Majorana Spin Liquids and Projective Spin Rotation Symmetry

Michael Hermele

Collaboration with
Gang Chen & Andrew Essin
(work in progress)

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Outline

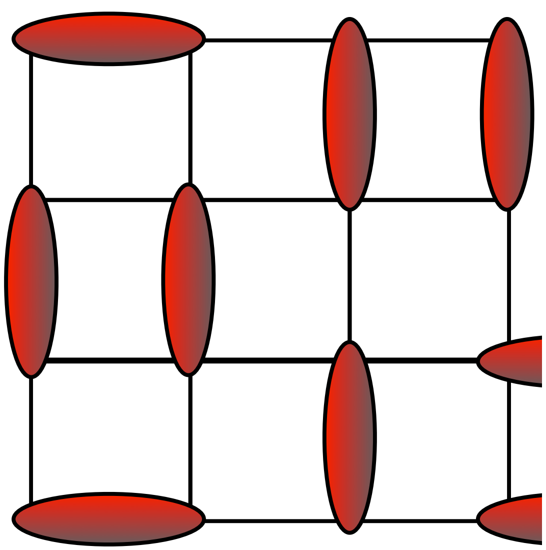
1. Spin liquids: Motivation
2. Review: S=1/2 fermionic parton approach
3. Projective spin symmetry and Majorana spin liquids
4. PSG classification and some example states

Quantum spin liquids

- Consider (generalized) S = 1/2 Heisenberg model (this talk: focus on two and higher dimensions)

$$\mathcal{H} = J \sum_{\langle ij \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j + \cdots$$

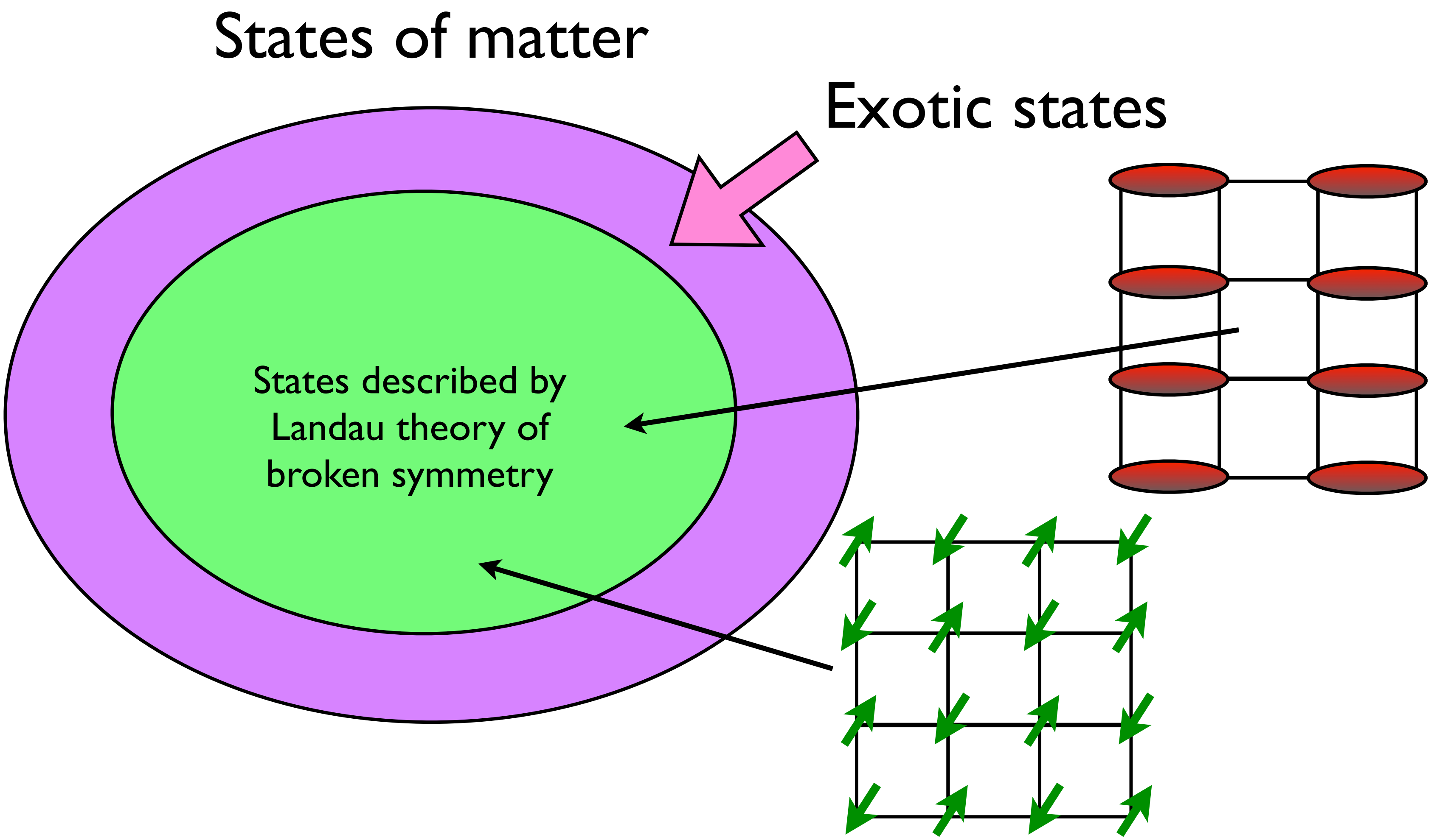
- Simplest definition of quantum spin liquid: ground state breaks *no* symmetries



Resonating Valence Bond Picture

Why care about spin liquids?

