD with the Euclidean path integral

- Fermion and quark action

$$S_F = \sum_{f=1}^{N_f} \int d^4x \, \overline{\psi}_f(x) \left[\gamma_\nu (\partial_\nu + iA_\nu(x)) + m \right] \psi_f(x)$$

$$S_G = \frac{1}{2g^2} \int d^4x \, \mathrm{Tr} F_{\mu\nu}(x) F_{\mu\nu}(x)$$

- Vacuum expectation values with Feynman's path integral

$$\langle O \rangle = \frac{1}{Z} \int D[\psi, \overline{\psi}, A] \; e^{-S_F[\psi,\overline{\psi},A] - S_G[A]} \; O[\psi, \overline{\psi}, A]$$

- Lattice QCD gives a mathematically sound definition of the path integral and Monte Carlo methods can be used to compute it.

Discretizing QCD on a space time lattice

- Replace space-time by a 4-D grid with lattice constant a.
- Quark fields live on the sites of the lattice. Derivatives turn into

$$\partial_\nu \psi(x) \;\rightarrow\; \frac{1}{2a}\Big[\,\psi(x+a\widehat{\nu}) \;-\; \psi(x-a\widehat{\nu})\,\Big]$$

- For the gluons we use gauge transporters attached to the links (x,ν) of the lattice

$$U_\nu(x) \;=\; \mathcal{P}\,\exp\left(\, i \int_x^{x+a\widehat{\nu}} ds_\tau A_\tau(s)\right)$$

- Integral over all field configurations

$$D[\psi,\overline{\psi},U] \;=\; \prod_x d\psi(x)\, d\overline{\psi}(x) \prod_{\nu=0}^{3} dU_\nu(x)$$

Discretizing QCD on a space time lattice

- Discretized fermion action (1 flavor)

$$S_F \;=\; \int d^4x\, \overline{\psi}(x)\Big[\gamma_\nu(\partial_\nu + iA_\nu) + m\Big]\psi(x) \;\;\rightarrow\;\; \sum_{x,y} \overline{\psi}(x)\; D_{x,y}[U]\; \psi(y)$$

- Partition sum

$$Z \;=\; \int D[\psi,\overline{\psi},U]\; e^{-\,S_G\,-\,S_F} \;=\; \int D[U]\; e^{-\,S_G[U]}\; \det D[U]^{N_f}$$

- Monte Carlo: Generate gauge configurations with probability

$$P[U] \;\propto\; e^{-\,S_G[U]}\; \det D[U]^{N_f}$$

and use these to approximate the path integral.

Lattice QCD with chemical potential: complex phase problem

- For non-zero chemical potential μ the fermion determinant is complex

$$\det D[U,\mu] \in \mathbb{C}$$

- The Boltzmann factor cannot be used as probability in a MC calculation

$$e^{-S_G[U]} \det D[U,\mu]^{N_f} \in \mathbb{C}$$

- So far very limited progress with expansions or reweighting techniques.

In this project we explore new strategies in effectice theories of QCD.

An effective theory for QCD

Center symmetry and deconfinement

- Observable for a single static quark: Polyakov loop

$$P(\vec{x}) \;=\; \mathcal{P}\,\exp\left(\, i\int_0^\beta dt A_0(t,\vec{x})\right) \;\;\rightarrow\;\; \prod_{t=1}^{\beta} U_0(t,\vec{x})$$

- Deconfinement transition at finite temperature

$$\langle \mathrm{Tr}\, P\rangle \;\;\propto\;\; e^{-\beta F_q} \quad \begin{array}{ll} = 0 & \text{for } T < T_c \\ \neq 0 & \text{for } T > T_c \end{array}$$

- An associated symmetry: center symmetry (only gauge sector)

$$U_0(t^*,\vec{x}) \;\rightarrow\; z\, U_0(t^*,\vec{x}) \quad , \qquad z \;\in\; \mathbb{Z}_3 \;=\; \{1, e^{i2\pi/3}, e^{-i2\pi/3}\}$$

Center symmetry and deconfinement

- The gauge action and the path integral measure are invariant.

