13th July 2011, Euler Symposium, Saint Petersburg

Localization of the Laplace operator eigenfunctions

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Plan of the talk

Introduction

- Historical overview and related problems
- Empirical observations

Low-frequency localization

High-frequency localization

Open problems and questions…

What do w
what do we

We know:

General spectral pr

Eigenfunctions for

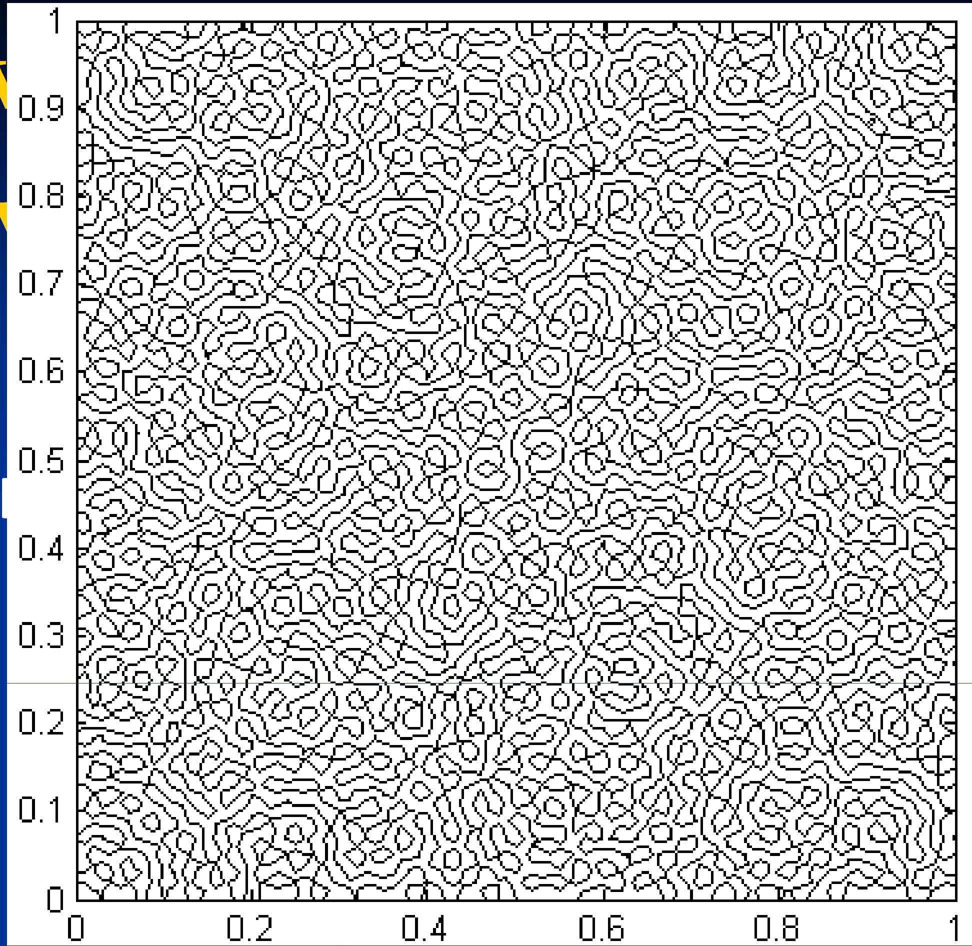
Asymptotic behavi

We want to know:

Relation eigenfunctions/geometry?

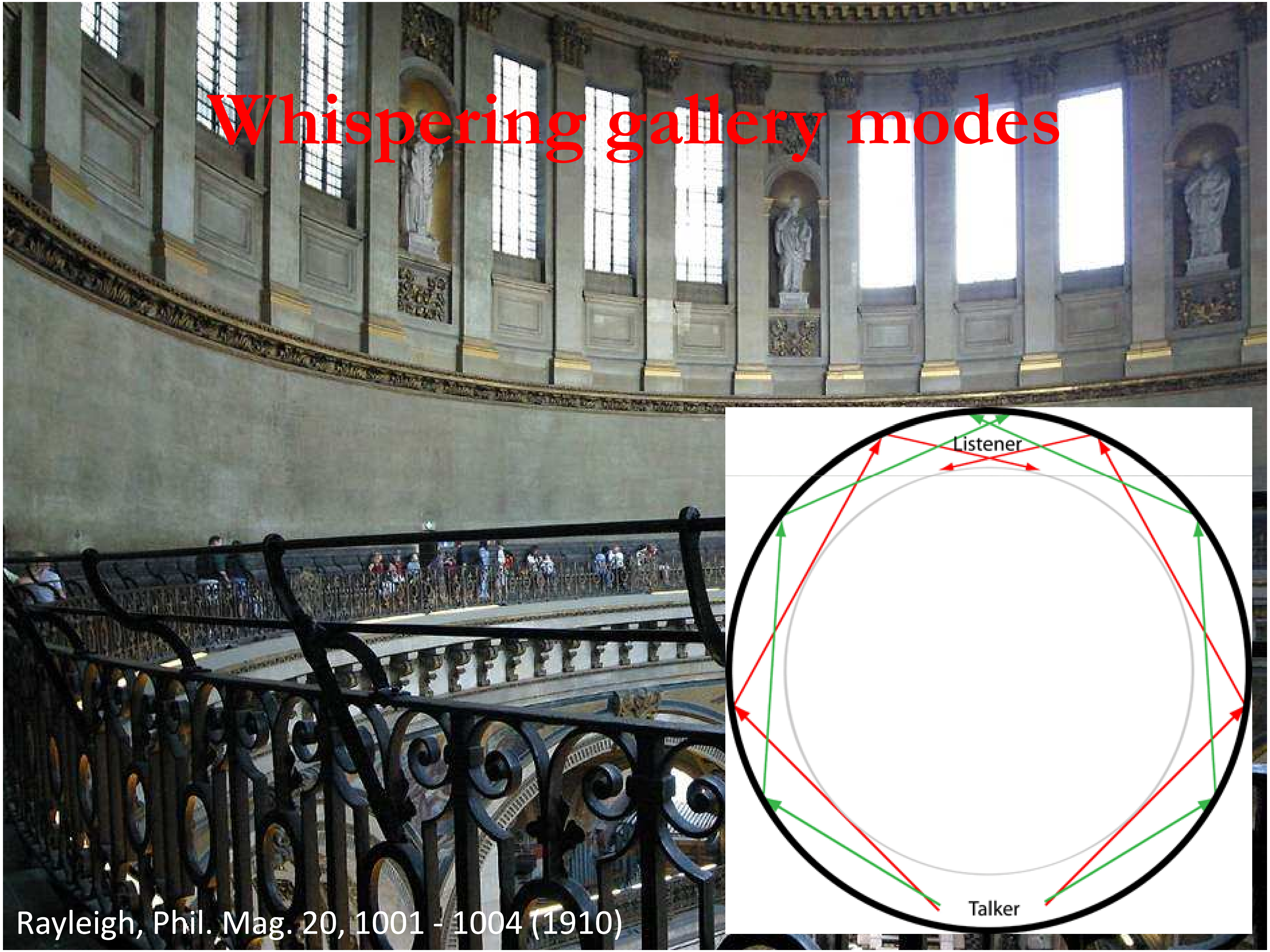
Localization of eigenfunctions

Physical consequences and applications



$\lambda = 5525\pi^2$

Whispering gallery modes



Rayleigh, Phil. Mag. 20, 1001 - 1004 (1910)

Localization by potential

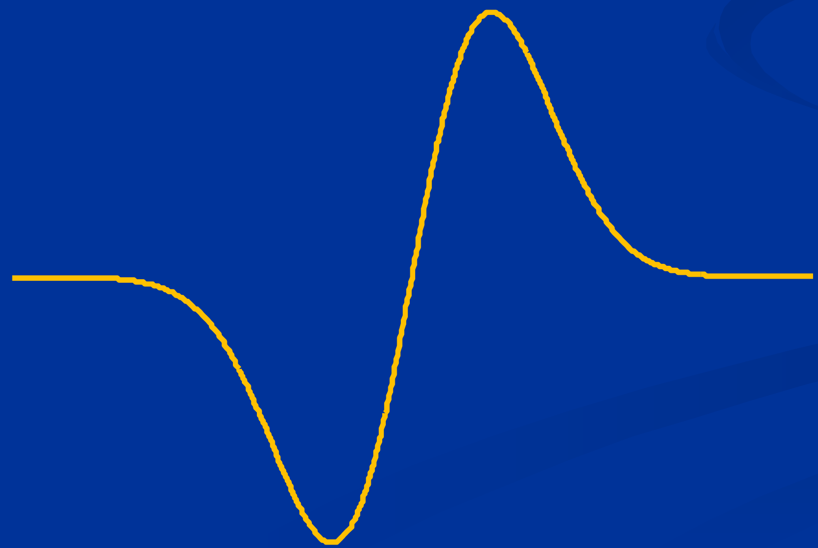
Example: quantum harmonic oscillator

$$H = -\frac{\hbar^2}{2m}\Delta + \frac{m\omega^2}{2}x^2 \qquad H\psi_n(x) = E_n\psi_n(x)$$

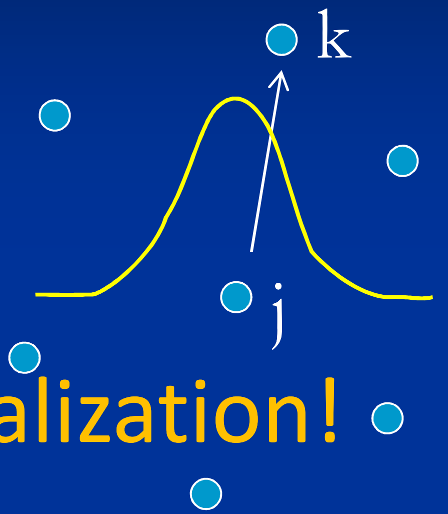
$$\psi_n(x) \propto e^{-(x/\ell)^2/2} H_n(x/\ell)$$

Hermite polynomials

$$\ell = \sqrt{\hbar/(m\omega)}$$



Anderson localization



Each site has its energy E_j,
which is a random variable with P(E)

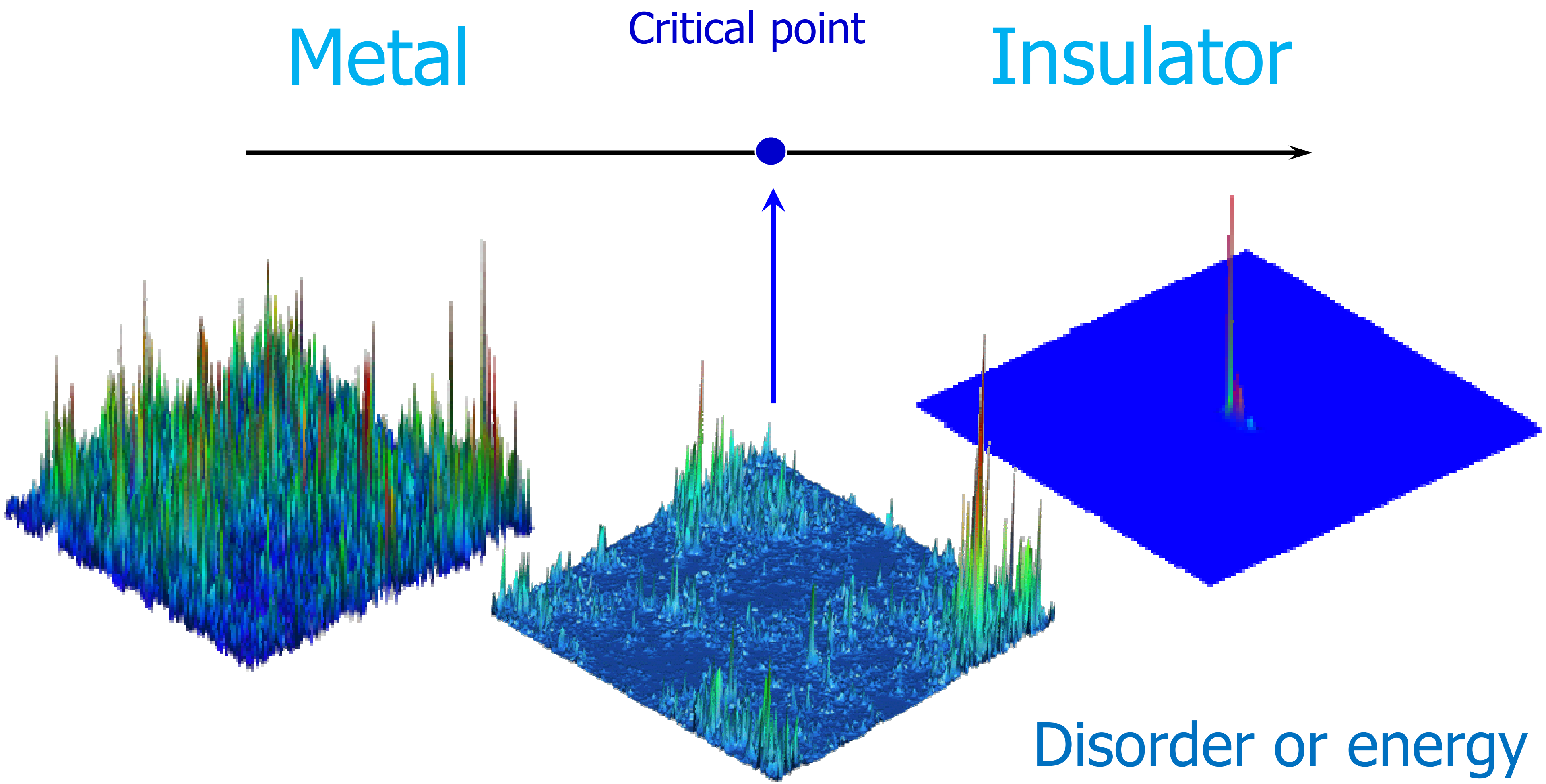
$V_{jk}(r_{jk})$ is the interaction between sites j and k
(i.e., a possibility of transfer between them)

Localization!