- Search for "exotic" states of matter



Why care about spin liquids?

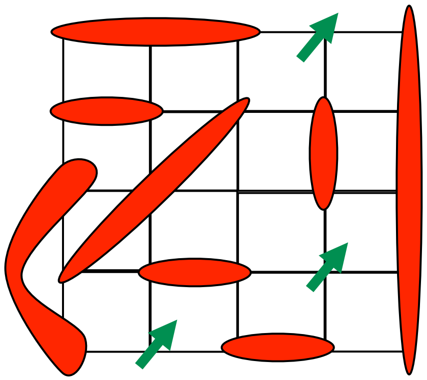
- Search for “exotic” states of matter
- Exotic states exist (model systems, fractional quantum Hall effect)
- We understand less about exotic states than about those described by Landau theory
- Therefore, we have to study exotic states if we are to improve our understanding of states of matter
- Exotic spin liquids are a (relatively) simple setting for building understanding, *and* there are interesting relevant experiments

How can spin liquids be exotic?

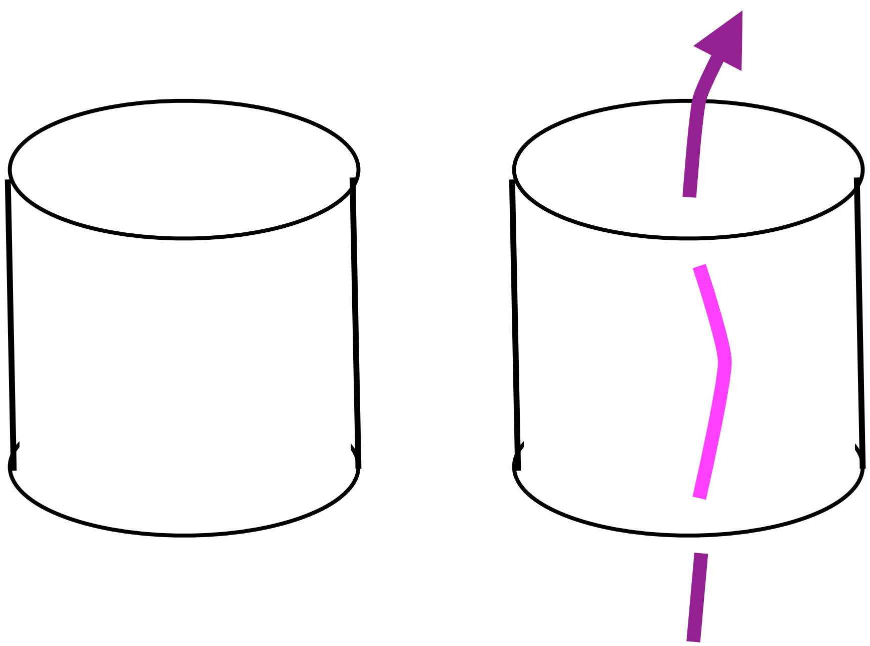
Spin gap?

Yes

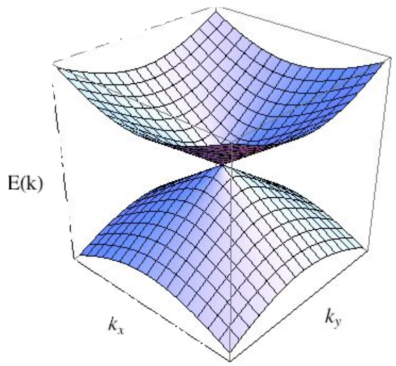
No

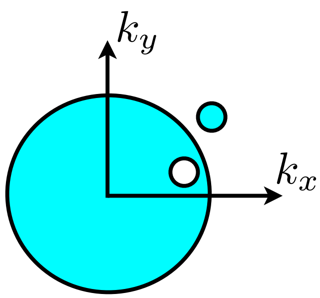


Topological order
(or, gapless photon-like mode)



