# Interaction-induced criticality in topological insulators

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# Outline

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# What is a topological insulator?

**Topological insulator**
**≡ bulk insulator with (topologically protected) metallic edge/surface**

## Examples

- **Quantum Hall effect** at the plateau (QHE insulator with edge states)
- Materials with extreme **spin-orbit** coupling **(inverted gap)**
  - 2D: Quantum spin-Hall effect (HgTe/CdHgTe quantum wells)
  - 3D: $Bi_xSb_{1-x}$, $Bi_2Te_3$, $Bi_2Se_3$, $Sb_2Te_3$, $TlBiSe_2$, ...

# Topological invariants

## Topological invariants

- **QHE**: time-reversal symmetry broken by magnetic field
  Chern number $=$ $\#$ of edge states $= \ldots -2, -1, 0, 1, 2, \ldots$ $(\mathbb{Z})$
  $\Longrightarrow$ $\mathbb{Z}$ topological insulator

- **QSHE**: time-reversal symmetry preserved, spin-rotational broken
  band structure topological invariant $(\mathbb{Z}_2)$: $n = 0$ or $n = 1$
  $\Longleftrightarrow$ odd vs. even number of Kramers pairs of edge states $(\mathbb{Z}_2)$
  $\Longrightarrow$ $\mathbb{Z}_2$ topological insulator

# Quantum Hall effect: $\mathbb{Z}$ topological insulator

## IQH transition

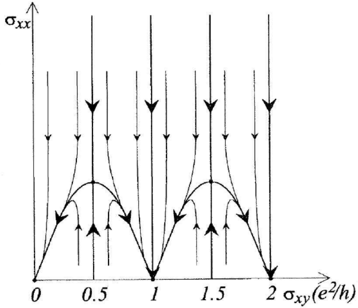
IQHE flow diagram

Khmelnitskii' 83, Pruisken' 84

N edge states — N+1 edge states

localized — localized

critical point




Field theory (Pruisken):

$\sigma$-model with topological term

$$S = \int d^2r \left\{ -\frac{\sigma_{xx}}{8} \mathrm{Tr}(\partial_\mu Q)^2 + \frac{\sigma_{xy}}{8} \mathrm{Tr}\epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q \right\}$$

QHE insulator: Z - topological insulator

# Classification of topological insulators

**Periodic table of Topological Insulators**

| | Symmetry classes | | | | Topological insulators | | | |
|---|---|---|---|---|---|---|---|---|
| $p$ | $H_p$ | $R_p$ | $S_p$ | $\pi_0(R_p)$ | d=1 | d=2 | d=3 | d=4 |
| 0 | AI | BDI | CII | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |
| 1 | BDI | BD | AII | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 |
| 2 | BD | DIII | DIII | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| 3 | DIII | AII | BD | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| 4 | AII | CII | BDI | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| 5 | CII | C | AI | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| 6 | C | CI | CI | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| 7 | CI | AI | C | 0 | 0 | 0 | $\mathbb{Z}$ | 0 |
| $0'$ | A | AIII | AIII | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| $1'$ | AIII | A | A | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |

$H_p$ – symmetry class of Hamiltonians

$R_p$ – sym. class of classifying space (of Hamiltonians with eigenvalues $\to \pm 1$)

$S_p$ – symmetry class of compact sector of $\sigma$-model manifold

Kitaev'09; Schnyder, Ryu, Furusaki, Ludwig '08-09

# Classification of topological insulators

Two ways to detect existence of TIs of class $p$ in $d$ dimensions:

(i) by inspecting the topology of classifying spaces $R_p$:

$$\begin{cases} \text{TI of type } \mathbb{Z} \\ \text{TI of type } \mathbb{Z}_2 \end{cases} \iff \pi_0(R_{p-d}) = \begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \end{cases}$$

(ii) by analyzing homotopy groups of the $\sigma$-model manifolds:

$$\begin{cases} \text{TI of type } \mathbb{Z} \iff \pi_d(S_p) = \mathbb{Z} & \text{Wess-Zumino term} \\ \text{TI of type } \mathbb{Z}_2 \iff \pi_{d-1}(S_p) = \mathbb{Z}_2 & \theta = \pi \ \ \text{topological term} \end{cases}$$