Random potential (disorder) may lead to localization of wave functions

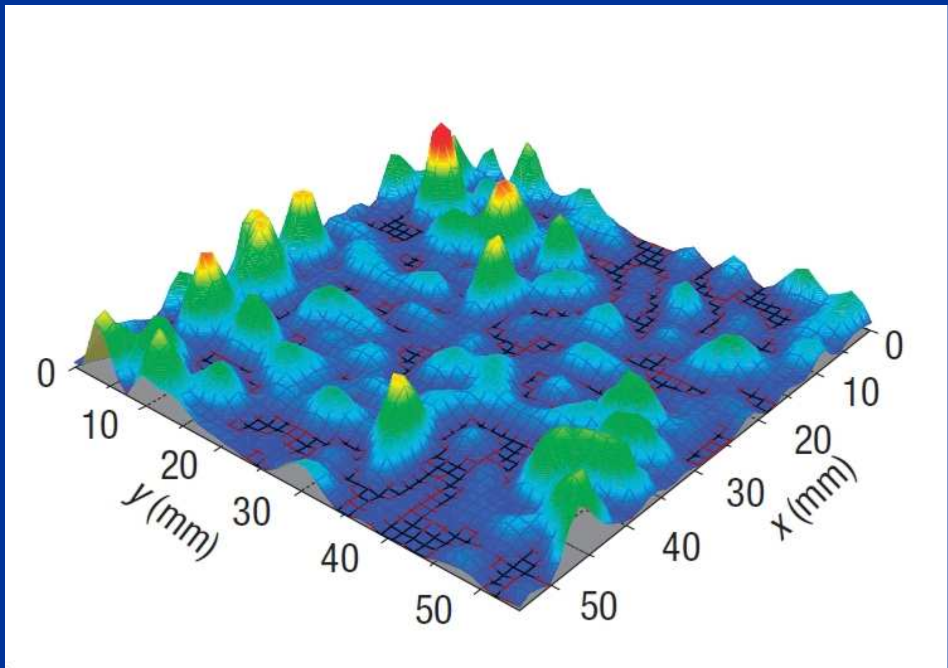
Anderson, Phys. Rev. 109, 1492 (1958) Evers, Mirlin, Rev. Mod. Phys. 80, 1355 (2008)

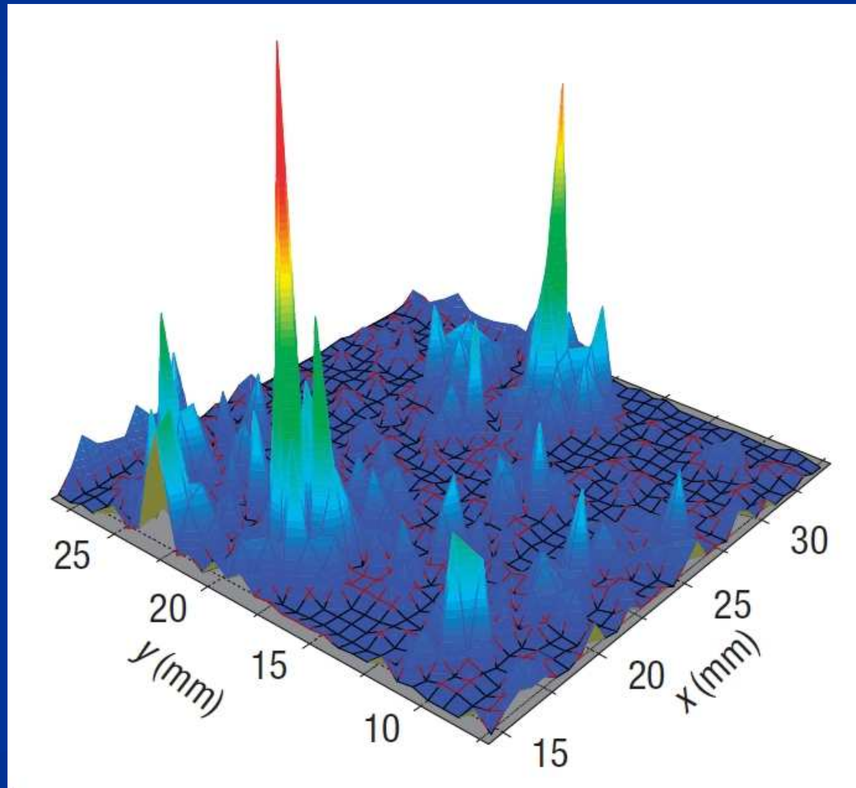
Wave functions across Anderson transition



(Slide from I. Gruzberg's talk)

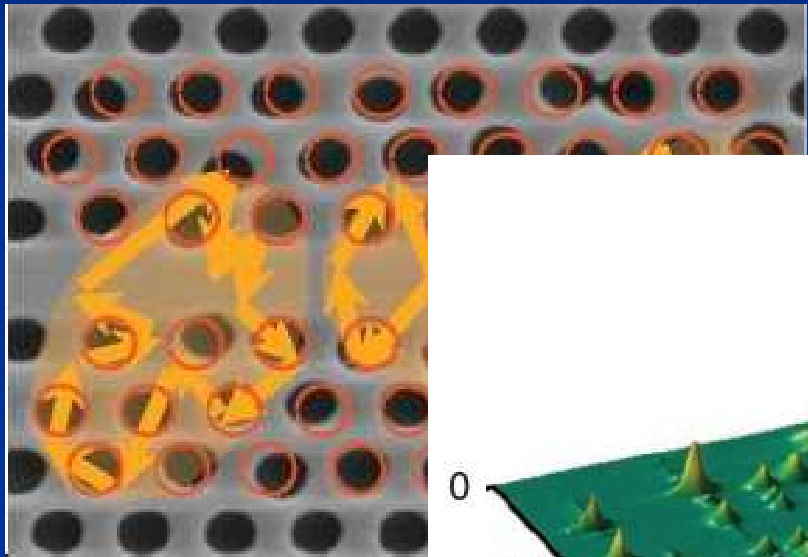
Localized waves functions in ultrasound experiments

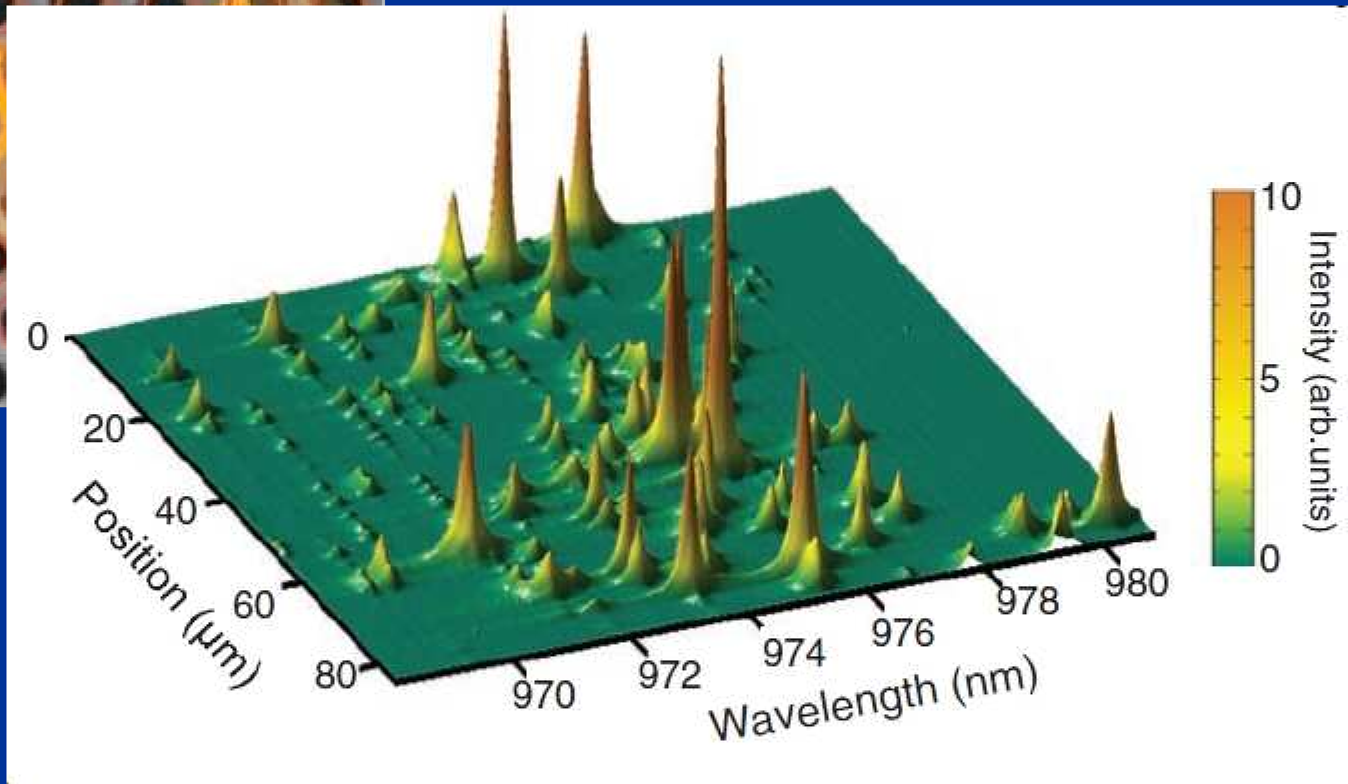




Hu et al., Nature Phys. 4, 945 (2008)

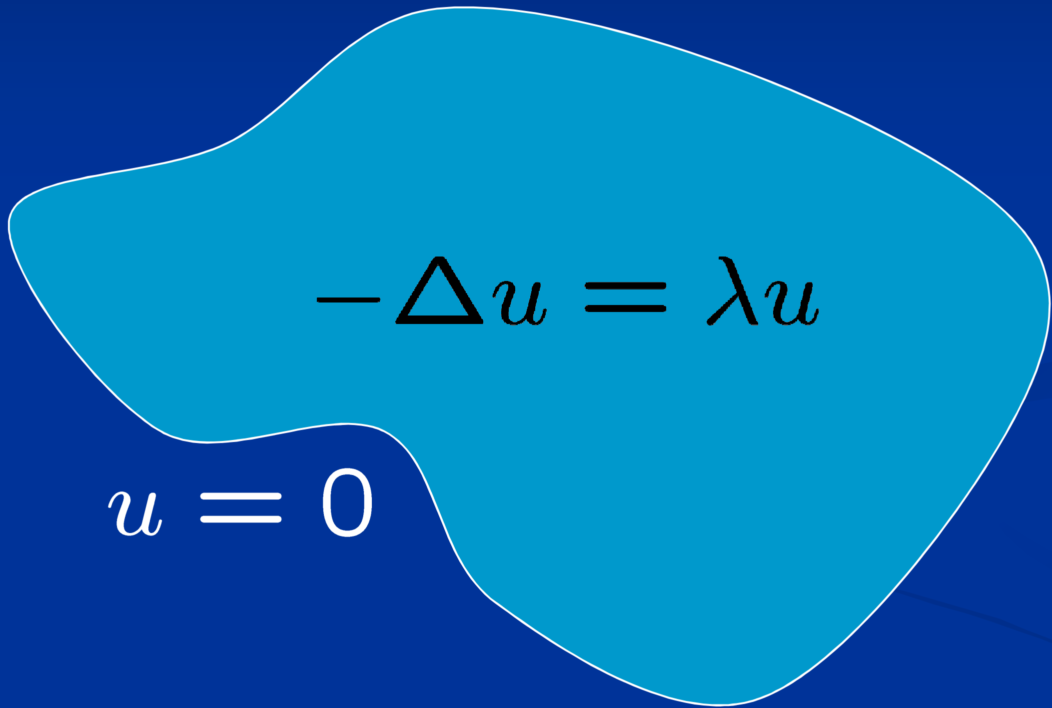
Localized waves functions in disordered photonic crystals