Critical spin liquids

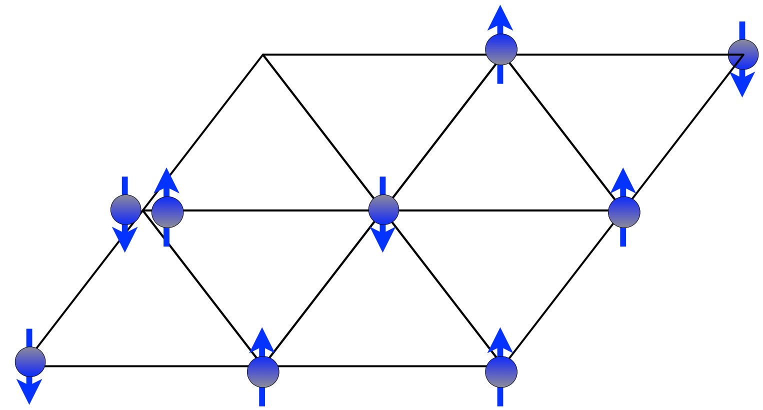




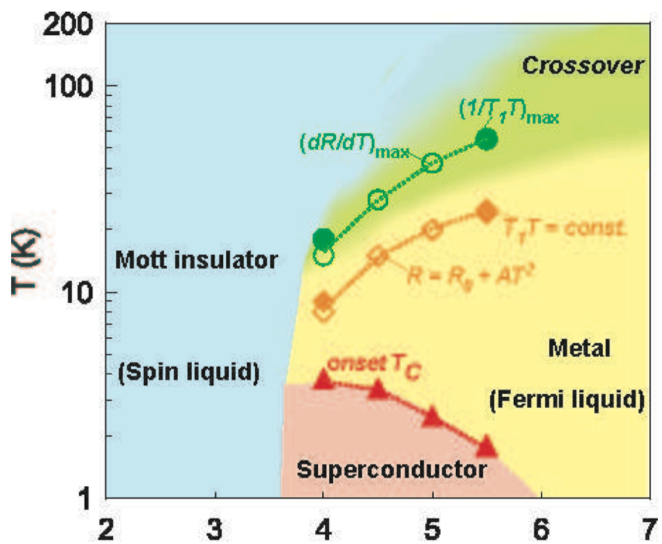
Gapless excitations may interact strongly at low energy (interesting fixed points)

Experimental candidates: Triangular lattice organic I

κ-BEDT-TTF$_2$Cu$_2$(CN)$_3$



- Layered organic material
- Triangular lattice Hubbard model (half-filling), near metal-insulator transition



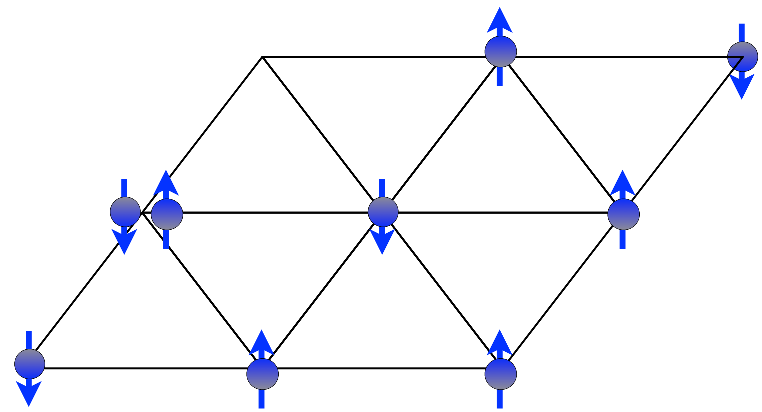
From Y. Kurosaki *et. al.* PRL 95, 177001 (2005).

