Effective action for Regge processes in gravity

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Content

1 Scattering amplitudes at high energies

High energy kinematics

$$s = 4E^2 \gg -t = \vec{q}^{\,2}\,,\ \theta \approx \frac{|q|}{E} \ll 1$$

t-channel partial wave expansion

$$A^p(s,t) = s\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\,\left((-s)^\omega - p s^\omega\right)\, f^p_\omega(t)\,,\ p = \pm 1$$

Regge pole hypothesis and Pomeron trajectory

$$f^p_\omega(t) = \frac{\gamma^2(t)}{\omega - \omega_p(t)}\,,\ \omega_P(-q^2) \approx \Delta_P - \alpha'_P q^2\,,\ p_P = 1$$

Asymptotics of elastic amplitudes and cross-sections

$$A(s,t) \approx is\,\gamma^2(-q^2)s^{\Delta_P - \alpha'_P q^2}\,,\ \sigma = \gamma^2(0)\,s^{\Delta_P}$$

2 Gribov Pomeron calculus

Multi-particle t-channel unitarity

$$\Im_t f_\omega(t) \sim \sum_n \int d\Omega_n |f_\omega^{(n)}|^2$$

Mandelstam cut contribution

$$A_M(s,t) \approx -i \int d^2k\, \Phi^2(k, q-k)\, s^{2\Delta - \alpha'_P k^2 - \alpha'_P (q-k)^2}$$

Separation of particles in their rapidities

$$0 < y_1 < ... < y_n < \ln s\,,\; y_k - y_{k-1} \gg 1\,,\; y = \frac{1}{2}\,\ln \frac{E+p}{E-p}$$

Gribov's Pomeron action

$$S = \int dy\, d^2\rho \left(\phi^* (\partial_y - \Delta)\phi + \frac{(\vec{\nabla}\phi)^2}{2m} + \lambda(\phi^*\phi^2 + \phi\phi^{*2}) + ... \right)$$

3 Gluon reggeization in QCD

QCD Born amplitude

$$M_{AB}^{A'B'}(s,t)|_{Born} = 2s\, g\, T_{A'A}^{c}\, \delta_{\lambda_{A'}\lambda_A}\, \frac{1}{t}\, g\, T_{B'B}^{c}\, \delta_{\lambda_{B'}\lambda_B}$$

Leading Logarithmic Approximation

$$M(s,t) = M_{Born}(s,t)\, s^{\omega(t)}\,,$$

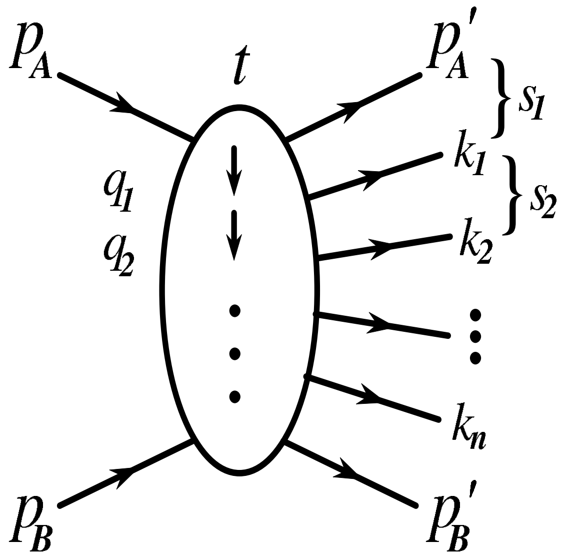
Its region of applicability

$$\alpha_s \ln s \sim 1\,,\ \ \alpha_s = \frac{g^2}{4\pi} \ll 1$$

Gluon trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2} \int d^2k\, \frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\, \ln\frac{|q^2|}{\lambda^2}$$

4 Amplitudes in multi-Regge kinematics



$$M_{2\to 2+n}^{BFKL} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT_{c_2c_1}^{d_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... gT_{c_{n+1}c_n}^{d_n}\, C(q_{n+1},q_n)\, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2}\,,$$

$$\omega_r = -\frac{\alpha_s N_c}{2\pi}\left(\ln\frac{|q_r^2|}{\mu^2} - \frac{1}{\epsilon}\right)\,,\ C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*}\,,\ \sigma_t = \sum_n \int d\Gamma_n\, |M_{2\to 2+n}|^2$$

5 BFKL equation (1975)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\;,\;\;\sigma_t \sim s^{\Delta}\,,\;\;\Delta = -\frac{\alpha_s N_c}{2\pi}\,E_0$$