Sapienza *et al.*, Science 327, 1352-1355 (2010)

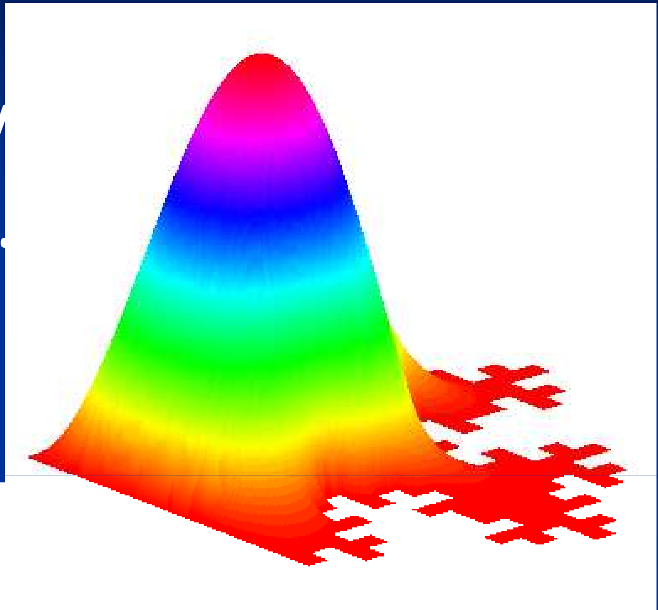
Laplacian eigenfunctions

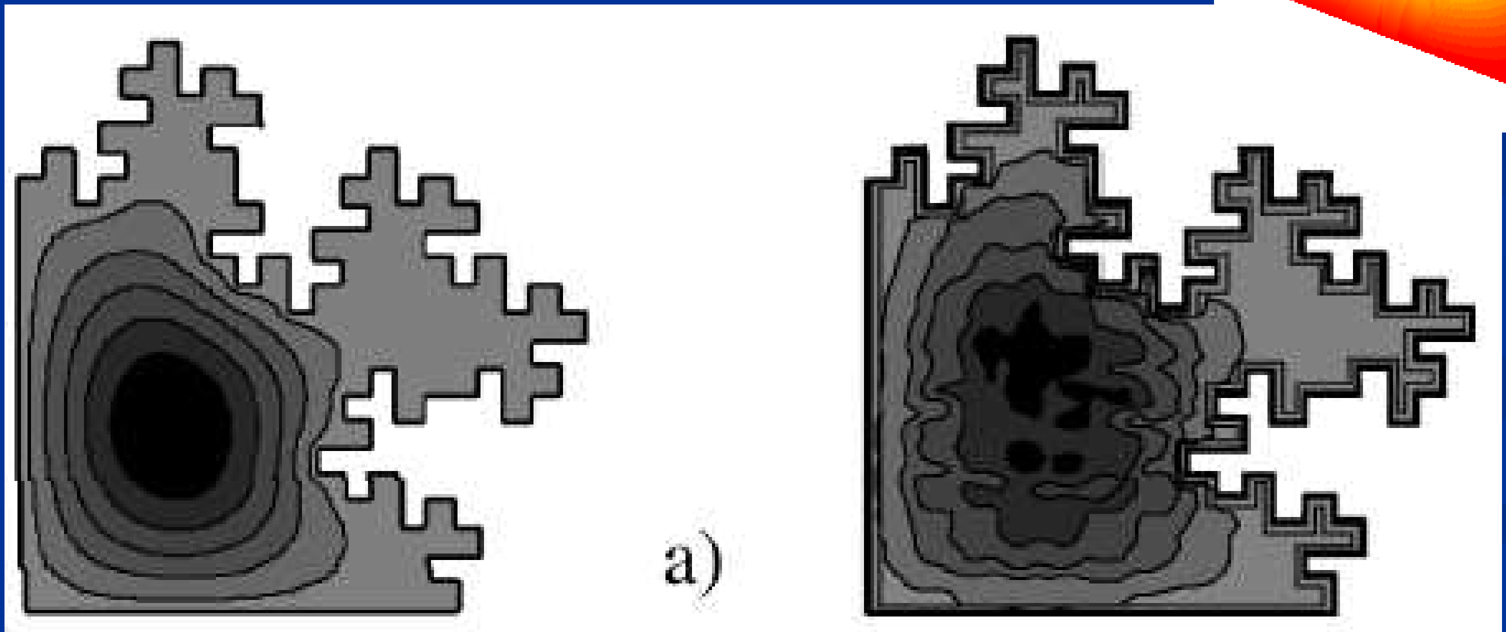


No potential, only the shape

Laplacian eigenfunctions

Since 1990s, many studies of v
irregular or fractal drums by B.

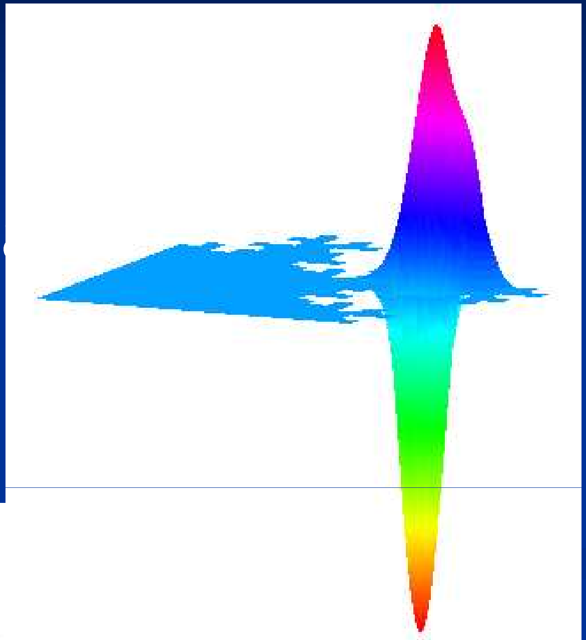


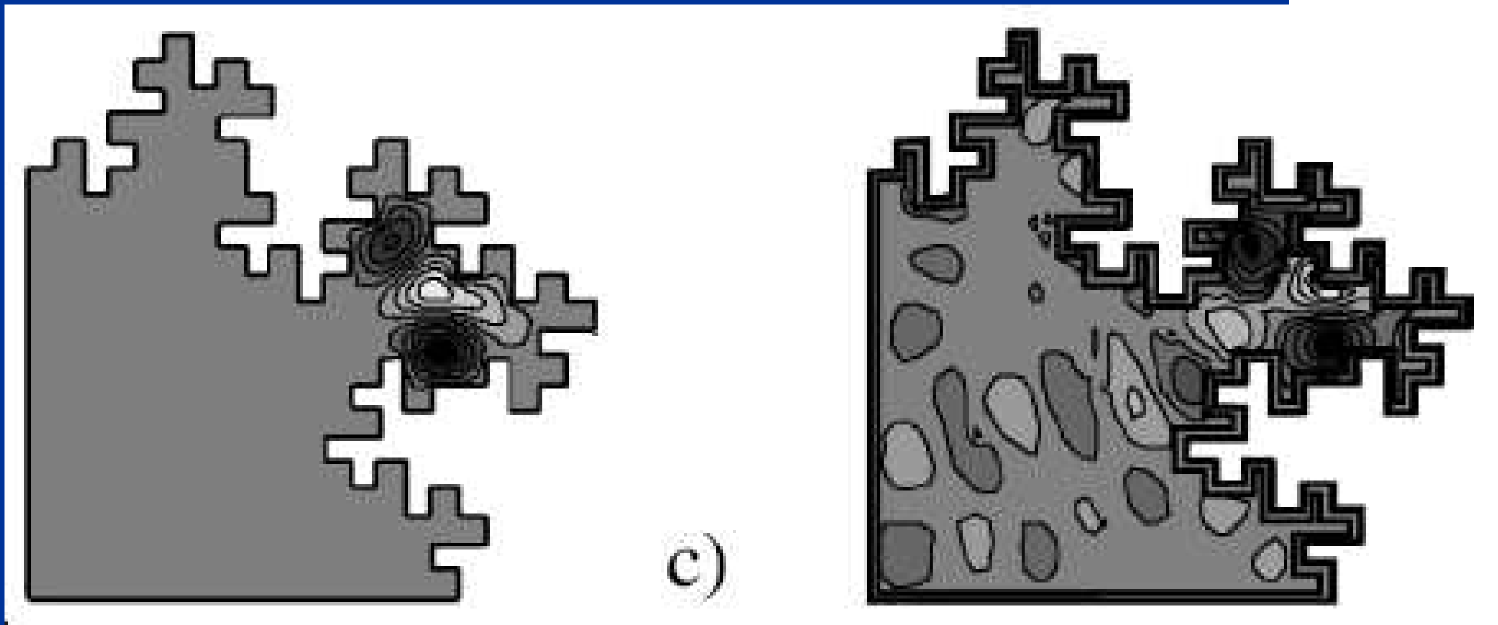


Even et al., Phys. Rev. Lett. 83, 726 (1999)

Laplacian eigenfunctions

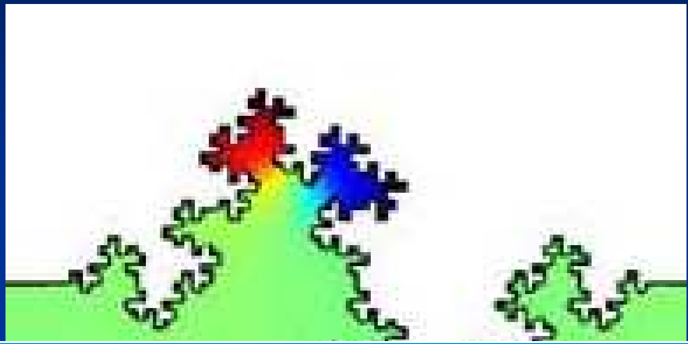
Since 1990s, many studies of vib
irregular or fractal drums by B. S



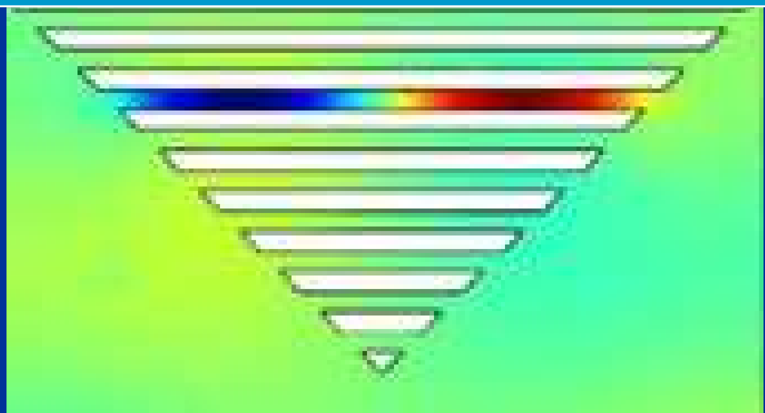


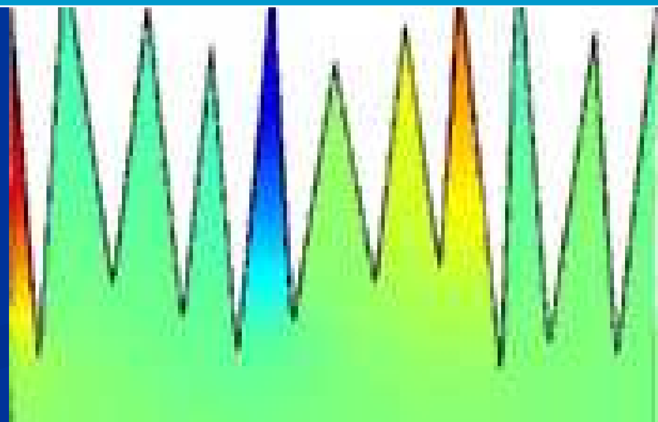
Even et al., Phys. Rev. Lett. 83, 726 (1999)

Laplacian eigenfunctions



Geometrical irregularity may lead to localization of eigenfunctions





S. Felix, et al., J. Sound. Vibr. 299, 965, 2007

Laplacian eigenfunctions

Since 1990s, many studies of vibrations of irregular or fractal drums by B. Sapoval *et al.*

… towards one of (many) practical applications

(Anti-noise wall)

