Interaction-induced criticality in topological insulators

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Outline

1 Introduction
   - Definitions
   - Quantum spin-Hall effect
   - 3D topological insulators

2 Symplectic symmetry class AII: no interaction
   - Scaling theory
   - Topological protection

3 Interaction-induced criticality
   - Surface of a 3D topological insulator
   - Quantum spin-Hall criticality
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- Surface of a 3D topological insulator
- Quantum spin-Hall criticality
What is a topological insulator?

Topological insulator
≡ bulk insulator with (topologically protected) metallic edge/surface

Examples

- **Quantum Hall effect** at the plateau (QHE insulator with edge states)

- Materials with extreme **spin-orbit** coupling (**inverted gap**)
  - 2D: Quantum spin-Hall effect (HgTe/CdHgTe quantum wells)
  - 3D: Bi$_x$Sb$_{1-x}$, Bi$_2$Te$_3$, Bi$_2$Se$_3$, Sb$_2$Te$_3$, TlBiSe$_2$, ...
Topological invariants

- **QHE**: time-reversal symmetry broken by magnetic field
  
  Chern number $= \#$ of edge states $= \ldots -2, -1, 0, 1, 2, \ldots$ ($\mathbb{Z}$)
  
  $\implies \mathbb{Z}$ topological insulator

- **QSHE**: time-reversal symmetry preserved, spin-rotational broken
  
  band structure topological invariant ($\mathbb{Z}_2$): $n = 0$ or $n = 1$
  
  $\iff$ odd vs. even number of Kramers pairs of edge states ($\mathbb{Z}_2$)
  
  $\implies \mathbb{Z}_2$ topological insulator
Introduction

Symplectic symmetry class AII: no interaction

Interaction-induced criticality

Definitions

Quantum spin-Hall effect
3D topological insulators

Quantum Hall effect: $\mathbb{Z}$ topological insulator

IQH transition

IQHE flow diagram

Khmelnitskii’ 83, Pruisken’ 84

Field theory (Pruiskens):

$\sigma$-model with topological term

$$S = \int d^2r \left\{ -\frac{\sigma_{xx}}{8} \text{Tr}(\partial_\mu Q)^2 + \frac{\sigma_{xy}}{8} \text{Tr}\epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q \right\}$$

QHE insulator: $\mathbb{Z}$ - topological insulator
### Periodic table of Topological Insulators

<table>
<thead>
<tr>
<th>$p$</th>
<th>$H_p$</th>
<th>$R_p$</th>
<th>$S_p$</th>
<th>$\pi_0(R_p)$</th>
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<th>$d=2$</th>
<th>$d=3$</th>
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<td>DIII</td>
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</tr>
<tr>
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<td>AII</td>
<td>BD</td>
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<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
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</tr>
<tr>
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<tr>
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<tr>
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<td>$\mathbb{Z}$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
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</tr>
</tbody>
</table>

- $H_p$ – symmetry class of Hamiltonians
- $R_p$ – sym. class of classifying space (of Hamiltonians with eigenvalues $\rightarrow \pm 1$)
- $S_p$ – symmetry class of compact sector of $\sigma$-model manifold

Kitaev’09; Schnyder, Ryu, Furusaki, Ludwig ’08-09
Classification of topological insulators

Two ways to detect existence of TIs of class $p$ in $d$ dimensions:

(i) by inspecting the topology of classifying spaces $R_p$:

$$
\begin{align*}
\text{TI of type } \mathbb{Z} & \iff \pi_0(R_{p-d}) = \mathbb{Z} \\
\text{TI of type } \mathbb{Z}_2 & \iff \pi_0(R_{p-d}) = \mathbb{Z}_2
\end{align*}
$$

(ii) by analyzing homotopy groups of the $\sigma$-model manifolds:

$$
\begin{align*}
\text{TI of type } \mathbb{Z} & \iff \pi_d(S_p) = \mathbb{Z} \quad \text{Wess-Zumino term} \\
\text{TI of type } \mathbb{Z}_2 & \iff \pi_{d-1}(S_p) = \mathbb{Z}_2 \quad \theta = \pi \quad \text{topological term}
\end{align*}
$$

WZ and $\theta = \pi$ terms make boundary excitations “non-localizable”

TI in $d \iff$ topological protection from localization in $d - 1$

Bott periodicity: $\pi_d(R_p) = \pi_0(R_{p+d})$, periodicity 8
Kane and Mele '05, Sheng et al. '05, Bernevig and Zhang '06.

- The QSHE state does not break the (symplectic) TR symmetry: $T^2 = -1$.

- **Simplified picture:** two copies of QH states for two spin components, each seeing the opposite magnetic field.