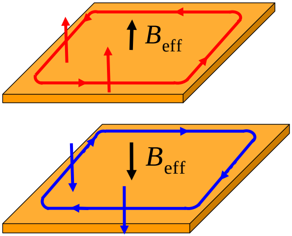
WZ and $\theta = \pi$ terms make boundary excitations “non-localizable”

TI in $d$ $\iff$ topological protection from localization in $d-1$

Bott periodicity: $\pi_d(R_p) = \pi_0(R_{p+d})$ , periodicity 8

# 2D topological insulators: Quantum Spin-Hall effect

Kane and Mele '05, Sheng et al. '05,
Bernevig and Zhang '06.




- The QSHE state does not break the (symplectic) TR symmetry: $T^2 = -1$.

- **Simplified picture:** two copies of QH states for two spin components, each seeing the opposite magnetic field.

$$\left.\begin{array}{l} \text{Spin} \uparrow \text{QHE} : \sigma_{xy} = n\, e^2/h \\ \text{Spin} \downarrow \text{QHE} : \sigma_{xy} = -n\, e^2/h \end{array}\right\} \quad \text{Zero net charge QHE}$$

$$\text{Spin Hall conductance} \;\; \sigma_{xy}(\uparrow) - \sigma_{xy}(\downarrow) = 2\, e^2/h$$

- **Generic SO:** spins are not conserved, but Kramers degeneracy still holds

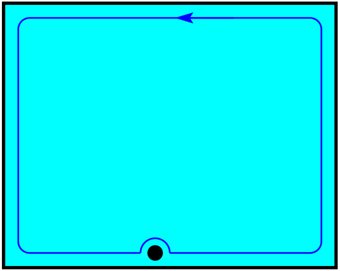
**TR invariance forbids backscattering**

**➔ topologically protected edge states**

# Quantum (spin-)Hall effect with disorder

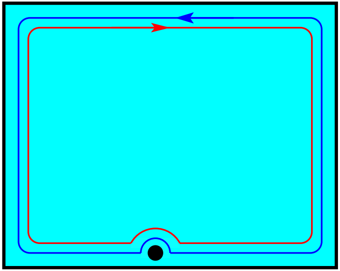
**Impurities do not localize the edge states:**

**QHE**



due to chirality of carriers
(class A)
any disorder

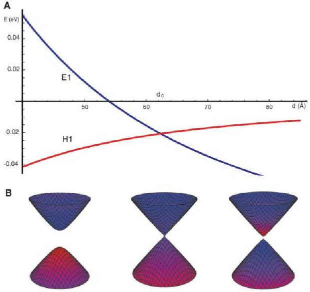
**QSHE**

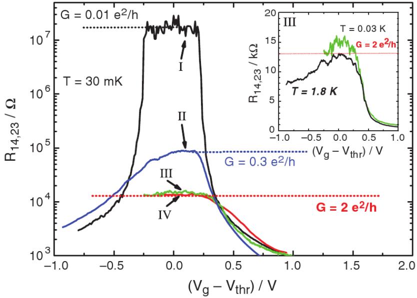


due to time-inversion symmetry
(class AII)
no magnetic impurities

# Quantum spin-Hall effect (HgTe/CdTe QW)

Theory: Bernevig, Hughes, Zhang '06; Experiment: Molenkamp group '07




2D Dirac Hamiltonian with tunable mass: $m \gtrless 0$ when $d \lessgtr d_c$

I — $d = 5.5$nm: normal insulator

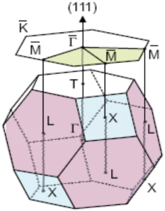
II, III, IV — $d = 7.3$nm: inverted band gap — topological insulator

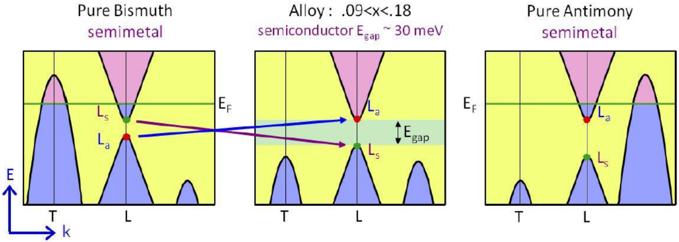
# 3D topological insulator: band structure