Plan of the talk

Introduction

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- Empirical observations

Low-frequency localization

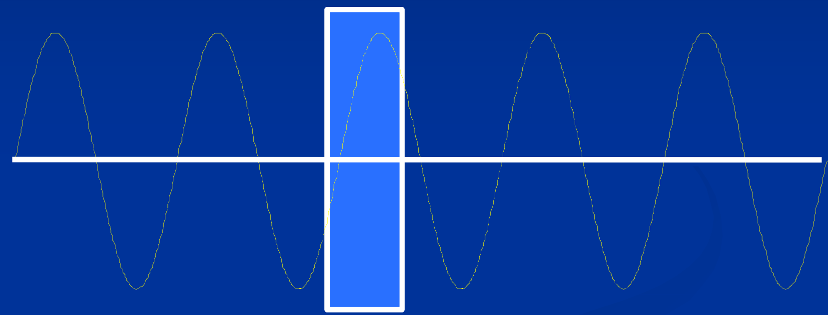
High-frequency localization

Open problems and questions…

Laplacian eigenfunctions

"Regular" domains
(interval, square, cube)

$$u_m(x) = \sin(\pi m x)$$

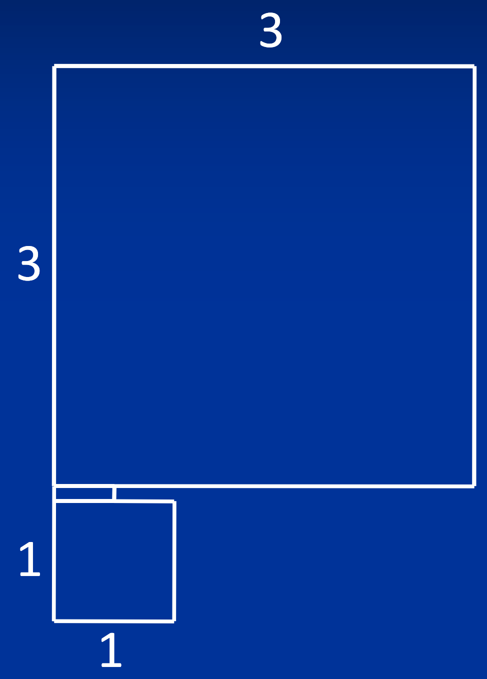


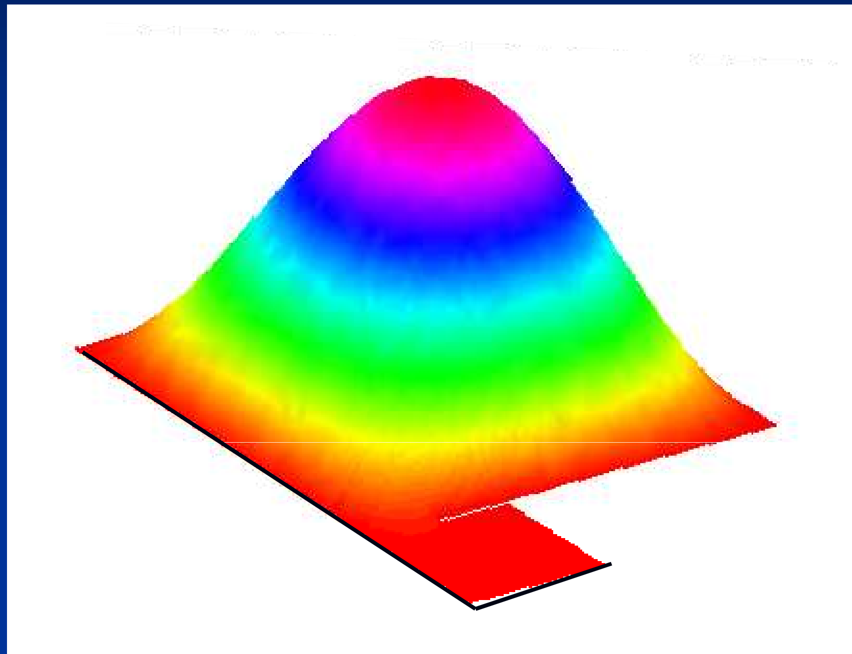
Nonlocalized (extended) eigenfunction:
Eigenfunction is supported by the whole domain

~~Localized eigenfunction:~~
~~There exists a small open subset that supports a large fraction of eigenfunction~~

?

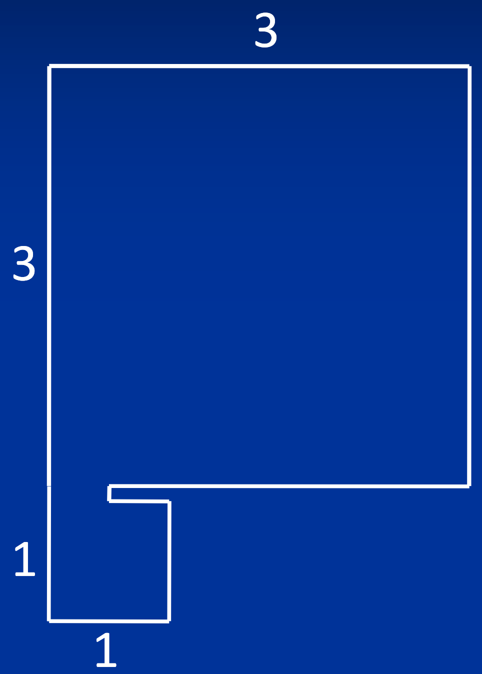
“Bottleneck” localization

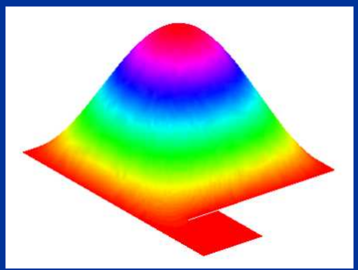


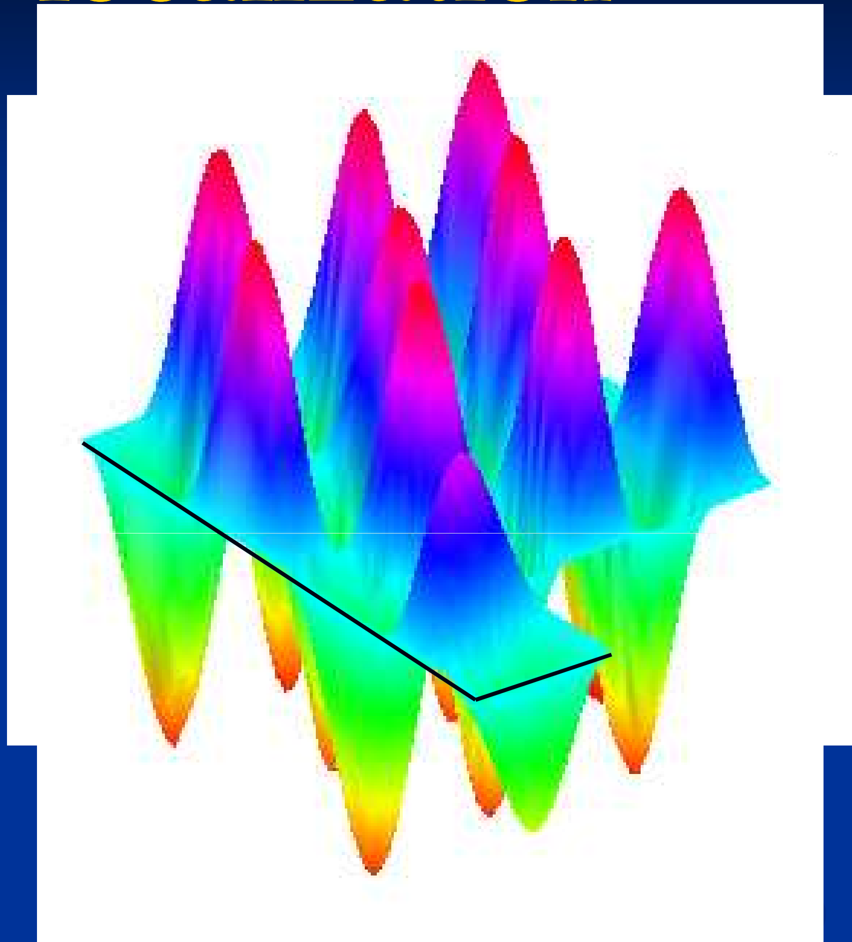


$$\lambda_1 \simeq 2.2$$

“Bottleneck” localization

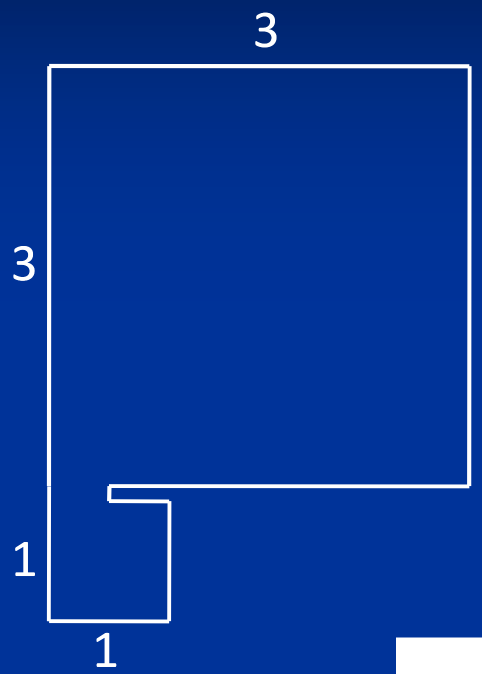


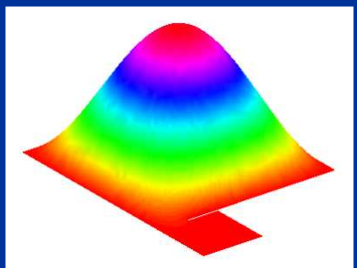


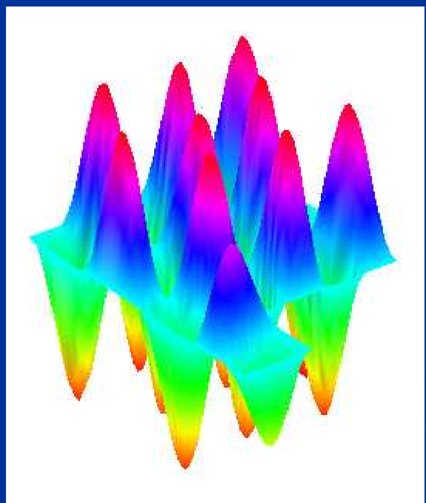


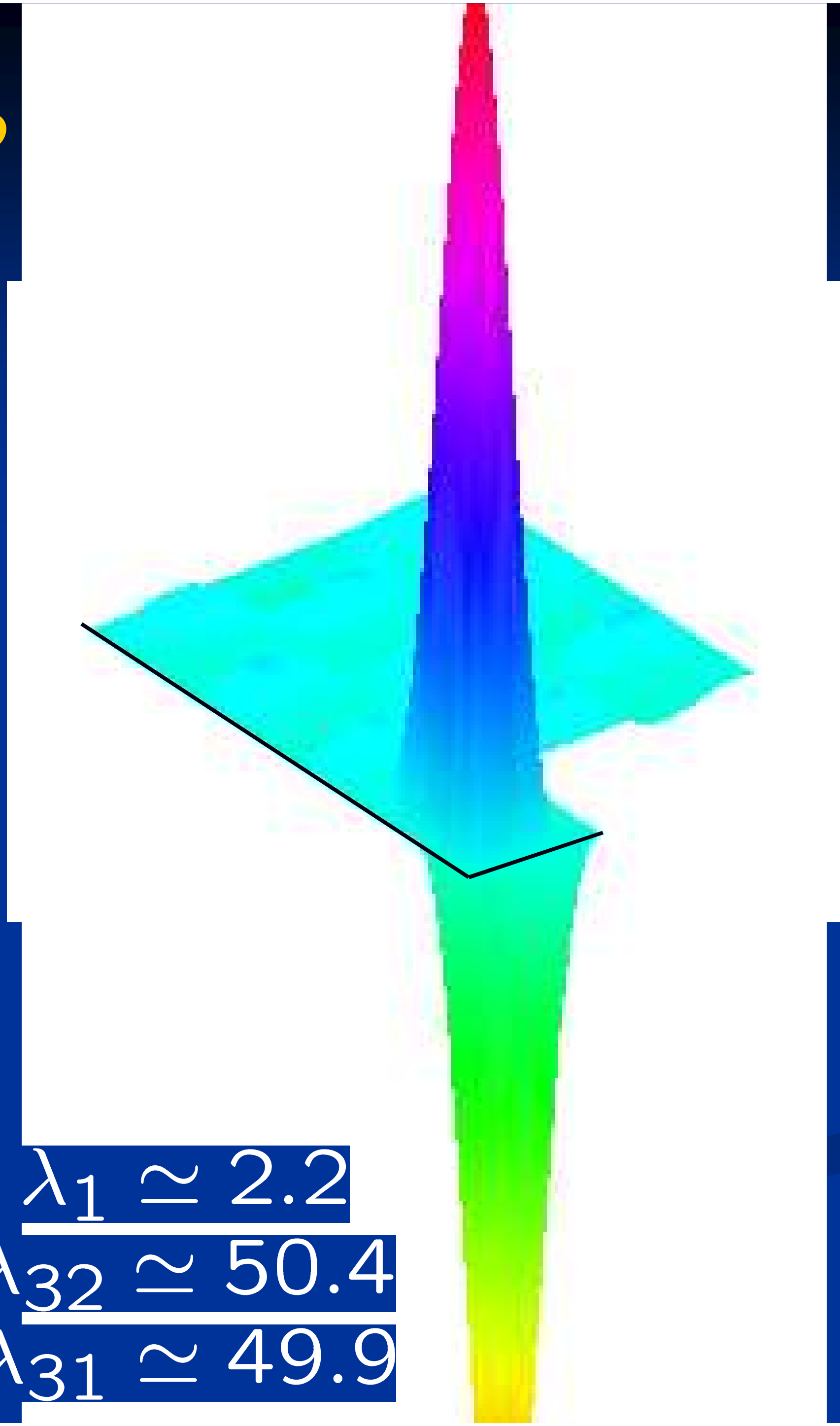
$$\lambda_1 \simeq 2.2$$
$$\lambda_{32} \simeq 50.4$$