- The Polyakov loop transforms non-trivially

$$P \;\rightarrow\; z\,P \quad , \qquad z \;\in\; \mathbb{Z}_3$$

- The deconfinement transition of pure gauge theory is understood as the spontaneous breaking of center symmetry

$$T < T_c: \qquad \langle \mathsf{Tr}\, P \rangle \;=\; 0 \qquad \text{center symmetry intact}$$

$$T > T_c: \qquad \langle \mathsf{Tr}\, P \rangle \;\neq\; 0 \qquad \text{center symmetry broken}$$

Svetitsky-Yaffe conjecture

- For pure gauge theory deconfinement can be understood as the spontaneous breaking of center symmetry. An influential idea for understanding this phase transition is the Svetistky-Yaffe conjecture (1981):

- At T_c the critical behavior of SU(3) gauge theory in $d+1$ dimensions can be described by a d - dimensional spin system with a $\mathbb{Z}_3$ - invariant effective action for the local Polyakov loop $P(x) \in$ SU(3).

- Leading term of the effective action from a strong coupling expansion

$$S_c[P] \;=\; -\tau \sum_{\langle x,y \rangle} \Big[\; \mathrm{Tr} P(x)\, \mathrm{Tr} P(y)^\dagger \;+\; \mathrm{Tr} P(y)\, \mathrm{Tr} P(x)^\dagger \;\Big]$$

Explicit center breaking terms from the fermion determinant

- The fermion determinant breaks the center symmetry explicitly.

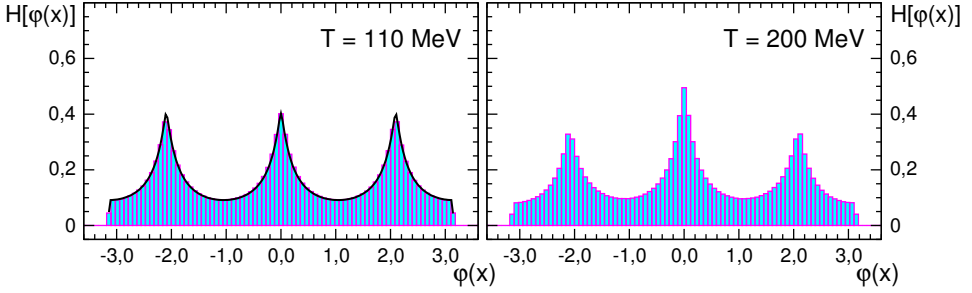
- The leading center symmetry breaking can be calculated using hopping expansion, including the chemical potential μ.

$$
\begin{aligned}
S[P] \;\; = \;\; & -\tau \sum_{\langle x,y \rangle} \Big[\; \mathrm{Tr} P(x) \, \mathrm{Tr} P(y)^\dagger \; + \; \mathrm{Tr} P(y) \, \mathrm{Tr} P(x)^\dagger \; \Big] \\
& -\kappa \sum_{x} \Big[\; e^{\,\mu} \, \mathrm{Tr} P(x) \; + \; e^{-\mu} \, \mathrm{Tr} P(x)^\dagger \; \Big]
\end{aligned}
$$

- For non-vanishing chemical potential μ the action is complex and the effective theory inherits the complex phase problem of QCD.

Reduction to the center degrees of freedom

- The distribution of $\mathrm{Tr}P(x)$ is dominated by the center.



- Reduction to center elements: $\mathrm{Tr}P(x) \rightarrow L(x) \in \mathbb{Z}_3 = \{1, e^{i2\pi/3}, e^{-i2\pi/3}\}$

$$S = -\tau \sum_{\langle x,y \rangle} \left[L(x)L(y)^\star + L(y)L(x)^\star \right] - \kappa \sum_x \left[e^\mu L(x) + e^{-\mu} L(x)^\star \right]$$

Contains leading center symmetric and center symmetry breaking terms.

Solving the complex phase problem: Flux representation

Flux representation - I

- Identity for the nearest neighbor term:

$$e^{\tau[L(x)L(x+\hat{\nu})^* + c.c.]} \;=\; C \sum_{b_{x,\nu}=-1}^{+1} B^{|b_{x,\nu}|} \; \left(L(x)L(x+\hat{\nu}) \right)^{b_{x,\nu}}$$

- Identity for the magnetic term:

$$e^{\kappa[e^{\mu} L(x) + e^{-\mu} L(x)^*]} \;=\; \sum_{s_x=-1}^{+1} M_{s_x} \, L(x)^{s_x}$$

- C, B and M_s are real and positive functions of τ, κ and μ.