## Hasan group '08

**$Bi_{1-x}Sb_x$**

**Bulk band structure:**



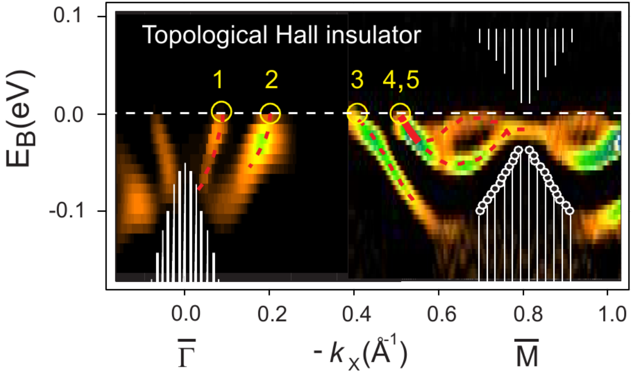

A7 **rhombohedral** crystal structure.

Other realizations: BiTe, BiSe ...

# 3D topological insulator: spectroscopy

Hasan group '08

ARPES measurement on $Bi_{0.9}Sb_{0.1}$

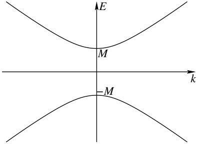



**Odd number of surface modes $\Longrightarrow$ nontrivial topology**

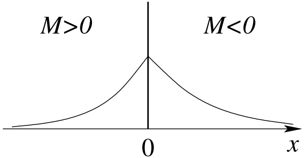
# 3D topological insulator: phenomenological description

cf. Volkov & Pankratov '85

Bulk Hamiltonian (Dirac): $H_b = \begin{pmatrix} M & \boldsymbol{\sigma}\mathbf{p} \\ \boldsymbol{\sigma}\mathbf{p} & -M \end{pmatrix}$




Domain wall:



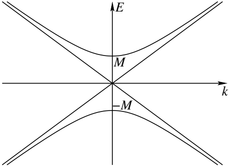

## Edge state

Decays into the bulk: $\Psi = e^{-|Mx|}\begin{pmatrix}\psi \\ \chi\end{pmatrix}$

Boundary condition: $\chi = -i\sigma_x\psi \Rightarrow -i\boldsymbol{\sigma}\mathbf{n}\psi$

## Surface Hamiltonian

$$H_s = \underbrace{\frac{\nabla\mathbf{n}}{2}}_{\text{curvature}} + \underbrace{\frac{1}{2}\left(\mathbf{n}[\mathbf{p}\times\boldsymbol{\sigma}] + [\mathbf{p}\times\boldsymbol{\sigma}]\mathbf{n}\right)}_{\text{Rashba}} \quad \Rightarrow \quad \mathbf{n}[\mathbf{p}\times\boldsymbol{\sigma}]$$



# Surface of 3D topological insulators of symmetry class AII

3D Topological Insulators

have 2D delocalized modes at the surface

surface of a 3D TI $=$ single-valley graphene

2D disordered Dirac fermions of symmetry class AII:

topological protection against localization

# Outline

# Scaling theory of localization

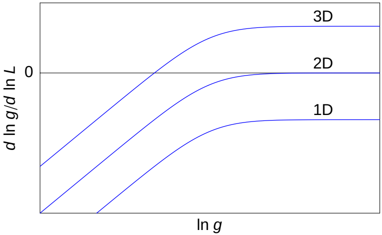
Abrahams, Anderson, Licciardello, Ramakrishnan '79

## Dimensionless conductance [in units $e^2/h$]

- Metallic sample (Ohm's law): $g \sim L^{d-2}$
- Insulating sample (tunneling): $g \sim e^{-L/\xi}$

## Universal scaling function

$$\frac{d \ln g}{d \ln L} = \beta(g) = \begin{cases} d-2, & g \gg 1, \quad \text{(metal)}, \\ \ln g, & g \ll 1, \quad \text{(insulator)}. \end{cases}$$