“Bottleneck”









$\lambda_1 \simeq 2.2$

$\lambda_{32} \simeq 50.4$

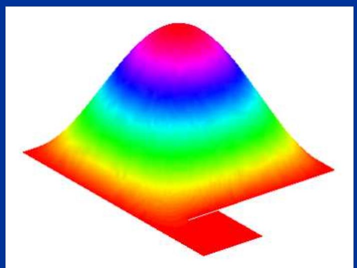
$\lambda_{31} \simeq 49.9$

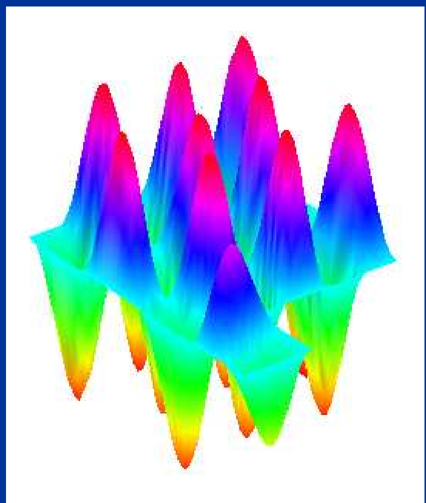
"Bottleneck" localization

"Separated" subsets

Not all but a fraction of eigenfuctions are localized

Localized eigenfunctions





$$\lambda_1 \simeq 2.2$$
$$\lambda_{32} \simeq 50.4$$
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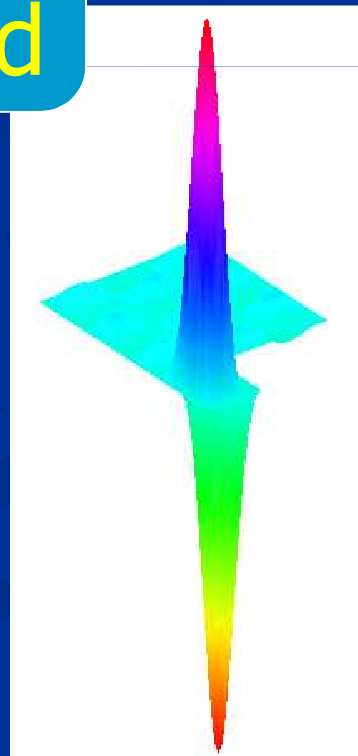
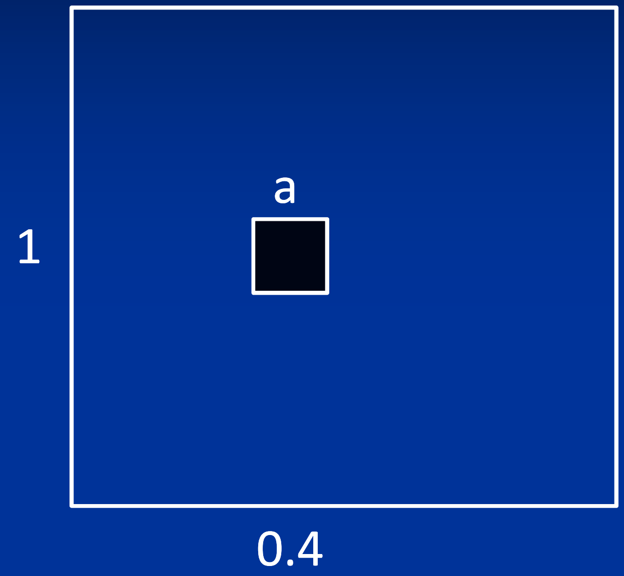
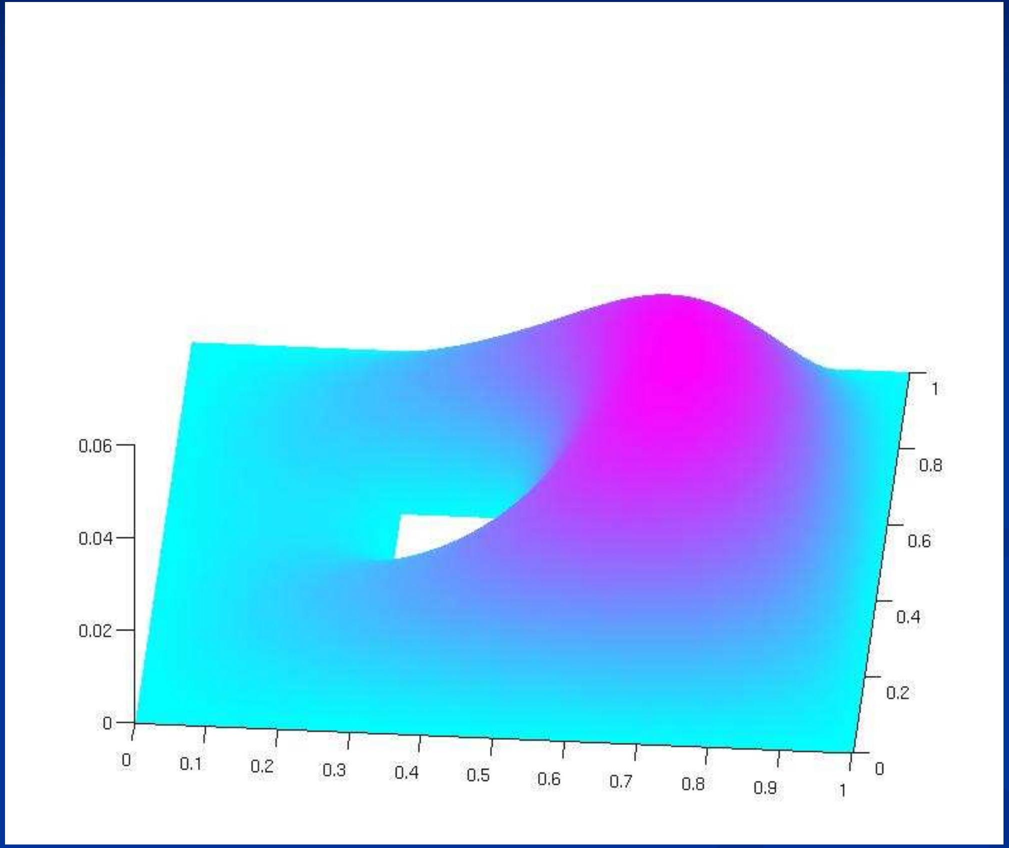
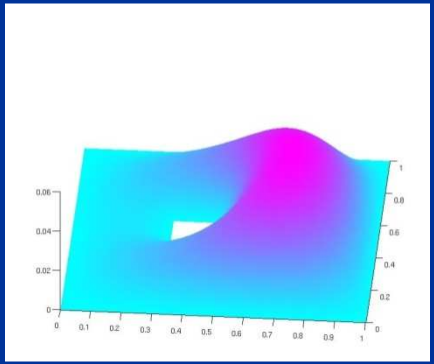


Localization by “dust”



L_{tot}=na = 0.15





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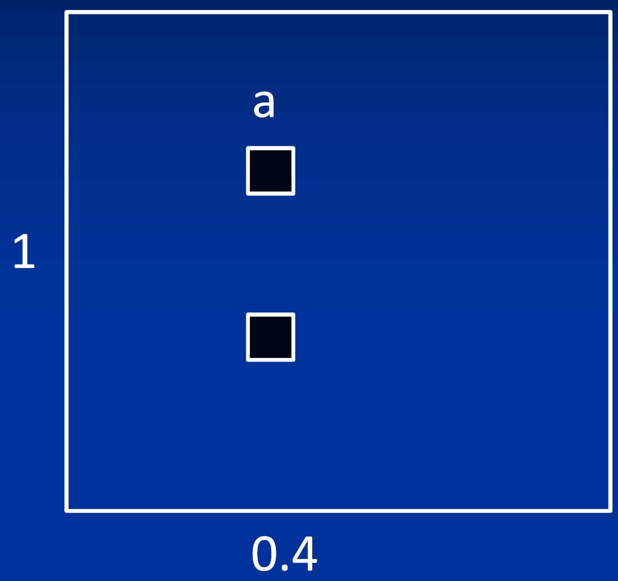

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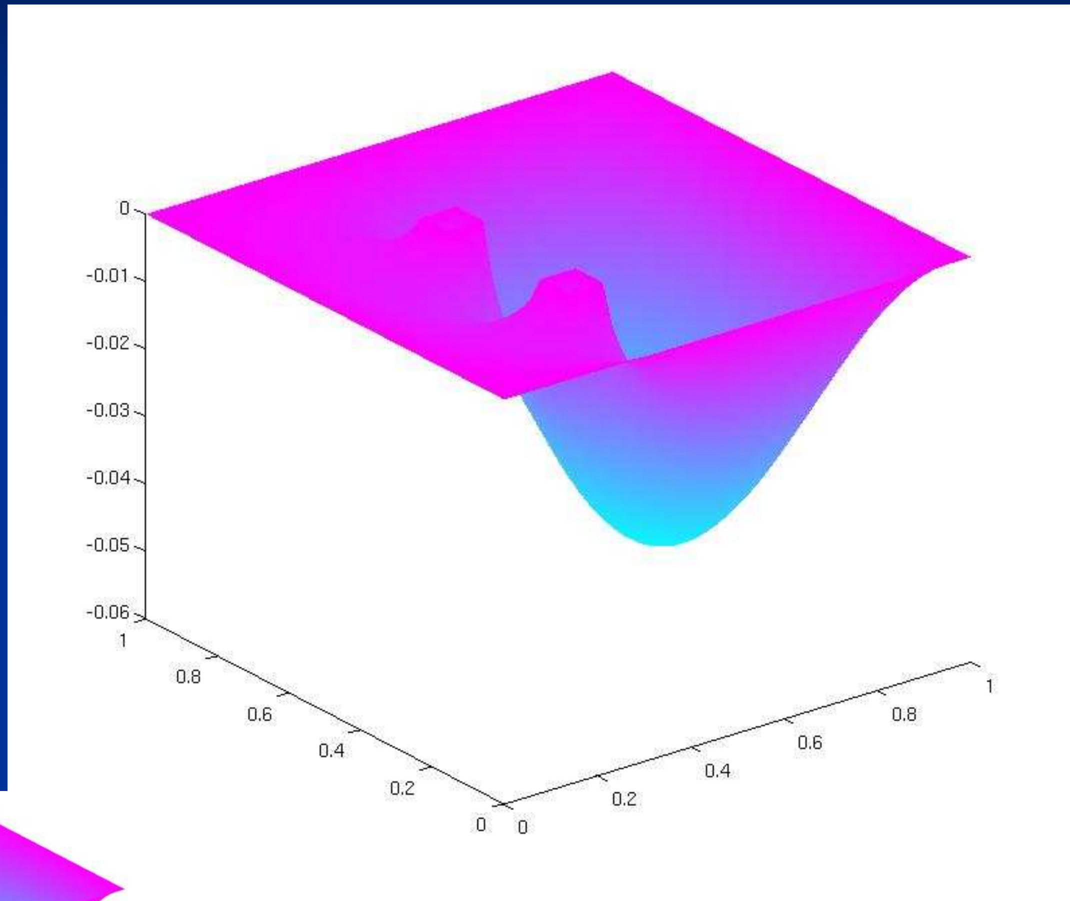
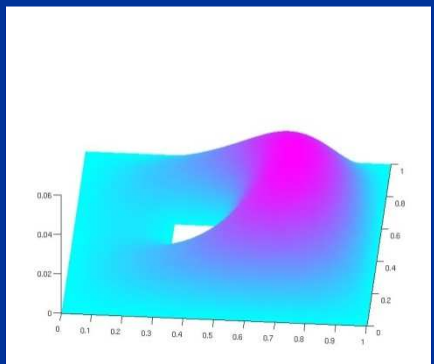
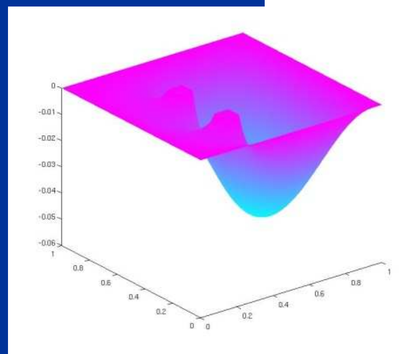
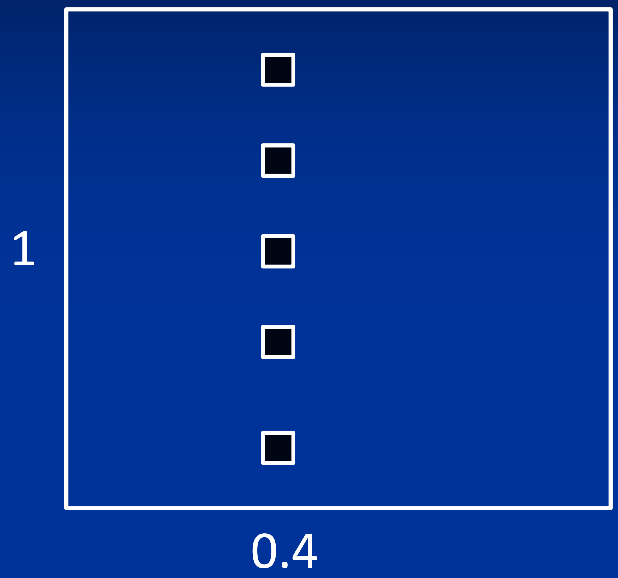