- **Generic SO:** spins are not conserved, but Kramers degeneracy still holds

**TR invariance forbids backscattering**

$$\Rightarrow \text{topologically protected edge states}$$
Quantum (spin-)Hall effect with disorder

Impurities do not localize the edge states:

QHE

due to chirality of carriers (class A)
any disorder

due to time-inversion symmetry (class AII)
no magnetic impurities
Quantum spin-Hall effect (HgTe/CdTe QW)

Theory: Bernevig, Hughes, Zhang '06; Experiment: Molenkamp group '07

2D Dirac Hamiltonian with tunable mass: \( m \geq 0 \text{ when } d \leq d_c \)

I — \( d = 5.5\text{nm} \): normal insulator
II, III, IV — \( d = 7.3\text{nm} \): inverted band gap — topological insulator
3D topological insulator: band structure
Hasan group ’08

**Bi$_{1-x}$Sb$_x$**

**Bulk band structure:**

- Pure Bismuth semimetal
- Alloy: 0.09 < x < 0.18 semiconductor $E_{\text{gap}} \sim 30$ meV
- Pure Antimony semimetal

A7 rhombohedral crystal structure.

Other realizations: BiTe, BiSe ...
3D topological insulator: spectroscopy
Hasan group '08

ARPES measurement on Bi$_{0.9}$Sb$_{0.1}$

Odd number of surface modes $\implies$ nontrivial topology
3D topological insulator: phenomenological description
cf. Volkov & Pankratov ’85

Bulk Hamiltonian (Dirac): \( H_b = \begin{pmatrix} M & \boldsymbol{\sigma} \boldsymbol{p} \\ \boldsymbol{\sigma} \boldsymbol{p} & -M \end{pmatrix} \)

Domain wall:
\( M > 0 \) & \( M < 0 \)

Edge state
Decays into the bulk: \( \Psi = e^{-|Mx|} \begin{pmatrix} \psi \\ \chi \end{pmatrix} \)
Boundary condition: \( \chi = -i\sigma_x \psi \Rightarrow -i\sigma n\psi \)

Surface Hamiltonian
\( H_s = \frac{\nabla n}{2} + \frac{1}{2} \left( n[p \times \boldsymbol{\sigma}] + [p \times \boldsymbol{\sigma}]n \right) \Rightarrow n[p \times \boldsymbol{\sigma}] \)
3D Topological Insulators
have 2D delocalized modes at the surface

surface of a 3D TI $=$ single-valley graphene

2D disordered Dirac fermions of symmetry class AII:
topological protection against localization
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Dimensionless conductance [in units $e^2/h$]

- Metallic sample (Ohm’s law): $g \sim L^{d-2}$
- Insulating sample (tunneling): $g \sim e^{-L/\xi}$

Universal scaling function

$$\frac{d \ln g}{d \ln L} = \beta(g) = \begin{cases} 
    d - 2, & g \gg 1, \text{ (metal),} \\
    \ln g, & g \ll 1, \text{ (insulator).}
\end{cases}$$
Weak localization correction in 2D
Gor’kov, Larkin, Khmelnitskii ’79; Hikami, Larkin, Nagaoka ’80

Scaling of conductivity in 2D (no e-e interaction)

\[
\frac{d \ln g}{d \ln L} = \begin{cases} 
-\frac{1}{g}, & \text{orthogonal (TR preserved)} \\
-\frac{1}{2}g^2, & \text{unitary (TR broken)} \\
\frac{1}{2}g, & \text{symplectic (TR preserved, spin-orbit) we are here!}
\end{cases}
\]

2D Dirac electrons: Metal or insulator?

MIT in symplectic class at \( \sigma_{Sp}^* \approx 1.4e^2/h \)
One-dimensional symplectic wire
Zirnbauer '92, Mirlin et al '94, Ando & Suzuura '02, Takane '04

Scattering matrix of a symplectic system

\[
\begin{pmatrix}
\Psi^L_{\text{out}} \\
\Psi^R_{\text{out}} \\
\Psi^L_{\text{in}} \\
\Psi^R_{\text{in}}
\end{pmatrix} =
\begin{pmatrix}
r & t' \\
t & r'
\end{pmatrix}
\begin{pmatrix}
\Psi^L_{\text{in}} \\
\Psi^R_{\text{in}}
\end{pmatrix}
\]

TI symmetry \implies \begin{align*}
r &= -r^T \\
r' &= -r'^T \\
t &= t'^T
\end{align*}

For N channels:

\[
\det r = (-1)^N \det r^T \implies \text{no localization if } N \text{ is odd !!}
\]
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Topological insulator: reduction to 1D

Hollow cylinder threaded with magnetic flux $\Phi$

Surface states:

$$E_n(p) = \pm \sqrt{p^2 + \left(n + \frac{1}{2} - \frac{e\Phi}{hc}\right)^2}$$

Time-inversion symmetry is preserved if $\frac{e\Phi}{hc}$ is integer or half-integer

no 1D localization $\implies$ no 2D localization
Dirac fermions in symplectic class: sigma model

Random potential: symplectic time-reversal symmetry $H = \sigma_y H^T \sigma_y$

Symplectic sigma model: topological $\theta$-term with $\theta = \pi$

$$S[Q] = \frac{\sigma_{xx}}{16} \text{Str}(\nabla Q)^2 + i\theta N[Q] \quad N[Q] = 0, 1$$

Similar to Pruisken sigma model for IQHE (instantons suppress localization)

**No localization! Criticality?**

- Minimal conductivity: $\sigma = 4\sigma_{Sp}^{**} \sim e^2 / h$, or
- Absolute antilocalization: $\sigma \rightarrow \infty$
Scaling of conductance: numerical results

- Absence of localization confirmed
- Supermetallic behaviour for microscopic models considered
Beta functions for symplectic system

\[ \beta(g) = \frac{d \log g}{d \log L} \]

Usual spin-orbit metal

Dirac fermions

Ostrovsky, Gornyi, Mirlin

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Surface of 3D $\mathbb{Z}_2$ TI: 2D massless Dirac mode $\equiv$ single-valley graphene!