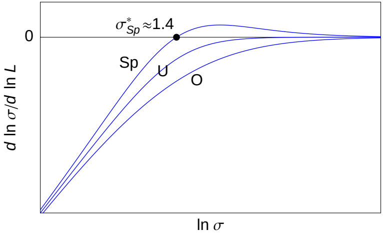


# Weak localization correction in 2D

Gor'kov, Larkin, Khmelnitskii '79; Hikami, Larkin, Nagaoka '80

## Scaling of conductivity in 2D (no e-e interaction)

$$\frac{d\ln g}{d\ln L} = \begin{cases} -1/g, & \text{orthogonal (TR preserved)} \\ -1/2g^2, & \text{unitary (TR broken)} \\ +1/2g, & \text{symplectic (TR preserved, spin-orbit)} \end{cases}$$ **we are here!**



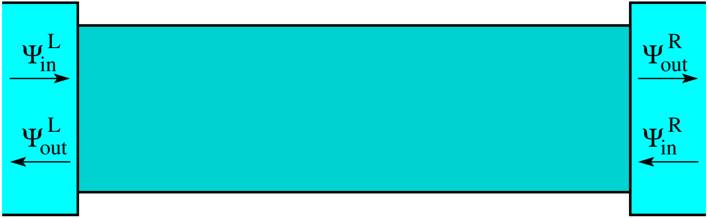

## 2D Dirac electrons:

Metal or insulator?

**MIT in symplectic class at $\sigma^*_{Sp} \approx 1.4 e^2/h$**

# One-dimensional symplectic wire

Zirnbauer '92, Mirlin et al '94, Ando & Suzuura '02, Takane '04




## Scattering matrix of a symplectic system

$$\begin{pmatrix} \Psi_{\text{out}}^{L} \\ \Psi_{\text{out}}^{R} \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} \Psi_{\text{in}}^{L} \\ \Psi_{\text{in}}^{R} \end{pmatrix}$$

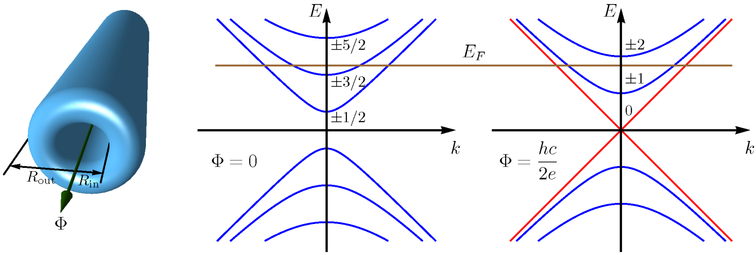
TI symmetry $\Longrightarrow$

$$r = -r^T$$
$$r' = -r'^T$$
$$t = t'^T$$

## For N channels:

$\det r = (-1)^N \det r^T$ $\Longrightarrow$ **no localization if $N$ is odd ! ! !**

# Topological insulator: reduction to 1D




## Hollow cylinder threaded with magnetic flux $\Phi$

Surface states: $E_n(p) = \pm\sqrt{p^2 + \left(n + \frac{1}{2} - \frac{e\Phi}{hc}\right)^2}$

Time-inversion symmetry is preserved if $\frac{e\Phi}{hc}$ is **integer** or **half-integer**

**no 1D localization** $\Longrightarrow$ **no 2D localization**

# Dirac fermions in symplectic class: sigma model

Random potential: symplectic time-reversal symmetry $H = \sigma_y H^T \sigma_y$

Symplectic sigma model: topological $\theta$-term with $\theta = \pi$

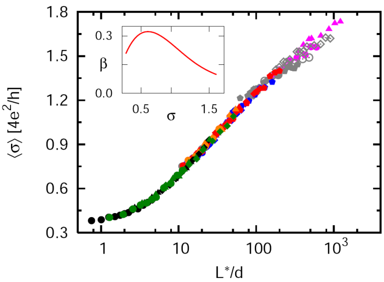
$$S[Q] = \frac{\sigma_{xx}}{16} \mathrm{Str}(\nabla Q)^2 + i\theta N[Q] \qquad N[Q] = 0, 1$$

Similar to Pruisken sigma model for IQHE (instantons suppress localization)