$$C(T) \sim T$$

$$\chi(T) \sim \text{constant}$$

Experimental candidates: Triangular lattice organic II

$EtMe_3Sb[Pd(dmit)_2]_2$

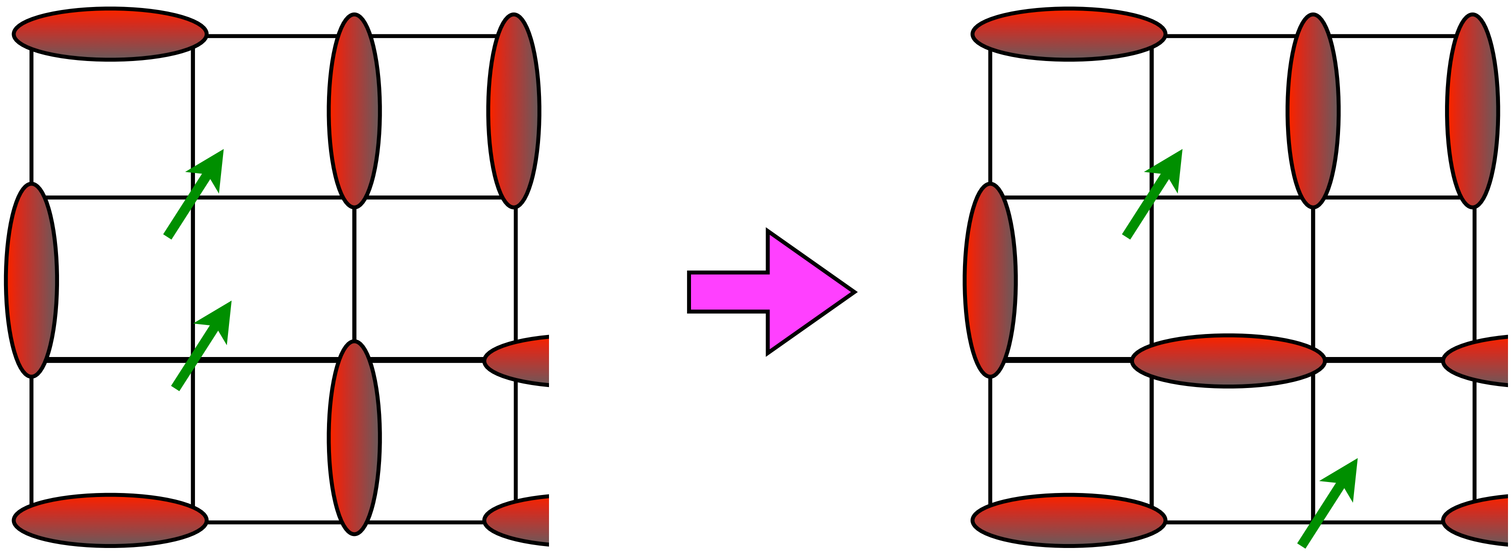


- Similar to BEDT material, with (roughly) similar phenomenology

- Difference from BEDT: significant *metallic*-like low-temperature thermal conductivity, $\kappa \propto T$

Excitations of exotic spin liquids

- Focus on case with SU(2) spin rotation symmetry
- Spin-carrying excitations: S=1/2 spinons



Break valence bond
→ S = 1 excitation

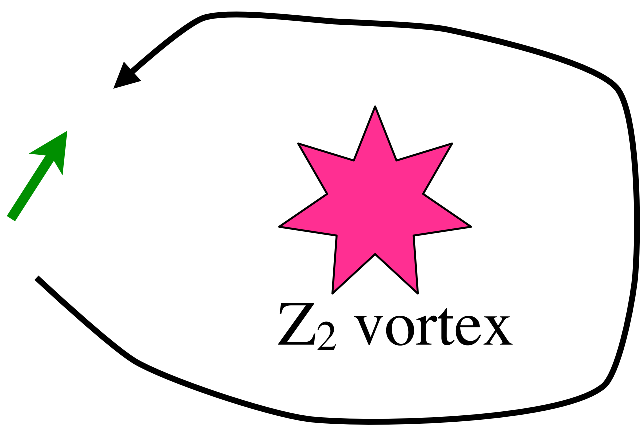
Separate into two
“halves” → S = 1/2
spinons

Excitations of exotic spin liquids

- Focus on case with SU(2) spin rotation symmetry
- Spin-carrying excitations: *S*=1/2 spinons
  - May be well-defined quasiparticles, or not
  - Also play a role as *formal objects*, e.g.

$$\vec{S}_i = \frac{1}{2} f_{i\alpha}^{\dagger} \vec{\sigma}_{\alpha\beta} f_{i\beta}$$

- Singlet excitations: gauge-field excitations (e.g. Z_2 vortices)



Statistical interaction

$$|\psi\rangle \to -|\psi\rangle$$

S = 1 Majorana spinons?

• Question: instead of S = 1/2 spinons, can we have S = 1 Majorana fermion excitations in a spin liquid?

$$\underset{\text{(boson)}}{S = 1} \Rightarrow \underset{\text{(fermion)}}{S = 1} + \underset{\text{(fermion)}}{S = 1}$$

• Happens in exactly solvable models - SU(2) invariant generalizations of Kitaev's honeycomb lattice model (F. Wang; H. Yao & D. H. Lee; H.-H. Lai & O. Motrunich)

• More general approach - applicable beyond special models - is desirable. (Some work in this direction by Biswas, Fu, Laumann & Sachdev.)

• This talk: Develop an approach to construct "Majorana spin liquids." Surprisingly, this approach *is part of* a well-known approach using S=1/2 fermionic partons, but was missed in prior work (to my knowledge).

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Single site

- Represent a single S=1/2 spin using S=1/2 fermions:

Spin operator: $\vec{S} = \frac{1}{2} f_\alpha^\dagger \vec{\sigma}_{\alpha\beta} f_\beta \quad (\alpha, \beta = \uparrow, \downarrow)$

Constraint: $f_\alpha^\dagger f_\alpha = 1$

$$\{|0\rangle, |\uparrow\rangle, |\downarrow\rangle, |\uparrow\downarrow\rangle\} \to \{|\uparrow\rangle, |\downarrow\rangle\}$$

Fermion Hilbert space	Constrained physical Hilbert space

Single site: SU(2) gauge redundancy

Baskaran & Anderson; I. Affleck; Dagotto, Fradkin & Moreo

- This representation has a local SU(2) redundancy
- This is nicely exposed using a matrix notation:

$$F = \begin{pmatrix} f_\uparrow & f_\downarrow^\dagger \\ f_\downarrow & -f_\uparrow^\dagger \end{pmatrix} \qquad \vec{S} = -\frac{1}{4}\,\mathrm{tr}(\vec{\sigma} F F^\dagger)$$