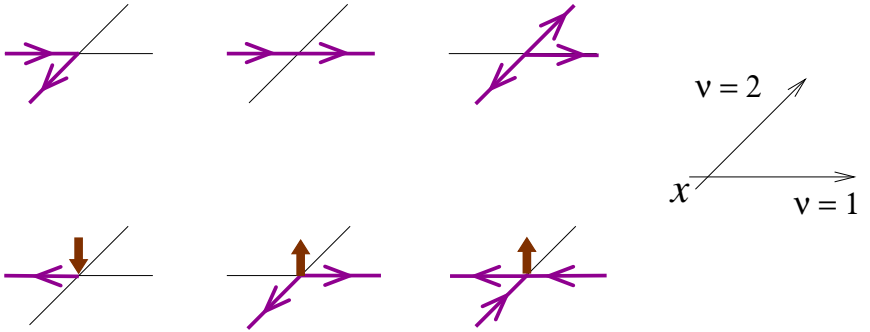
- Remaining sums over the center elements at each site:

$$\sum_{L \in \{1, e^{\pm i 2\pi/3}\}} L^n \;=\; 3\,T(n) \qquad \text{with} \qquad T(n) \;=\; \begin{cases} 1 & n \bmod 3 \;=\; 0 \\ 0 & n \bmod 3 \;\neq\; 0 \end{cases}$$

Flux representation - II

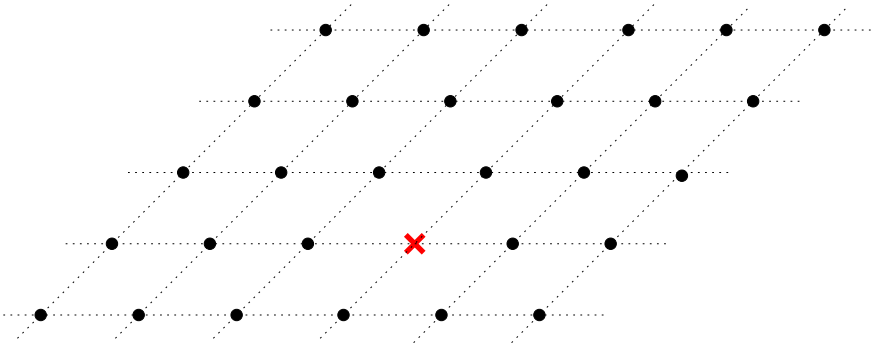
- Representation with dimers $b_{x,\nu} \in \{-1, 0, 1\}$ and monomers $s_x \in \{-1, 0, 1\}$

$$Z = \sum_{\{b,s\}} W(b,s) \prod_x T\left(\sum_\nu [b_{x,\nu} - b_{x-\hat{\nu},\nu}] + s_x\right) \quad \text{with} \quad W(b,s) > 0$$



Update with a Prokof'ev – Svistunov worm algorithm

- The worm starts at a random position on the lattice.
- The worm may decide to change dimers or insert monomers. Each change is accepted or rejected in a Metropolis step.
- Insertion of a monomer is followed by a random hop and another monomer insertion.
- The worm closes when it reaches its starting point.



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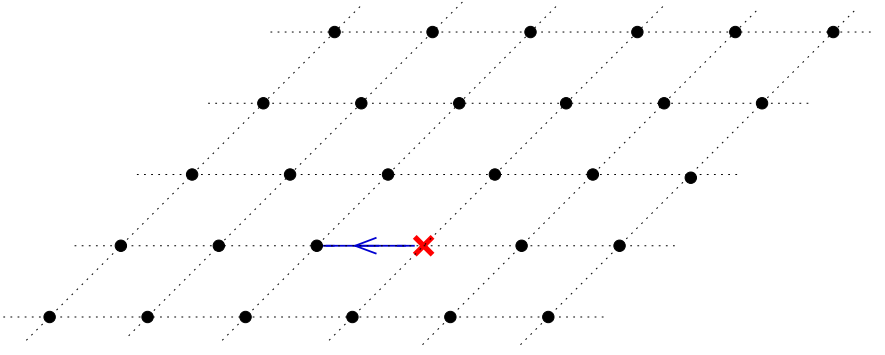