Localization by “dust”



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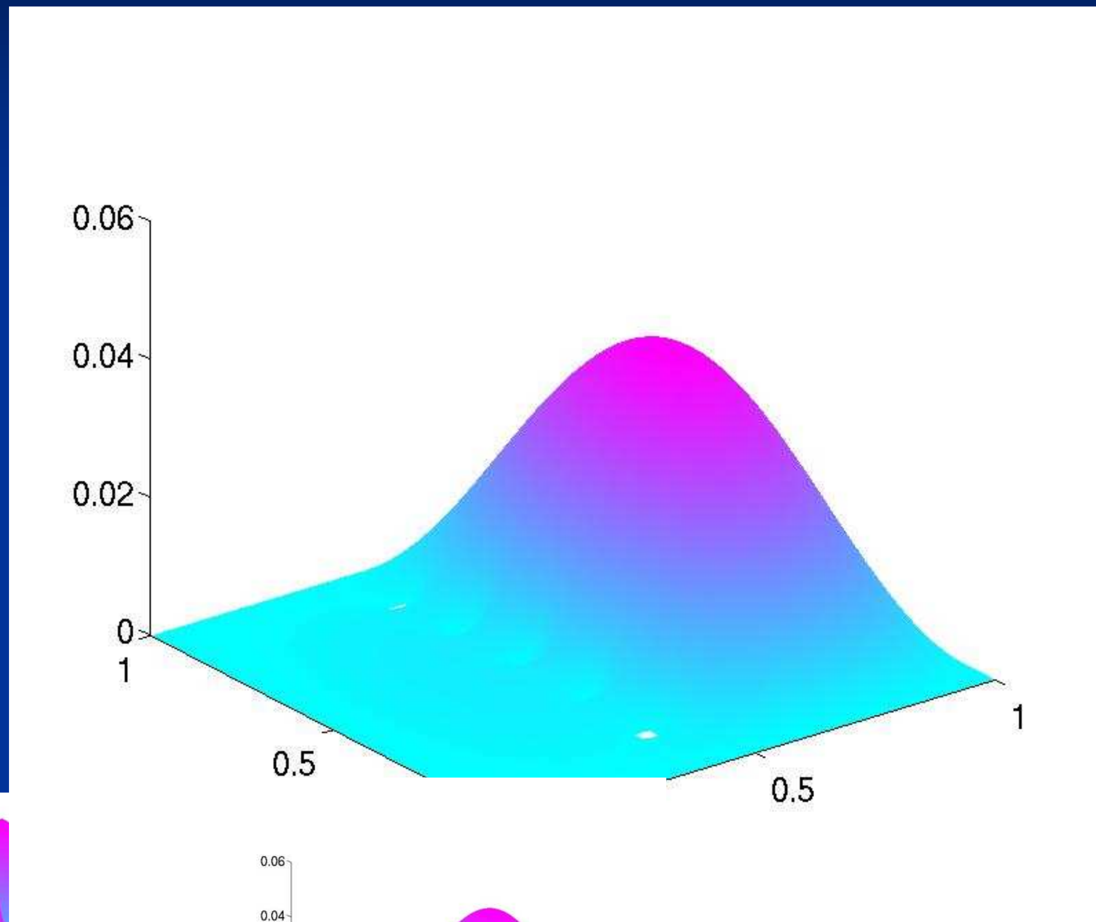

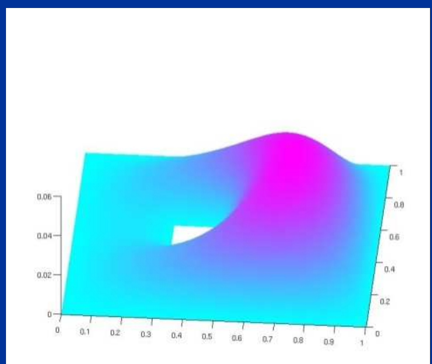

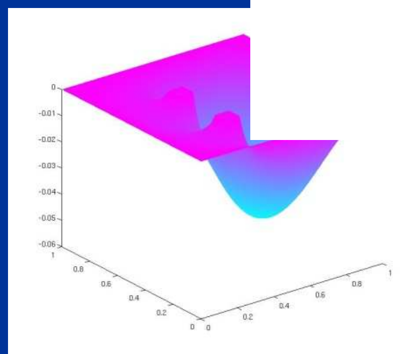

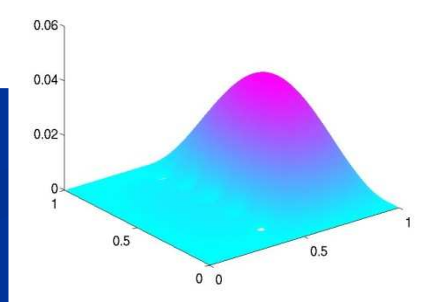

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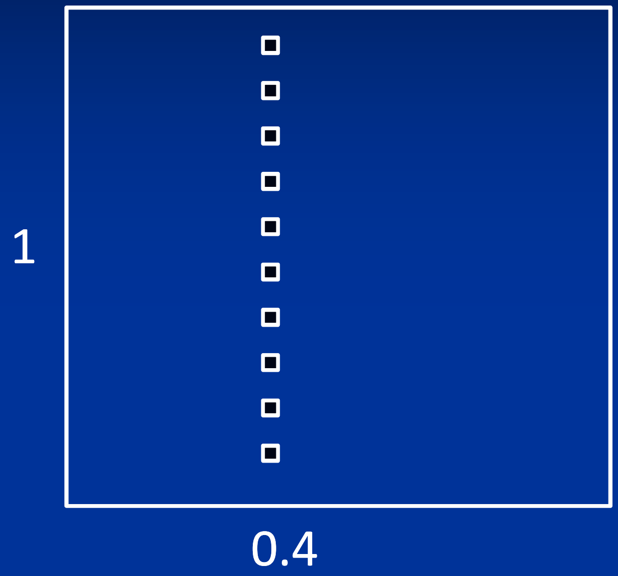

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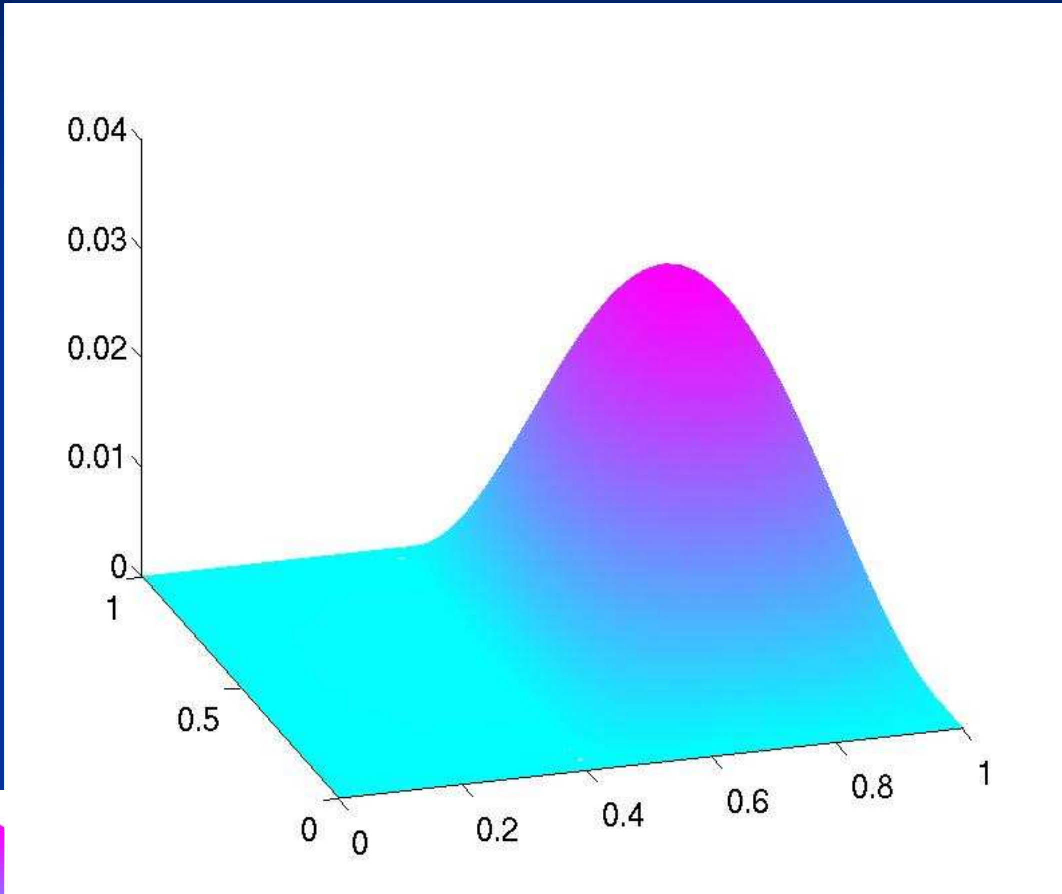
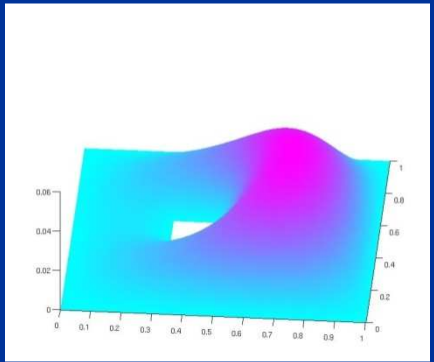
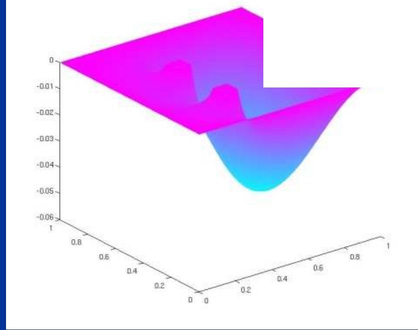
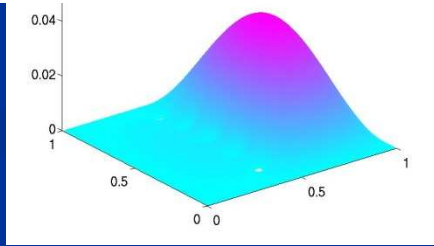