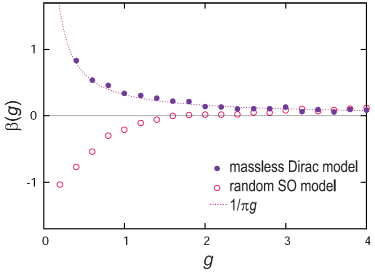
**No localization! Criticality?**

- Minimal conductivity: $\sigma = 4\sigma_{Sp}^{**} \sim e^2/h$, **or**
- Absolute antilocalization: $\sigma \to \infty$

# Scaling of conductance: numerical results




Bardarson, Tworzydło, Brouwer, Beenakker '07

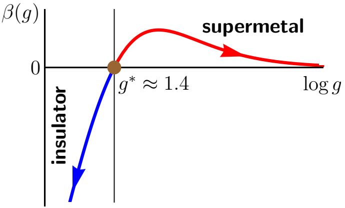



Nomura, Koshino, Ryu '07

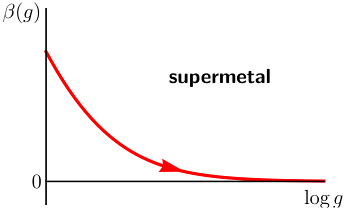
- Absence of localization confirmed
- Supermetallic behaviour for microscopic models considered

# Beta functions for symplectic system

$$\beta(g) = \frac{d \log g}{d \log L}$$




Usual spin-orbit metal




Dirac fermions

# Outline

# 2D surface states of a 3D TI: Disorder and interaction

Surface of 3D $\mathbb{Z}_2$ TI: 2D massless Dirac mode $\equiv$ single-valley graphene!

With disorder:

Topological protection from localization, RG flow towards supermetal

## What is the effect of Coulomb interaction?

assume not too strong interaction $\quad r_s = \sqrt{2}e^2/\epsilon\hbar v_F \lesssim 1$

$\Longrightarrow$ no instabilities, no symmetry-breaking

$\Longrightarrow$ topological protection from localization persists

But interaction may destroy the supermetal phase!

# Effect of Coulomb interaction

Altshuler, Aronov '79; Finkelstein '83

## Any 2D metallic sample $g \gg 1$

Diffusion + Coulomb repulsion $\Rightarrow$ Altshuler-Aronov correction

## Include correction into symplectic beta function

$$\beta(g) = \frac{d \log g}{d \log L} = \frac{1}{g} \left[ \frac{N}{2} - 1 + (N^2 - 1)\mathcal{F} \right]$$

$N$ = number of independent equivalent species (spin, valleys etc.)

Surface of a 3D topological insulator: $N = 1$

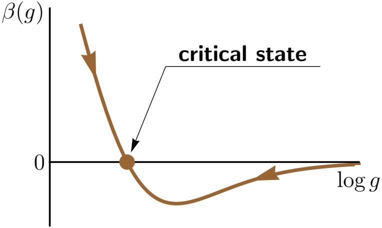
$$\beta(g) \Rightarrow -1/2g$$

**Coulomb repulsion destroys supermetallic phase!**

# Interaction-induced quantum critical state

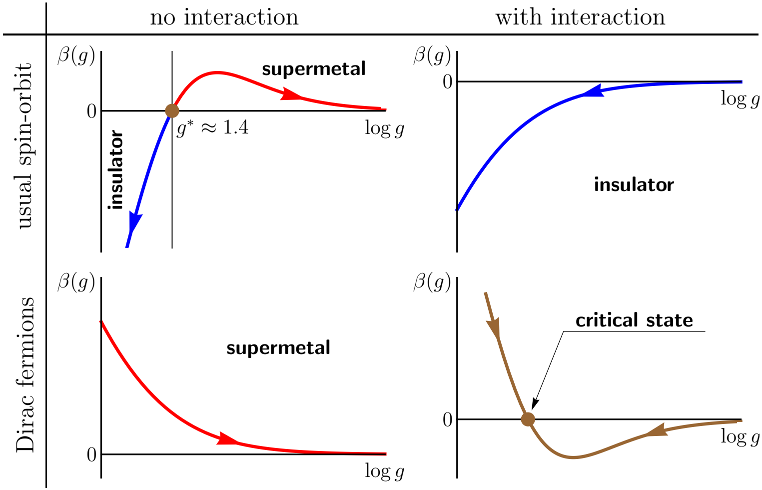
- Interaction $\Longrightarrow$ tendency to localization at $g \gg 1$
- Topology $\Longrightarrow$ prevents strong localization ($g \ll 1$ forbidden)

Result: Interaction induces a **novel quantum critical state** with universal conductivity $g \sim 1$ on the surface of a 3D topological insulator.