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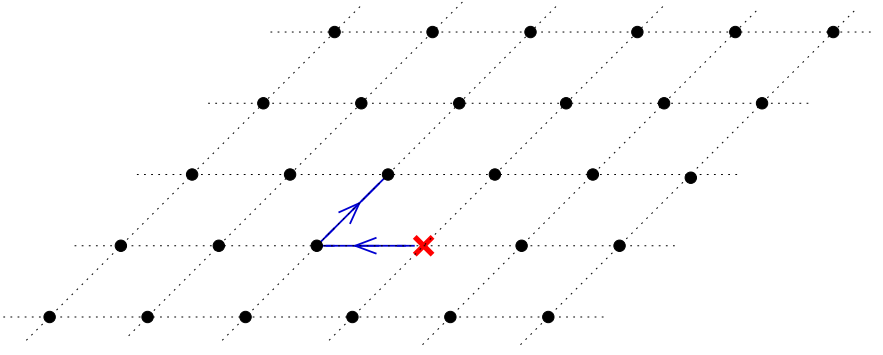

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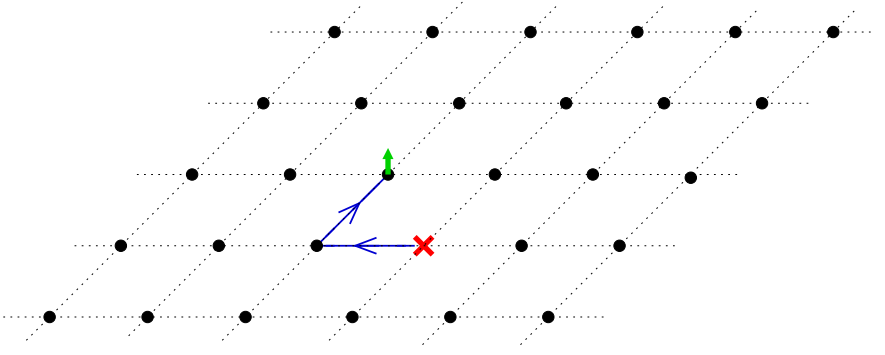

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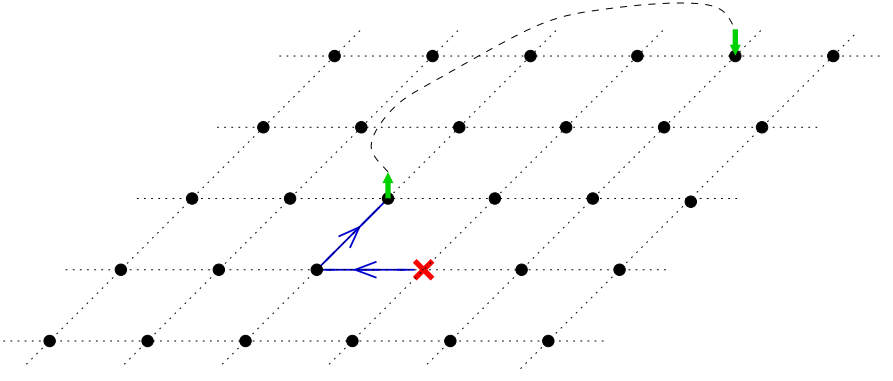

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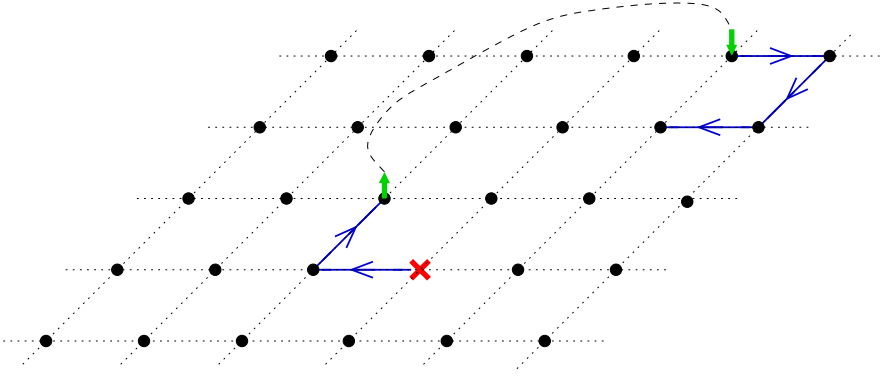