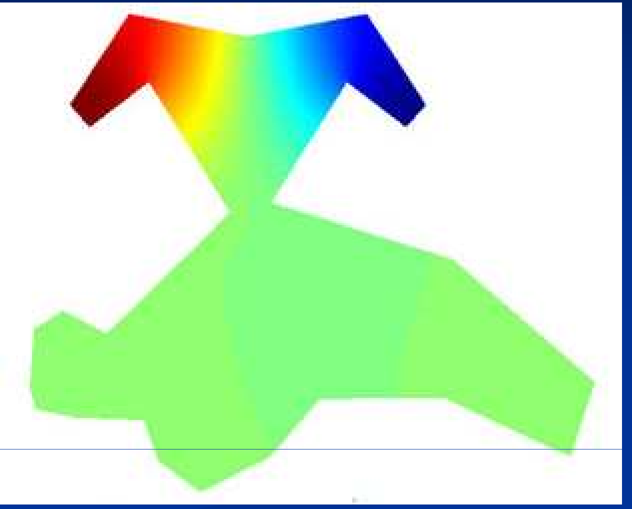




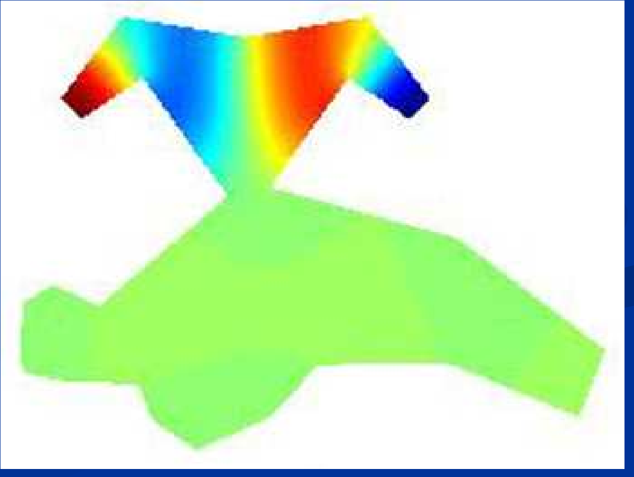
Localization and symmetry

5th eigenfunction

12th eigenfunction



4th eigenfunction



11th eigenfunction

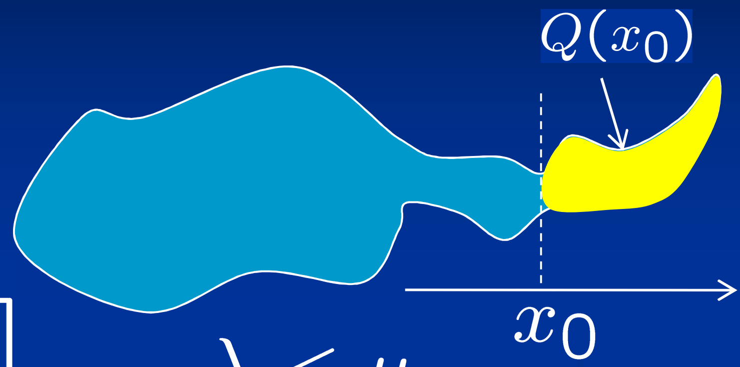
Heilman, Strichartz, Notices Amer. Math. Soc. 57, 624-629 (2010)

How to understand?

Domains with branches

Joint work with A. Delitsyn

$$\int_{Q(x_0)} u^2 \le C \exp[-2\sqrt{\mu - \lambda}\, x_0]$$

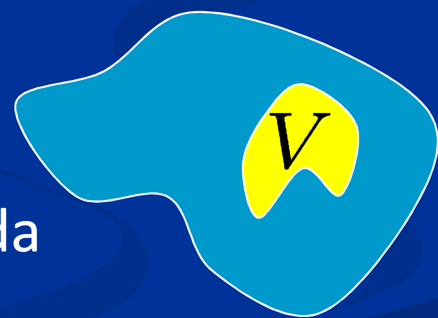


$\lambda < \mu$

μ is the smallest eigenvalue among all cross-sections

for eigenfunctions

work of M. Filoche and S. Mayboroda



$$c \ldots (\partial V) \le ||u||_{L_2(V)} \le C_{\lambda,V} ||u||_{L_\infty(\partial V)}$$

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High-frequency localization

Open problems and questions…

Two ways of thinking about localization

- A property of an individual eigenfunction:

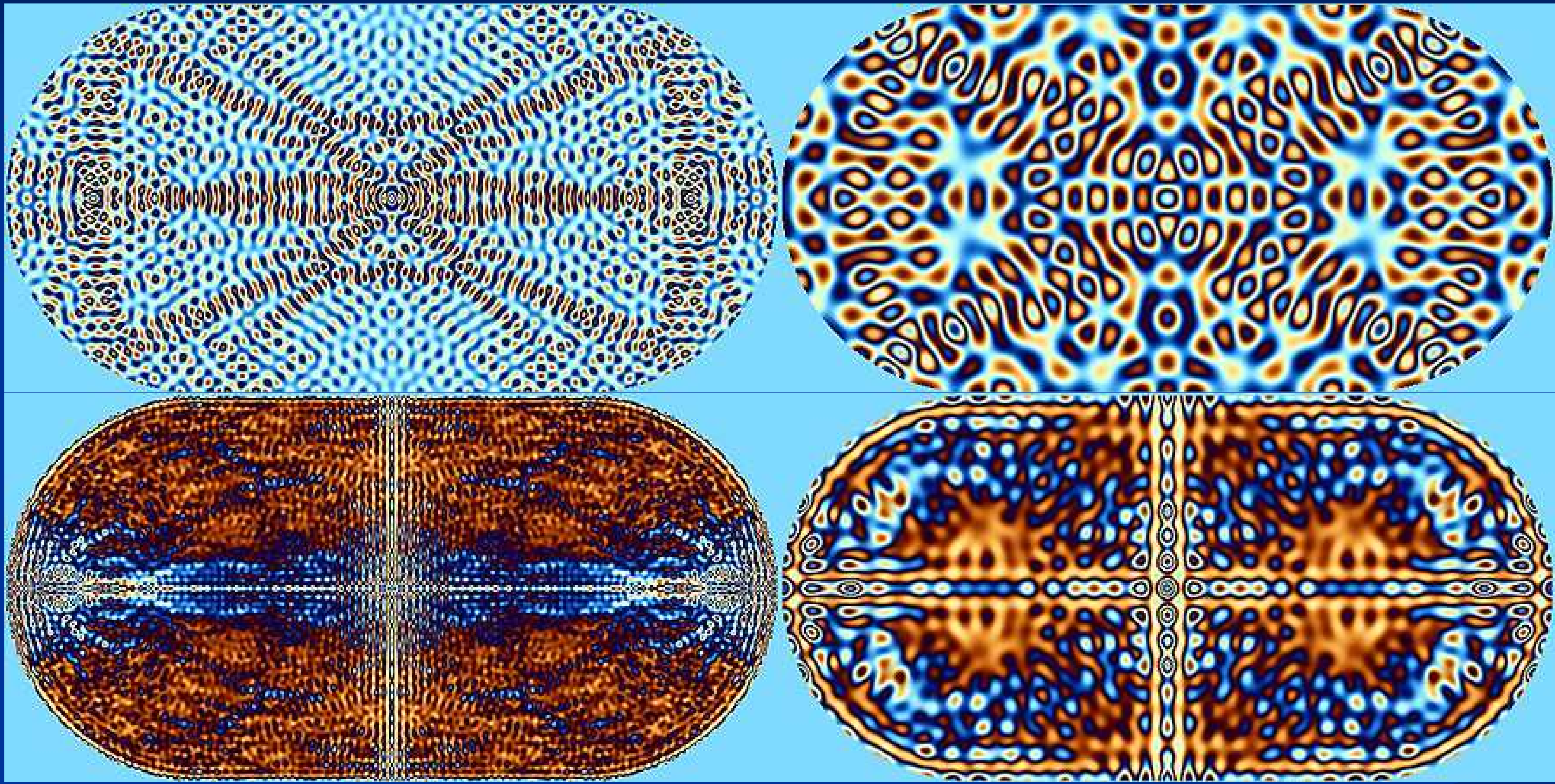
  Is a given eigenfunction localized or not?

  One needs fine analysis to distinguish localized and non-localized eigenfunctions

- A property of the domain:

  Do localized eigenfunctions exist at all?

Quantum billiards

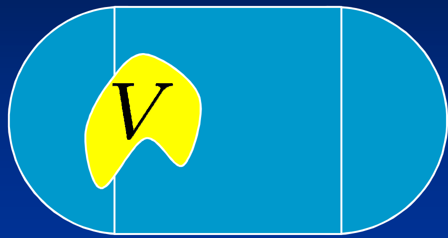


Scarring

(by Chris King, University of Auckland)

From quantum billiards…

Shnirelman theorem:



For a bounded domain with ergodic flow, there is a dense subsequence $u_{j_k}(x)$ of normalized eigenfunctions such that for any open subset V

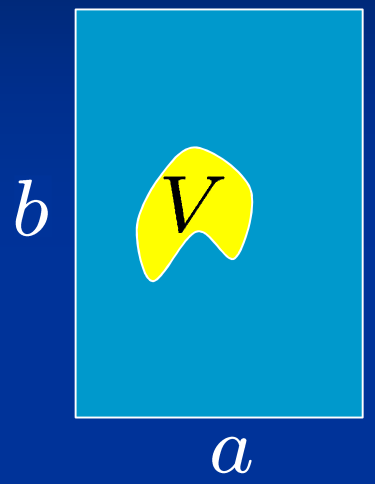
$$\lim_{k\to\infty} \int_V dx \; u_{j_k}^2(x) = \frac{\mu(V)}{\mu(\Omega)} > 0$$

Most of eigenfunctions are then not localized.

BUT, there still may exist localized states!

Zelditch, Zworski, Commun. Math. Phys. 175, 673 (1996)

No localization in rectangle



Consider a rectangle with sides a and b such that a/b is not rational. Then the eigenvalues are simple and

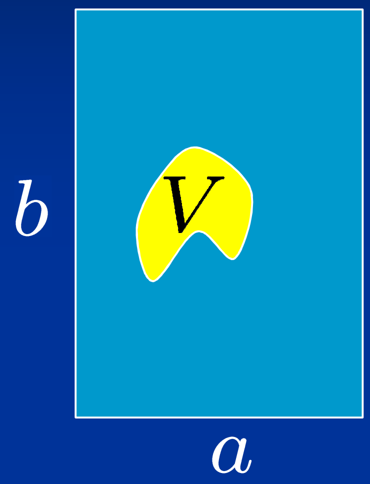
$$u_{m,n}(x,y) = \sin(\pi m x/a)\sin(\pi n y/b)$$

Then for any open subset V

$$C_V = \inf_k \int_V dx \; u_k^2(x) > 0$$

Consequently, there is no localized eigenfunctions which could “avoid” some regions of the rectangle

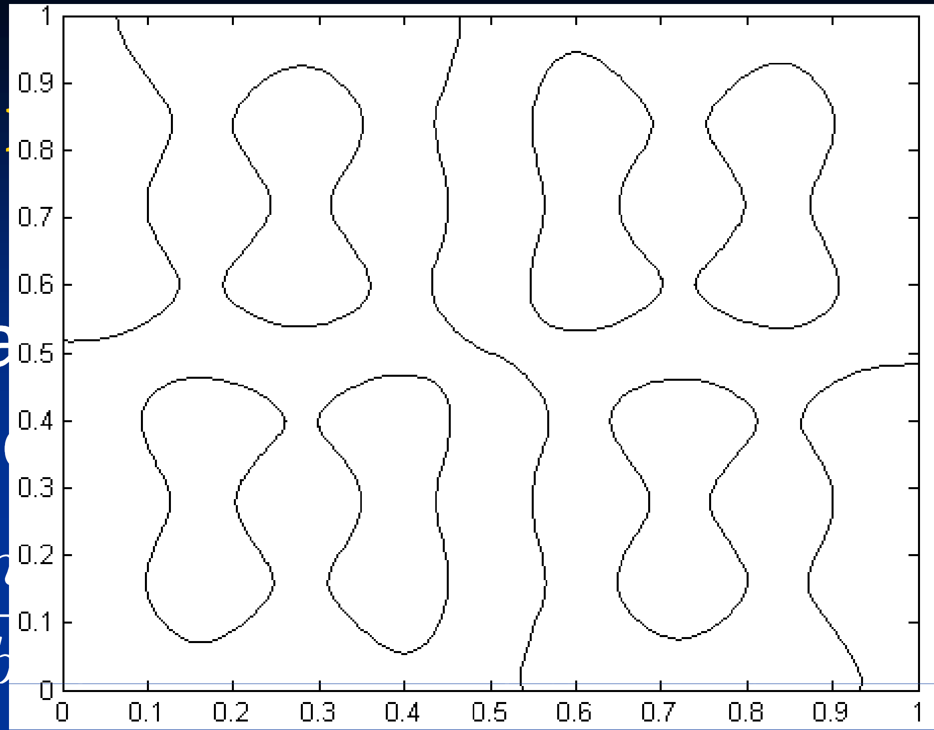
No localizatio



When a/b is ra

may be degen

$$\lambda = \pi^2 \left(\frac{m_1^2}{a^2} + \frac{n}{b} \right.$$



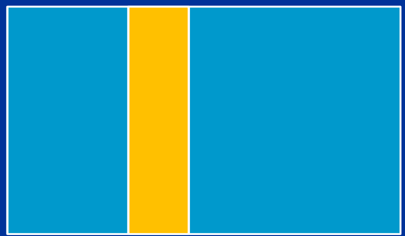
$$u_\lambda(x,y) = \sum_{j=1}^{k} c_j \sin(\pi m_j x/a) \sin(\pi n_j y/b)$$

The existence of the lower bound $C_V>0$ is related to the structure of nodal lines of such eigenfunctions

Example: unit square, $m_1=2$, $n_1=9$, $m_2=9$, $n_2=2$, linear combination of two eigenfunctions

No localization in rectangle?