Generates left-SU(2) rotations of *F*

$\vec{S}$ invariant under right-SU(2) rotations of *F*, generated by

$$\vec{G} = \frac{1}{4}\,\mathrm{tr}(F \vec{\sigma} F^\dagger)$$

$$G^1 = \frac{1}{2}(f_\uparrow f_\downarrow + f_\downarrow^\dagger f_\uparrow^\dagger)$$

$$G^2 = -\frac{i}{2}(f_\uparrow f_\downarrow - f_\downarrow^\dagger f_\uparrow^\dagger)$$

$$G^3 = \frac{1}{2}(1 - f_\alpha^\dagger f_\alpha)$$

$$\{|\uparrow\rangle, |\downarrow\rangle\} \rightarrow S = 1/2; G = 0$$

$$\{|0\rangle, |\uparrow\downarrow\rangle\} \rightarrow S = 0; G = 1/2$$

Hilbert space

Constraint $\vec{G} = 0$

- On lattice, $\vec{G}_r$ generates a local SU(2) symmetry, and we have the local constraint $\vec{G}_r = 0$

Mean-field Hamiltonian

• Most general quadratic Hamiltonian invariant under left-SU(2) (*i.e.* spin) rotations:

$$H_0 = \sum_{(r,r')} \left[i\chi^0_{rr'} \,\mathrm{tr}(F_r F^\dagger_{r'}) + \chi^i_{rr'} \,\mathrm{tr}(F_r \sigma^i F^\dagger_{r'}) \right] + \sum_r a^i_0(r) G^i_r$$

• Such Hamiltonians can be obtained as saddle points of an appropriate mean-field decoupling, starting from microscopic spin model

Beyond mean-field I: Projected wavefunction

• Start with mean-field ground state $|\psi_0\rangle$, apply projection operator to enforce constraint and obtain a spin wavefunction:

$$\mathcal{P} = \prod_r \mathcal{P}_r \qquad \mathcal{P}_r = \frac{4}{3}\vec{S}_r^2 = \frac{4}{3}(3/4 - \vec{G}_r^2)$$

$$|\psi\rangle = \mathcal{P}|\psi_0\rangle$$

• Note that $[\vec{S}_r, \mathcal{P}] = [\vec{G}_r, \mathcal{P}] = 0$

•Can also use the same mean-field starting point to construct a low-energy effective theory - will discuss shortly

Symmetries

X.-G. Wen

- Spin singlet: $\vec{S}|\psi\rangle = 0, \qquad \vec{S} = \sum_r \vec{S}_r$
- Space group operation $S : r \to S(r)$

$$\mathcal{U}_S F_r \mathcal{U}_S^\dagger = F_{S(r)} U_r^S$$

May need this gauge transformation in order to satisfy $\mathcal{U}_S H_0 \mathcal{U}_S^\dagger = H_0$

$$\mathcal{U}_S \vec{S}_r \mathcal{U}_S^\dagger = \vec{S}_{S(r)}$$

Physical requirement

- Time-reversal

$$\mathcal{T} : F_r \to (i\sigma^2) F_r U_r^{\mathcal{T}}$$
$$\mathcal{T} : \vec{S}_r \to -\vec{S}_r$$

- We say space group and time reversal operations are realized *projectively*, resulting algebraic structure is dubbed "projective symmetry group" (PSG)

Projective symmetry group

X.-G. Wen

- Call the group of pure gauge transformations leaving H_0 invariant the *invariant gauge group* (IGG).

- A symmetry operation can be composed with any element of IGG and remain a symmetry. So we write...

$$\mathrm{SG} = \mathrm{PSG}/\mathrm{IGG}$$

- IGG can be Z_2, U(1) or SU(2) (or products of these).

- Given a space group and a fixed IGG, gauge-inequivalent PSG's can be classified.

- So, mean-field Hamiltonians (also corresponding wavefunctions and/or effective theories) can be classified according to PSG.

- *Note*: PSG classification is *not* a classification of distinct quantum phases. (It's still useful, though.)

Beyond mean-field II: Effective lattice gauge theory

Senthil & M. P.A. Fisher; X.-G. Wen

• Prescription: Minimally couple the fermions to an IGG lattice gauge field. Resulting theory (generically) has precisely the symmetries of microscopic model. In strong coupling limit of gauge theory, reduces to Heisenberg spin model.

• I will give an explicit example later!

• Essentially equivalent to studying fluctuations about mean-field saddle point

• *Note*: At best rough correspondence between effective theory and projected wavefunction. (Work by D. Ivanov & Senthil; A. Paramekanti and coworkers; Y. Ran, MH, P.A. Lee and X.-G. Wen; T. Tay and O. Motrunich)

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Can SU(2) spin symmetry be realized projectively?

- Yes!

- We will see there are only two gauge-inequivalent possibilities:

1. "Naive" spin symmetry

$$F_r \to U F_r$$

Generator: $\vec{S}$

2. Projective spin symmetry

$$F_r \to U F_r U^\dagger$$

Generator: $\vec{T} = \vec{S} + \vec{G}$

$$\vec{G} \equiv \sum_r \vec{G}_r$$

- Before showing these are the only possibilities, let's consider the consequences of projective spin symmetry.