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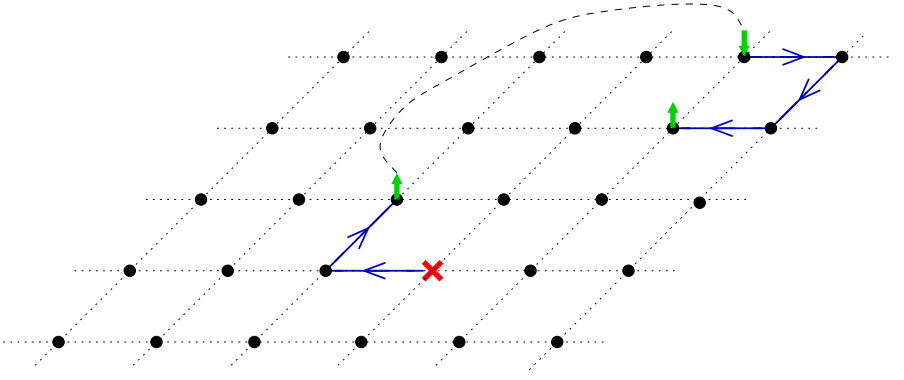

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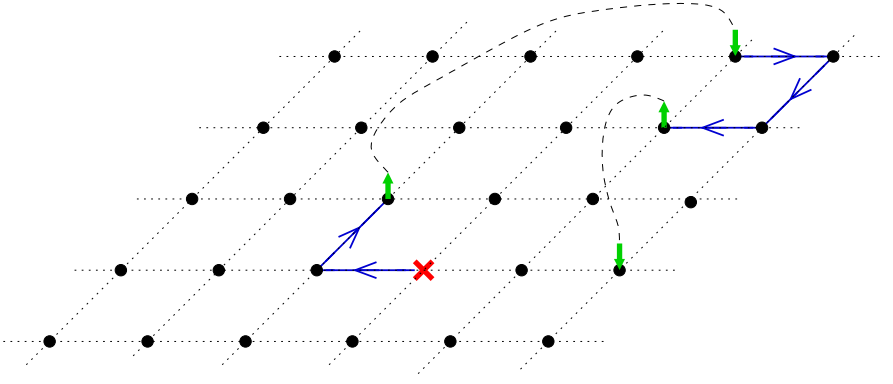

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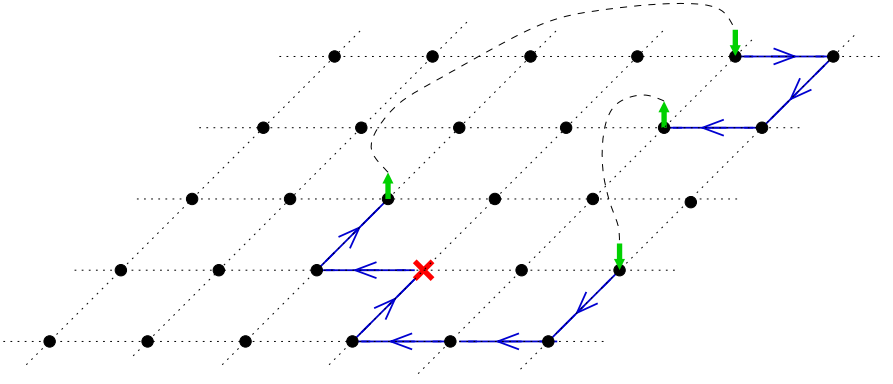