With disorder:
Topological protection from localization, RG flow towards supermetal

What is the effect of **Coulomb interaction**?

Assume not too strong interaction $r_s = \sqrt{2}e^2/\epsilon \hbar v_F \lesssim 1$

$\Rightarrow$ no instabilities, no symmetry-breaking

$\Rightarrow$ topological protection from localization persists

But interaction may destroy the supermetal phase!
Effect of Coulomb interaction
Altshuler, Aronov ’79; Finkelstein ’83

Any 2D metallic sample $g \gg 1$

Diffusion + Coulomb repulsion $\Rightarrow$ Altshuler-Aronov correction

Include correction into symplectic beta function

$$\beta(g) = \frac{d \log g}{d \log L} = \frac{1}{g} \left[ \frac{N}{2} - 1 + (N^2 - 1)F \right]$$

$N =$ number of independent equivalent species (spin, valleys etc.)
Surface of a 3D topological insulator: $N = 1$

$$\beta(g) \Rightarrow -1/2g$$

Coulomb repulsion destroys supermetallic phase!
Interaction-induced quantum critical state

- Interaction $\implies$ tendency to localization at $g \gg 1$
- Topology $\implies$ prevents strong localization ($g \ll 1$ forbidden)

Result: Interaction induces a novel quantum critical state with universal conductivity $g \sim 1$ on the surface of a 3D topological insulator.

"Self-organized" criticality: no adjustable parameters
Beta functions for 2D spin-orbit systems

<table>
<thead>
<tr>
<th>no interaction</th>
<th>with interaction</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>usual spin-orbit</strong></td>
<td><strong>supermetal</strong></td>
</tr>
<tr>
<td>( \beta(g) )</td>
<td>( \beta(g) )</td>
</tr>
<tr>
<td>( g^* \approx 1.4 )</td>
<td>( \log g )</td>
</tr>
<tr>
<td><strong>Dirac fermions</strong></td>
<td><strong>critical state</strong></td>
</tr>
<tr>
<td>( \beta(g) )</td>
<td>( \beta(g) )</td>
</tr>
<tr>
<td>( \log g )</td>
<td>( \log g )</td>
</tr>
</tbody>
</table>

Ostrovsky, Gornyi, Mirlin

Interaction-induced criticality in topological insulators
In the presence of disorder, normal and topological insulating phases are separated by the supermetal phase.

Transitions between them are conventional symplectic MIT.

No quantum spin-Hall transition.
**QSHE: Stability w.r.t. Coulomb interaction**

**$\mathbb{Z}_2$ edge in the presence of Coulomb interaction**

- **Edge of 2D TI:** single propagating mode in each direction
- **Impurity backscattering prohibited (symplectic TR invariance)**
- **Coulomb interaction $\rightarrow$ Luttinger liquid, conductance $e^2/h$**

Xu, Moore ’06; Wu, Bernevig, Zhang ’06:

**Disorder + interaction:** random Umklapp $\leftrightarrow$ 2-particle backscattering

$$\frac{\partial D_2}{\partial \ln L} = (3 - 8K)D_2$$

$K$ – Luttinger liquid parameter

2PB processes become relevant for $K < 3/8$
Random Umklapp for not too strong Coulomb interaction

Coulomb 1/r interaction:

\[ K(q) = \frac{1}{[1 + \alpha \ln(q_0/q)]^{1/2}} \quad \alpha = \frac{e^2}{\pi \epsilon \hbar v_F} \]

\( D_2 \) processes negligible up to the scale

\[ L_0 \sim q_0^{-1} \exp \frac{160}{9\alpha} \]

What happens with TI beyond this scale is an interesting but purely academic question for not too strong interaction:

\( r_s = 5 \quad \rightarrow \quad L_0 \sim 10 \text{ m} \)

TI phase persists in the presence of not too strong Coulomb interaction
Quantum spin-Hall effect: Coulomb interaction

- Edge modes are protected w.r.t. not too strong Coulomb interaction
- Distinction between normal and topological insulator is robust
- Coulomb interaction “kills” supermetal phase

Interaction restores direct quantum spin-Hall transition via a novel critical state

(a) no interaction
(b) with interaction

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Interaction-induced criticality in topological insulators
Conclusions

We have identified two critical states

- on the surface of 3D topological insulator
- at the quantum spin-Hall transition

Common features:

- symplectic symmetry
- topological protection
- interaction-induced criticality
- conductivity of order $\frac{e^2}{h}$

Maybe these two critical points are equivalent...