Mean-field Hamiltonian

• Most general quadratic Hamiltonian invariant under projective spin rotation:

$$H_0 = \sum_{(r,r')} \left[i\chi^1_{rr'} \, \mathrm{tr}(F_r F^\dagger_{r'}) + i\chi^2_{rr'} \, \mathrm{tr}(\sigma^i F_r \sigma^i F^\dagger_{r'}) \right]$$

Dot product of spin and gauge vectors

• IGG is Z_2 [unless $\chi_2 = 0$, in which case it's SU(2)]

• Projected wavefunction is a spin singlet:

$$\vec{T}|\psi_0\rangle = 0 \;\Rightarrow\; \vec{S}\mathcal{P}|\psi_0\rangle = (\vec{S} + \vec{G})\mathcal{P}|\psi_0\rangle = \vec{T}\mathcal{P}|\psi_0\rangle = \mathcal{P}\vec{T}|\psi_0\rangle = 0$$

Majorana fermions

• $f_{r\alpha}$ fermions have complicated transformations under projective spin rotations

• Projective spin symmetry leads us to define Majorana fermions s_r and t_r^i, where

$$F_r = \frac{1}{2}\left(is_r + \vec{t}_r \cdot \vec{\sigma}\right)$$

• Clearly s_r is a singlet while t_r^i is a triplet

• In terms of original fermions:

$$\begin{aligned} s_r &= -i(f_{r\uparrow} - f_{r\uparrow}^\dagger) \\ t_r^1 &= f_{r\downarrow} + f_{r\downarrow}^\dagger \\ t_r^2 &= -i(f_{r\downarrow} - f_{r\downarrow}^\dagger) \\ t_r^3 &= f_{r\uparrow} + f_{r\uparrow}^\dagger \end{aligned}$$

(Same mapping used recently by Burnell and Nayak to study Kitaev's honeycomb lattice model - problem without spin rotation symmetry.)

Mean-field Hamiltonian

- Re-express in terms of Majorana fermions:

$$H_0 = \sum_{(r,r')} \left[i\chi^s_{rr'} s_r s_{r'} + i\chi^t_{rr'} \vec{t}_r \cdot \vec{t}_{r'} \right]$$

$$\chi^s_{rr'} = \frac{1}{2}(\chi^1_{rr'} + 3\chi^2_{rr'})$$

$$\chi^t_{rr'} = \frac{1}{2}(\chi^1_{rr'} - \chi^2_{rr'})$$

IGG=SU(2) point for $\chi^s_{rr'} = \chi^t_{rr'}$

$$S^i_r = -\frac{1}{4}(is_r t^i_r + \frac{i}{2}\epsilon^{ijk} t^j_r t^k_r)$$

$$G^i_r = \frac{1}{4}(is_r t^i_r - \frac{i}{2}\epsilon^{ijk} t^j_r t^k_r)$$

$$T^i_r = -\frac{i}{4}\epsilon^{ijk} t^j_r t^k_r$$

Clear from this form that $[H_0, \vec{T}] = 0$

Constraint $\vec{G}_r = 0$ $\Longleftrightarrow$ $D_r \equiv t^1_r t^2_r t^3_r s_r = 1$

Note: all spin operators of following form are the same on physical Hilbert space:

$$S^i_r = -\frac{1}{4}\left[(1-x) is_r t_{ri} + (1+x)\frac{i}{2}\epsilon_{ijk} t_{rj} t_{rk}\right]$$

x = -1 $\Rightarrow$ $S^i_r = -\frac{i}{2} s_r t^i_r$ Kitaev representation

x = 1 $\Rightarrow$ $\vec{S}_r = \vec{T}_r$ Looks like Shastry-Sen representation - but it is *not* the same

Did we *need* S=1/2 partons at all?

- Not really - we could have used Majorana partons from the beginning - but present approach helps to clarify role of spin rotation symmetry
  - Unless we choose $\vec{S}_r = \vec{T}_r$, we have $[H_0, \vec{S}] \neq 0$
  - Can understand in terms of projective spin symmetry

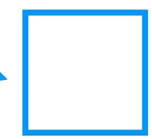
- Moreover, these results *complete* the *S*=1/2 fermionic parton approach.

Effective Z_2 gauge theory

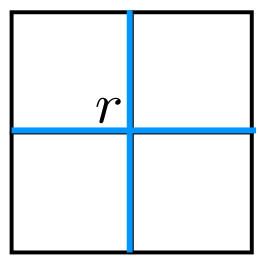
- Write down low-energy effective theory (on square lattice with nearest-neighbor fermion hopping, for simplicity)

- Ising degree of freedom on *links* of square lattice, acted on by $\sigma^x_{rr'}$ and $\sigma^z_{rr'}$ Pauli matrices

$$H = \sum_{\langle rr' \rangle} \sigma^z_{rr'} \left[i\chi^s s_r s_{r'} + i\chi^t \vec{t}_r \cdot \vec{t}_{r'} \right] - h \sum_{\langle rr' \rangle} \sigma^x_{rr'} - K \sum_{\Box} \prod_{rr' \in \Box} \sigma^z_{rr'}$$



Constraint: $\displaystyle\prod_{rr' \in +} \sigma^x_{rr'} = D_r$



- Limit $h \to \infty$, $\sigma^x_{rr'} = 1$ and $D_r = 1$ → back to spin model in this limit.