Numerical analysis

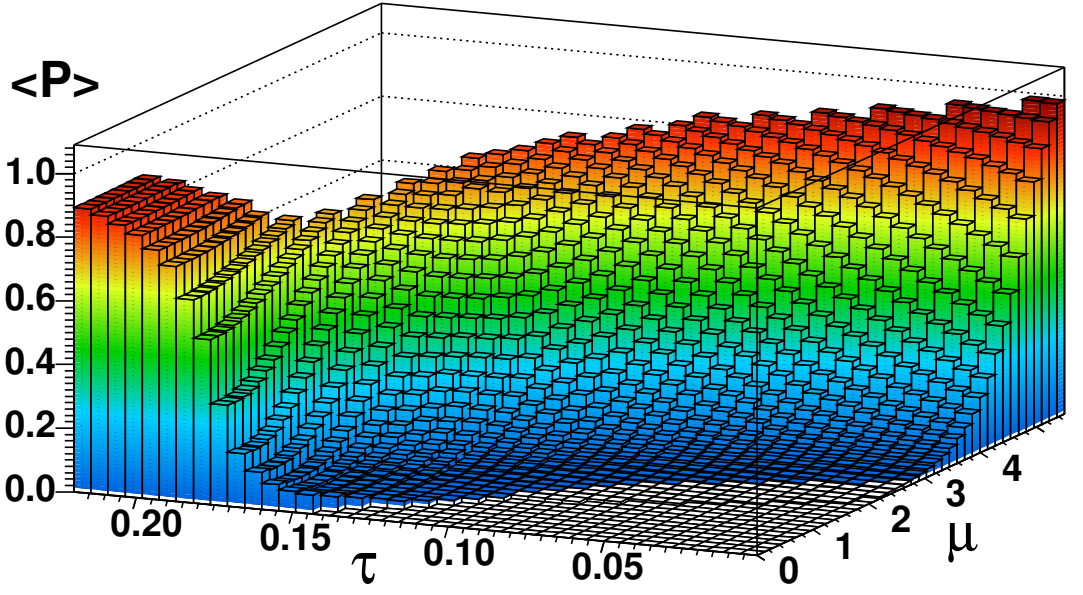
- The worm algorithm is ergodic and efficient also for large κ and μ.
- As observables we consider the internal energy E, the heat capacity C, the expectation value of the Polyakov loop $\langle P \rangle$ and the corresponding susceptibility χ_P.
- All our observables can be expressed in terms of the dimer- and monomer occupation numbers and their fluctuations.
- We work on lattices with sizes 16^3 to 72^3 with statistics of 10^5 to 10^6 worms.
- Comparison of the results for small τ with low temperature expansion.

The QCD phase diagram according to the center group

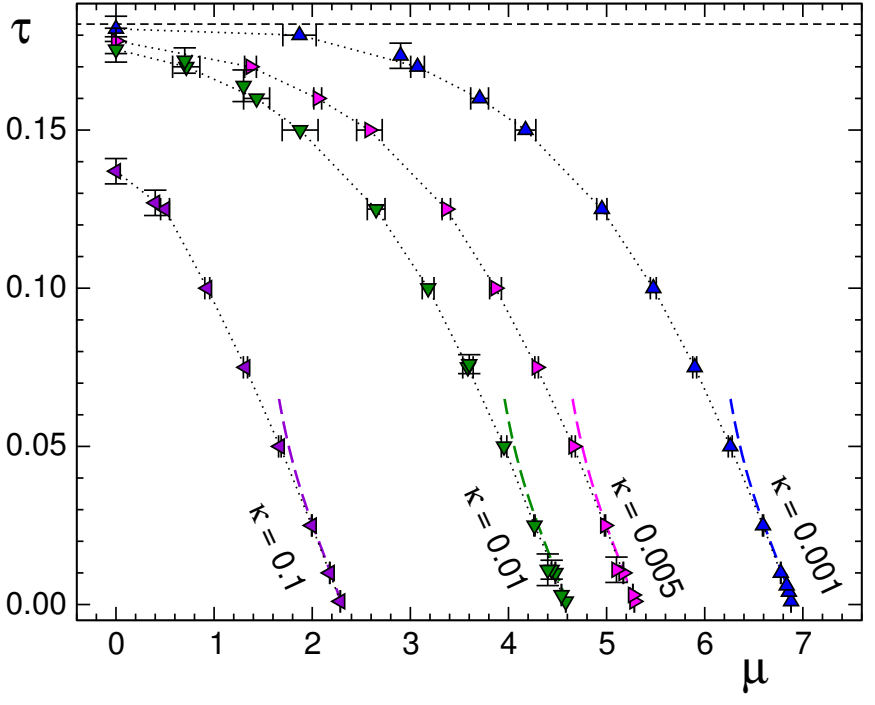
Ydalia Delgado Mercado, Hans Gerd Evertz, Christof Gattringer
Phys. Rev. Lett. 106 (2011) 222001, arXiv:1102.3096

Ydalia Delgado Mercado, Hans Gerd Evertz, Christof Gattringer, work in preparation.