Let Ω be a rectangle [A,B]x[0,1]. For any open subset V in the form ωx[0,1], there exists C_V>0 such that for any eigenfunction



$$\int_V dx\ u_k^2(x) > C_V$$

So, there is no localization in V.

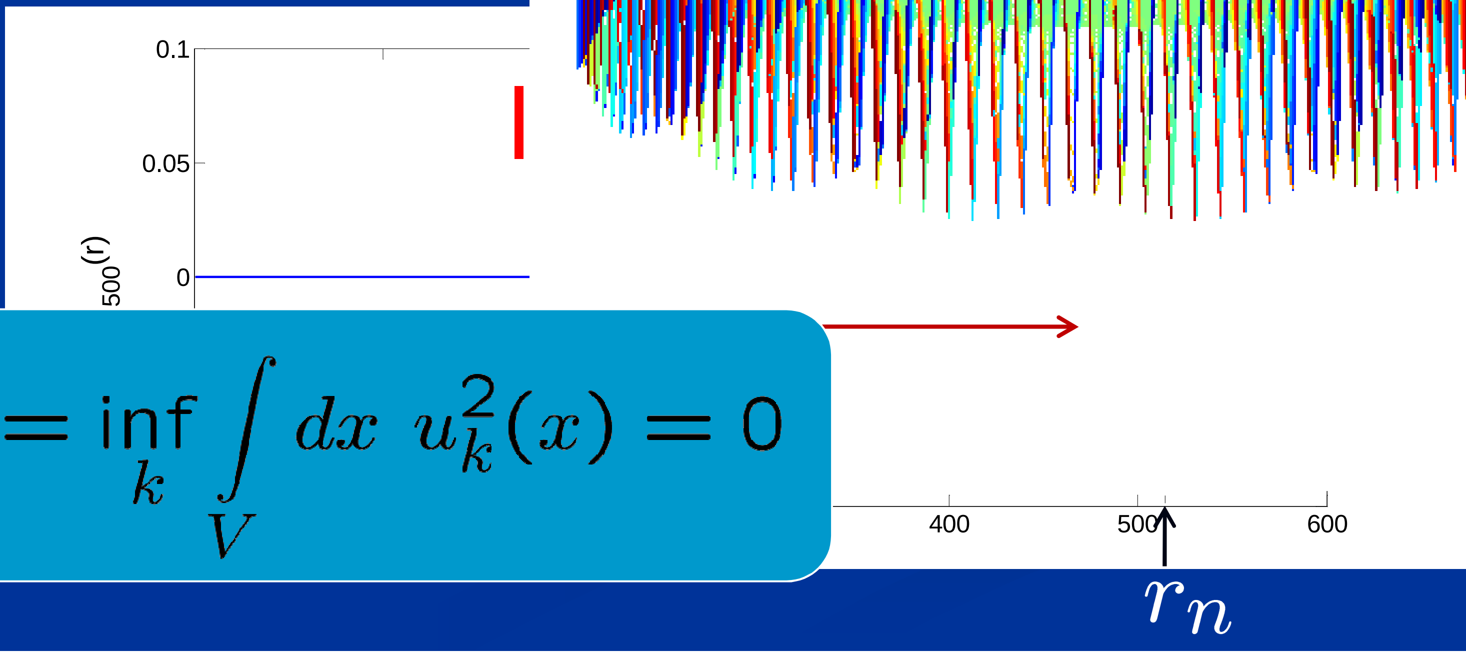
BUT, can one prove it for any open subset V???

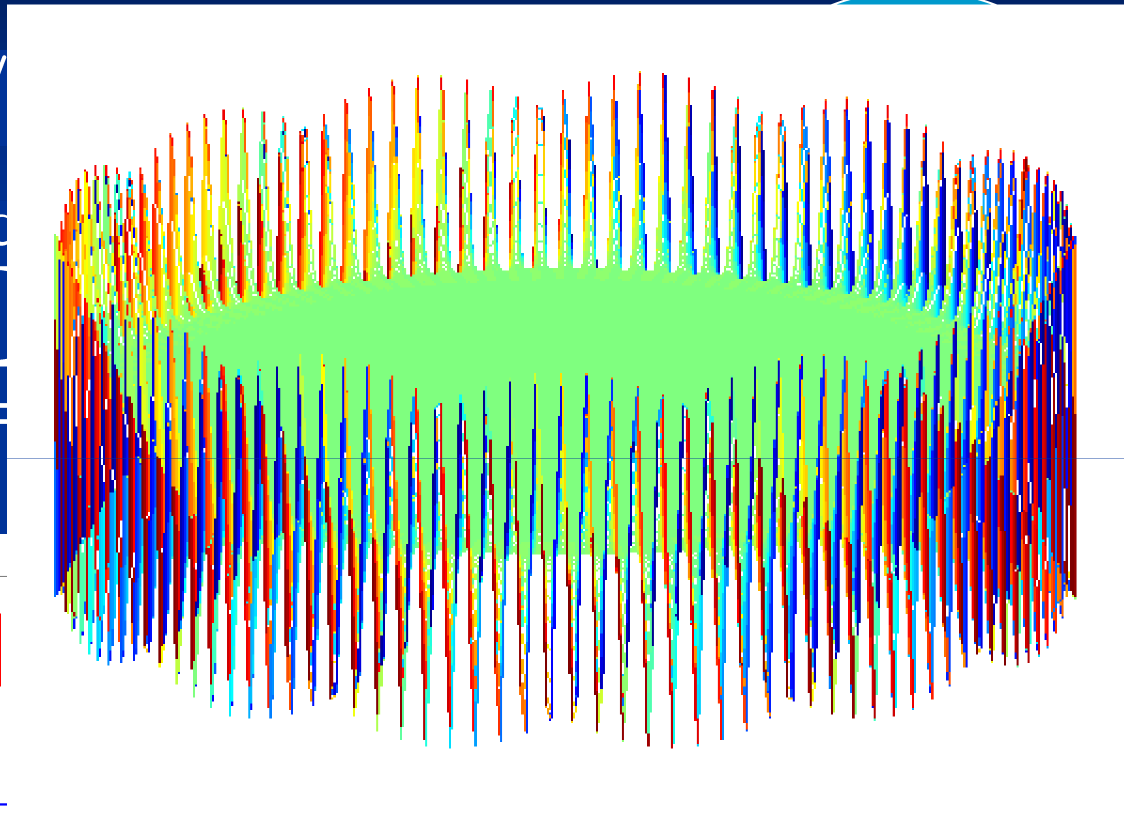
Burq, Zworski, SIAM Rev. 47, 43 (2005)

Localization in a disk

$$u_n(r,\theta) = J_n(r/$$

$$J_n(r) = (r/2)^n \sum_{k=}^{\infty}$$



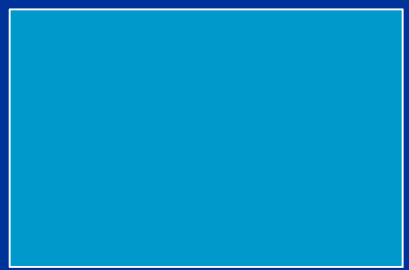


$$C_V = \inf_k \int_V dx \; u_k^2(x) = 0$$

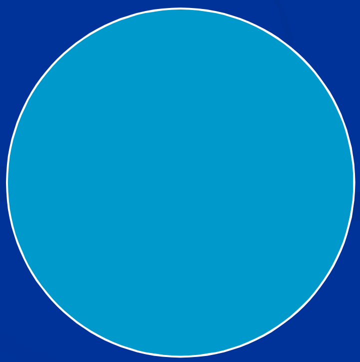
Localization in polygons?

$$C_V = \inf_k \int_V dx \ u_k^2(x)$$

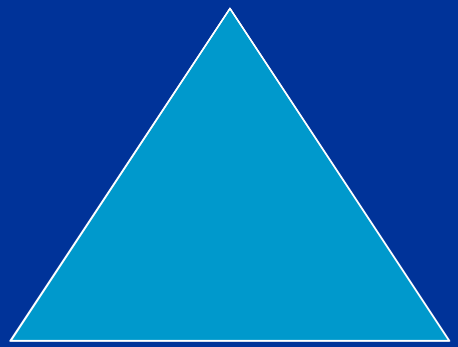
No localization: $C_V>0$

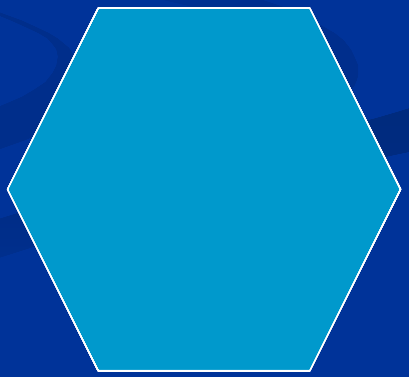


Localization: $C_V=0$



Other shapes???





Plan of the talk

Introduction

- Historical overview and related problems
- Empirical observations

Low-frequency localization

High-frequency localization

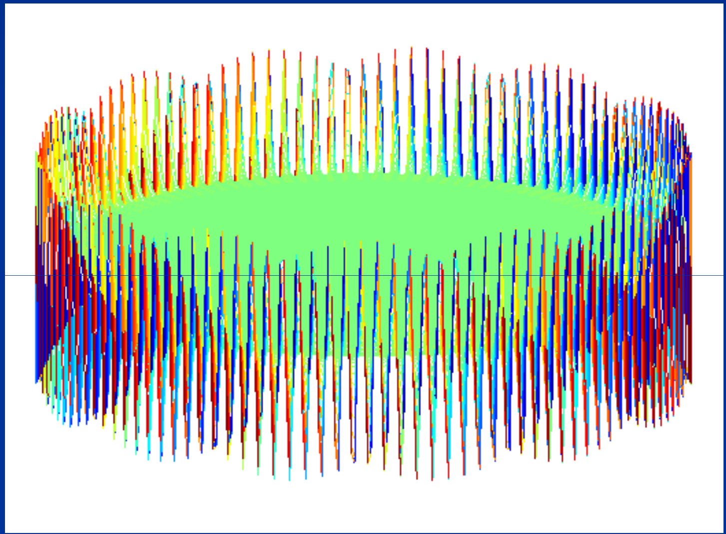
Open problems and questions…

Summary

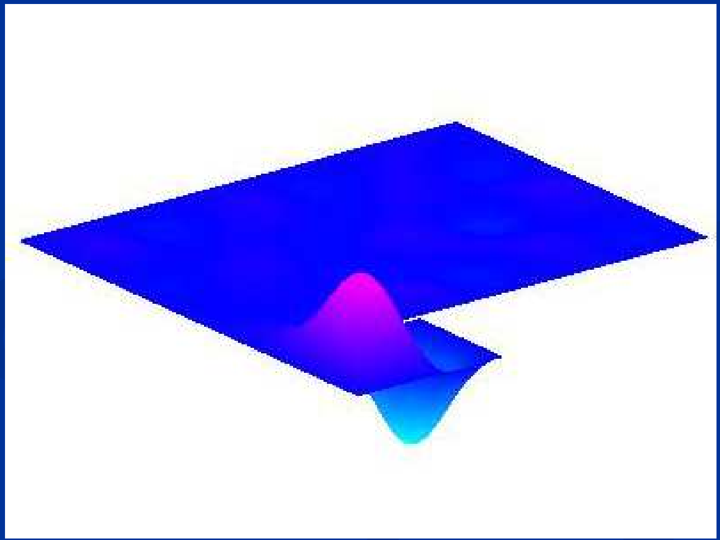
High-frequency

Low-frequency

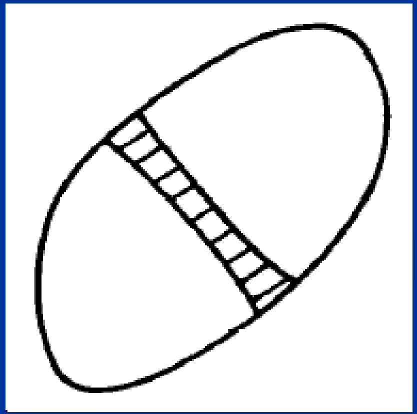
"Whispering" eigenfunctions



"Bottleneck" eigenfunctions



"Bouncing ball" eigenfunctions



Keller, Rubinow,
Ann. Phys 9, 24-75 (1960)

Questions to answer

"Whispering" eigenfunctions

- What are the necessary/sufficient conditions?
- Do they exist for equilateral polygons?
- Is there a relation to curvature of the boundary?
- Is it related to scarring and chaotic systems?

Questions to answer

"Whispering" eigenfunctions

- What are the necessary/sufficient conditions?
- Do …
- Is … ndary?
- Is it related to scarring and chaotic systems?

WHAT IS LOCALIZATION?

"Bottleneck" eigenfunctions

- What are the necessary/sufficient conditions?
- Do they exist in convex domains?
- How many localized eigenfunctions do exist?
- What is the relation to the geometry?