If $K = 0$, find $H = J \displaystyle\sum_{\langle rr' \rangle} \vec{S}_r \cdot \vec{S}_{r'}$, $\quad J = \dfrac{4}{h} \chi^t (\chi^t + \chi^s)$

(leading order degenerate perturbation theory)

- Limit of K large describes spin liquid state (deconfined phase of Z_2 gauge theory)

Distinct realizations of projective spin symmetry

- Will now argue for earlier claim that there are only two ways to realize spin symmetry

- Assume spin rotations are generated by $\vec{T}$, where $[H_0, \vec{T}] = 0$ for some quadratic Hamiltonian H_0. Noether's theorem implies $\vec{T}$ is bilinear in fermion operators.

- Further assumptions:

1. $\vec{T} = \sum_r T_r^i$
2. $\vec{T}|\psi\rangle = \vec{S}|\psi\rangle$ for gauge-invariant $|\psi\rangle$ (i.e. $\vec{G}_r|\psi\rangle = 0$)
3. $[T^i, T^j] = i\epsilon^{ijk}T^k$

- Most general T_r^i satisfying #2: $T_r^i = S_r^i + M_r^{ij}G_r^j$

- #3 implies either $M_r = 0$ or $M_r \in \mathrm{SO}(3)$

- So, make gauge transformation so that either $M_r = 0$ or $M_r^{ij} = \delta^{ij}$

Distinct realizations of projective spin symmetry

• So far we've shown $\vec{T}_r = \vec{S}_r$ or $\vec{T}_r = \vec{S}_r + \vec{G}_r$, might be different on different lattice sites

• But, suppose we have two sites, one of each type. There is no spin-rotation invariant fermion bilinear that can join these two sites. But we want to restrict to H_0 that fully connect the lattice.

• Therefore we can only have the two possibilities claimed

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PSG classification

- General result: Majorana spin liquid Z_2 PSGs are in one-to-one correspondence with SU(2) PSGs

- There are not many SU(2) PSGs (e.g. only four on the square lattice).

- Focus on square lattice, a PSG (with projective spin symmetry) is specified by:

$$T_x : s_r \to \pi_r^x s_{r+\hat{x}}$$

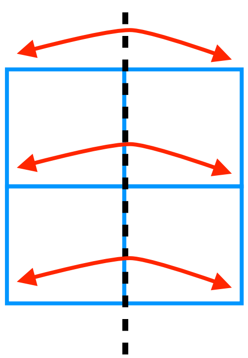
$$T_y : s_r \to \pi_r^y s_{r+\hat{y}}$$

$$P_x : s_r \to \pi_r^{P_x} s_{r'}$$

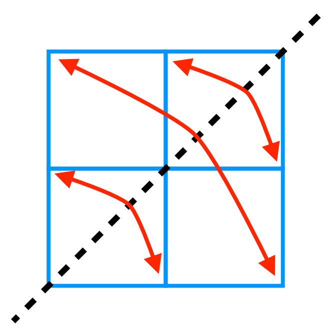
$$P_{xy} : s_r \to \pi_r^{P_{xy}} s_{r'}$$