Polyakov loop (= order parameter)

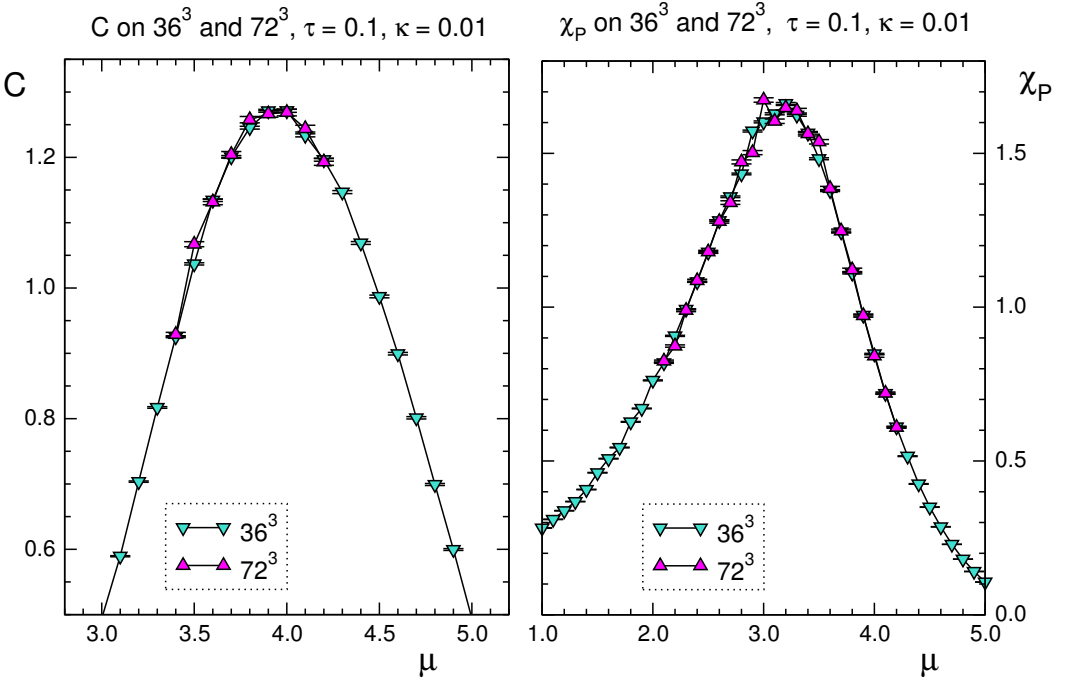


Exploring the phase diagram (maxima of χ_P)

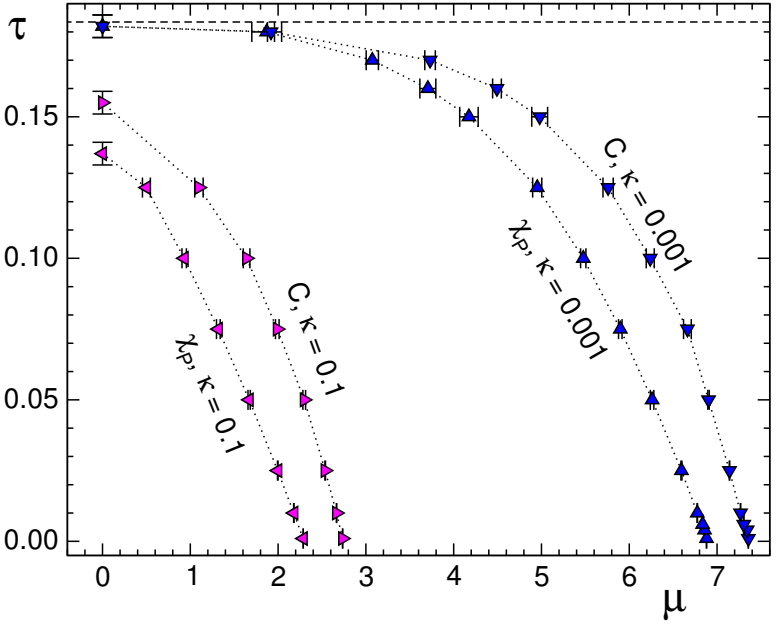


Volume scaling

For all values of κ we studied no volume scaling of the second derivatives of the free energy C and χ_P was observed $\Rightarrow$ Crossover.