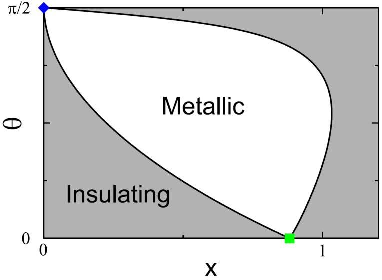



"Self-organized" criticality: no adjustable parameters

# Beta functions for 2D spin-orbit systems



# Quantum spin-Hall effect: phase diagram




Obuse, Furusaki, Ryu, Mudry '07

- In the presence of disorder, normal and topological insulating phases are separated by the supermetal phase
- Transitions between them are conventional symplectic MIT
- **No quantum spin-Hall transition**

# QSHE: Stability w.r.t. Coulomb interaction

## $\mathbb{Z}_2$ edge in the presence of Coulomb interaction

- Edge of 2D TI: single propagating mode in each direction
- Impurity backscattering prohibited (symplectic TR invariance)
- Coulomb interaction $\longrightarrow$ Luttinger liquid, conductance $e^2/h$

Xu, Moore '06; Wu, Bernevig, Zhang '06:

Disorder + interaction: random Umklapp $\longleftrightarrow$ 2-particle backscattering

$\partial \mathcal{D}_2/\partial \ln L = (3 - 8K)\mathcal{D}_2$ $\qquad$ $K$ – Luttinger liquid parameter

2PB processes become relevant for $K < 3/8$

# Random Umklapp for not too strong Coulomb interaction

## Coulomb 1/r interaction:

$$K(q) = \frac{1}{[1+\alpha \ln(q_0/q)]^{1/2}} \qquad \alpha = e^2/\pi\epsilon\hbar v_F$$

$\mathcal{D}_2$ processes negligible up to the scale $\quad L_0 \sim q_0^{-1} \exp \dfrac{160}{9\alpha}$

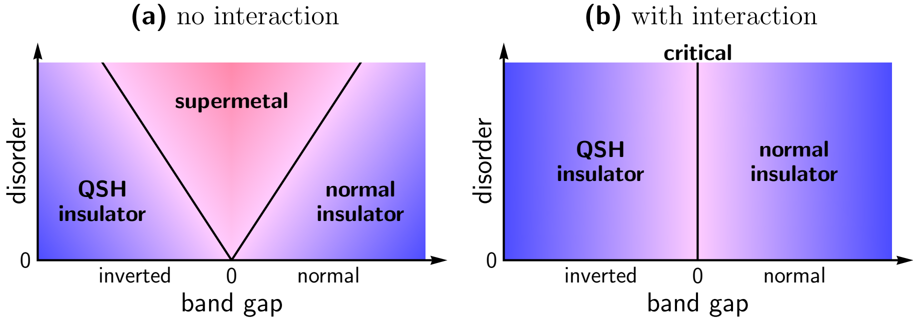
What happens with TI beyond this scale is an interesting but purely academic question for not too strong interaction:

$r_s = 5 \quad \longrightarrow \quad L_0 \sim 10$ m

TI phase persists in the presence of not too strong Coulomb interaction

# Quantum spin-Hall effect: Coulomb interaction

- Edge modes are protected w.r.t. not too strong Coulomb interaction
- Distinction between normal and topological insulator is robust
- Coulomb interaction “kills” supermetal phase




Interaction restores direct **quantum spin-Hall transition**
via a **novel critical state**

# Conclusions

## We have identified two critical states

- on the surface of 3D topological insulator
- at the quantum spin-Hall transition

## Common features:

- symplectic symmetry
- topological protection
- interaction-induced criticality
- conductivity of order $e^2/h$

Maybe these two critical points are equivalent...