$$\mathcal{T} : s_r \to \pi_r^{\mathcal{T}} s_r$$

Transformations of t_r^i are same as above

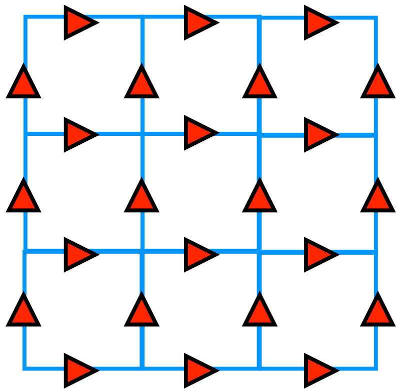


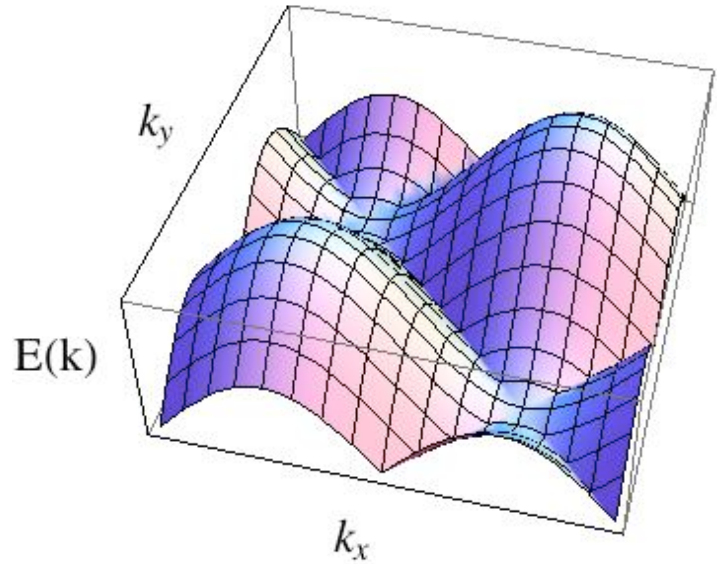
P_x



P_{xy}

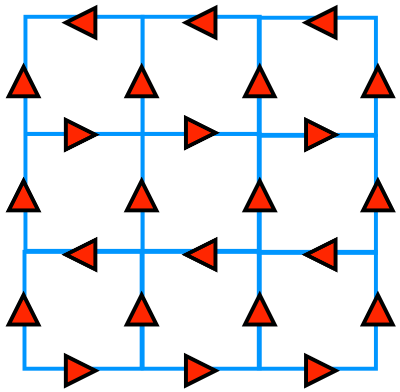
“Uniform” state

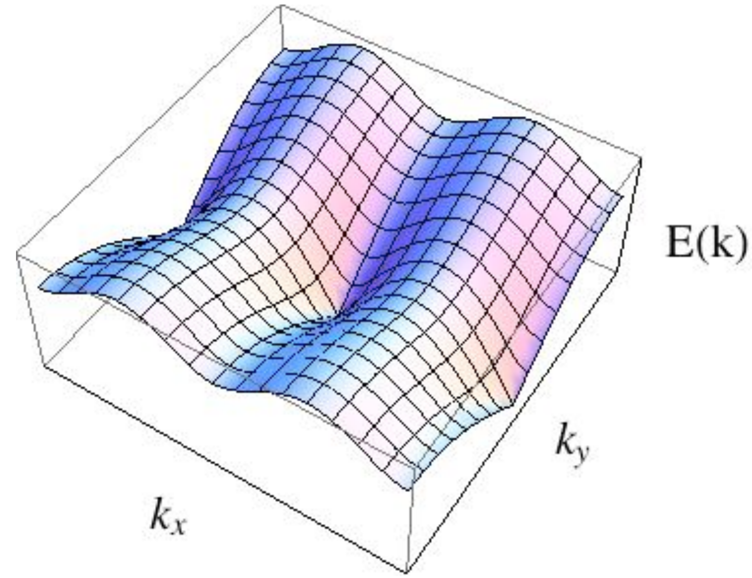




- When $\chi^s = \chi^t$, H_0 describes pure imaginary hopping of *f*-fermions
- This state is actually unstable to Neel order (based on mean-field calculation in presence of on-site fermion interactions)
- Time reversal: $\mathcal{T} : s_r \rightarrow (-1)^{(r_x+r_y)} s_r$
- In general, H_0 must have a bipartite structure to respect time reversal symmetry.

“π-flux” state



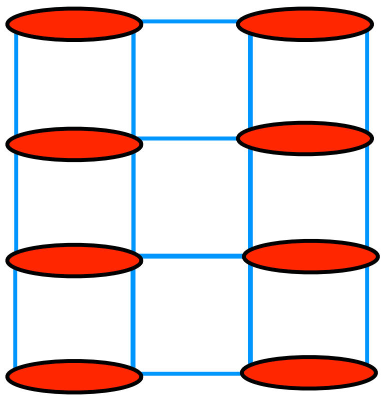


• This state has gapless Dirac fermions (gapless nature is protected by lattice symmetries).

• Moreover, this is a stable spin liquid phase. (Quartic interactions are RG irrelevant.)

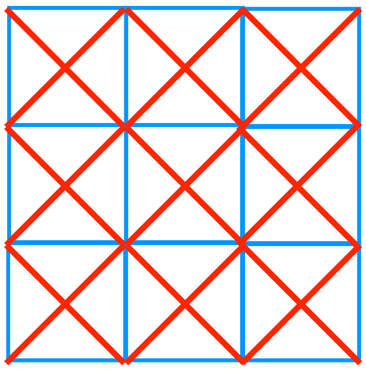
• Natural question: to what extent can we realize similar physics to Kitaev’s model by perturbing the π-flux state? In particular, if we break symmetries to gap out the nodes, can we realize Z_2 vortices with non-Abelian statistics?

Dimerized "π-flux" state



- Make some hoppings stronger, in pattern shown
- This opens a gap at Dirac points, corresponding to an achiral mass
- In particular, resulting band structure is topologically trivial

T-breaking "π-flux" state



- Add second-neighbor hopping so that translation and rotation symmetries are preserved (P and T are broken)

- This corresponds to chiral mass for fermions

- Topologically nontrivial band structure with gapless chiral edge modes - one Majorana edge mode for each bulk Majorana fermion.

- Either *s* and *t*'s co-propagate, or counter-propagate

- In both cases, Z_2 vortices *lack* non-Abelian statistics

Combine T-breaking and dimerization

- Relative strength of T-breaking and dimerization perturbations may be different for s- and t-fermions.

- For example: if t-fermions have dominant dimerization, and s-fermion has dominant T-breaking, only the s-fermion has a chiral edge mode (or vice-versa)

- In this way we can get non-Abelian excitations ... but we had to break a lot of symmetry to do it

Conclusions / Open questions

• Results of this talk: tied up a loose end in the S=1/2 fermionic parton approach to spin liquids, and in the process developed an approach to construct/study spin liquids with S=1 Majorana fermion excitations

• Strong restrictions limit the number of states with projective spin symmetry

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• More detailed connection with exact solutions?

• Doping - holons should be charged S=1/2 bosons

• Are there Z_2 spin liquids where S=1 Majorana fermions arise as Z_2 vortices (or bound states of vortices and electric gauge charges)? If so, is this just another way of describing the states discussed here?

• Possible relevance to experiments???