Strips of crossover behavior



If QCD is reduced to center degrees of freedom $\Rightarrow$ Crossover transitions.

Developments and perspectives

Christof Gattringer, Nucl. Phys. B 850 (2011) 242, arXiv:1104.2503

Ydalia Delgado Mercado, Hans Gerd Evertz, Christof Gattringer, work in preparation.

More realistic models:

- Effective theory with SU(3)-valued Polykov loops ($P(x) \in \mathsf{SU(3)}$):

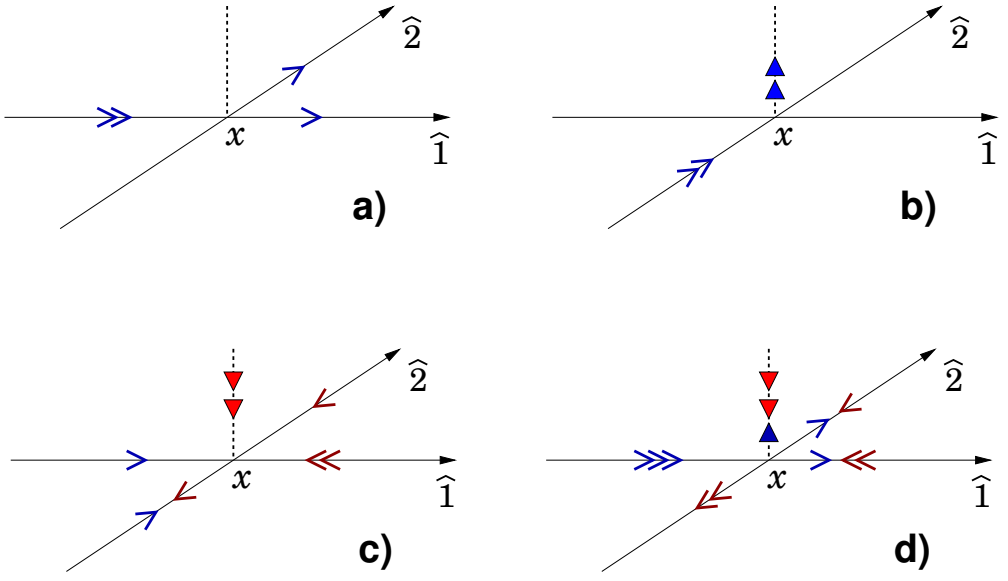
$$\begin{aligned} S[P] \;\; = \;\; & -\tau \sum_{\langle x,y \rangle} \Big[\; \mathsf{Tr} P(x) \, \mathsf{Tr} P(y)^\dagger \; + \; \mathsf{Tr} P(y) \, \mathsf{Tr} P(x)^\dagger \; \Big] \\ & -\kappa \sum_{x} \Big[\; e^{\,\mu} \, \mathsf{Tr} P(x) \; + \; e^{-\mu} \, \mathsf{Tr} P(x)^\dagger \; \Big] \end{aligned}$$

- Flux representation:

$$Z \; = \sum_{\{l,\bar{l},s,\bar{s}\}} \mathcal{W}[l,\bar{l},s,\bar{s}] \; \prod_x T\Big(\sum_\nu [(l_{x,\nu} - \bar{l}_{x,\nu}) - (l_{x-\hat{\nu},\nu} - \bar{l}_{x-\hat{\nu},\nu})] + (s_x - \bar{s}_x) \Big)$$

$$\mathcal{W}[l,\bar{l},s,\bar{s}] \;\; \text{real and positive} \;\; ; \quad l_{x,\nu} \, , \, \bar{l}_{x,\nu} \, , \, s_x \, , \, \bar{s}_x \; \in \; \mathbb{N}_0$$

Examples of admissible flux and monomer vertices



Generalized PS worm algorithms can be applied for a Monte Carlo calculation.

Summary

- We study QCD reduced to the leading center symmetric and center symmetry breaking terms.
- The complex phase problem is solved by using a flux representation with dimers and monomers, which is suitable for a PS worm algorithm.
- We map out the phase diagram using E, $\langle P \rangle$, C and χ_P.
- If QCD is reduced to the center degrees of freedom only crossover transitions remain.
- Work in progress: Generalizations to theories closer to QCD.
- The new models can and should be used for testing various approaches to finite μ (expansions, complex Langevin, reweighting ...).