Hamiltonian for the Pomeron wave function

$$H_{12} = \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^* + \frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 + \ln\,|p_1 p_2|^2 - 4\psi(1)\,,$$

$$\rho_{12} = \rho_1 - \rho_2\,,\;\;\rho_r = x_r + iy_r\,,\;\;\Delta = 4\alpha N_c\,\ln 2\,/\pi$$

Möbius invariance and Pomeron intercept

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,\;\;m = \gamma + n/2\,,\;\;\widetilde{m} = \gamma - n/2\,,\;\;\gamma = 1/2 + i\nu\,,$$

$$E = \epsilon_m + \epsilon_{\widetilde{m}}\,,\;\;\epsilon_m = \psi(m) + \psi(1-m) - 2\psi(1)\,,\;\;\Delta = \frac{g^2 N_c}{\pi^2}\ln 2$$

6 Effective action in QCD

Locality of the theory in the rapidity space

$$y = \frac{1}{2}\ln\frac{\epsilon_k + |k|}{\epsilon_k - |k|}\,, \;\; |y - y_0| < \eta\,, \;\; \eta \ll \ln\, s$$

Gluon and Reggeized gluon fields

$$v_\mu(x) = -iT^a v_\mu^a(x)\,, \;\; A_\pm(x) = -iT^a A_\pm^a(x)\,, \; \delta A_\pm(x) = 0$$

Effective action for their interactions (L., 1995)

$$S = \int d^4x \left(L_{QCD} + Tr(V_+\partial_\mu^2 A_- + V_-\partial_\mu^2 A_+)\right)\,,$$

$$V_+ = -\frac{1}{g}\partial_+ \, P\exp\left(-g\int_{-\infty}^{x^+} v_+(x')d(x')^+\right) = v_+ - g v_+ \frac{1}{\partial_+} v_+ + \ldots$$

7 Pomeron and graviton in $N = 4$ SUSY

BFKL Pomeron in a diffusion approximation

$$j = 2 - \Delta - D\,\nu^2\,,\;\; \gamma = 1 + \frac{j-2}{2} + i\nu$$

Constraint from the energy-momentum conservation

$$D = \Delta$$

AdS/CFT relation for the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\;\; t = E^2/R^2\,,\;\; \alpha' = \frac{R^2}{2}\,\Delta$$

Large coupling asymptotics for γ and Δ (KLOV, BPST)

$$\gamma = -\Delta\sqrt{j-2+\Delta}\,,\;\; \Delta = \frac{1}{\sqrt{g^2 N_c}}$$

8 Perturbation theory in gravity

Einstein-Hilbert action

$$S_{EH} = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, R\,, \;\; R = R_{\mu\nu}\, g^{\mu\nu}$$

Riemann tensor

$$R_{\mu\nu} = R^{\sigma}_{\mu,\sigma\nu}\,, \;\; R^{\sigma}_{\mu,\alpha\beta} = \partial_\beta \Gamma^{\sigma}_{\mu\alpha} - \partial_\alpha \Gamma^{\sigma}_{\mu\beta} + \Gamma^{\rho}_{\mu\alpha}\, \Gamma^{\sigma}_{\rho\beta} - \Gamma^{\rho}_{\mu\beta}\, \Gamma^{\sigma}_{\rho\alpha}$$

Christophel symbol and gravity field

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\, g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)\,, \;\; g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

General coordinate transformation

$$\delta h_{\mu\nu} = D_\mu \chi_\nu + D_\nu \chi_\mu\,, \;\; D_\mu \chi_\nu = \partial_\mu \chi_\nu - \Gamma^{\rho}_{\mu\nu} \chi_\rho$$

9 High energy amplitudes in gravity

Production amplitudes in LLA (L.L. (1982))

$$A_{2\to n} = -s^2 \Gamma^{\mu'\nu'}_{\mu\nu} \frac{s_1^{\omega(q_1^2)}}{q_1^2} \Gamma_{\rho_1\sigma_1} \frac{s_2^{\omega(q_2^2)}}{q_2^2} \Gamma_{\rho_2\sigma_2} ... \Gamma^{\rho'\sigma'}_{\rho\sigma}$$

Graviton-graviton-reggeon vertex

$$\Gamma^{\mu'\nu'}_{\mu\nu} = \frac{\kappa}{4} \left(\Gamma_{\mu\mu'} \Gamma_{\nu\nu'} + \Gamma_{\mu\nu'} \Gamma_{\nu\mu'} \right)$$

Gluon-gluon-reggeized gluon vertex

$$\Gamma_{\mu\mu'} = -\delta_{\mu\mu'} + \frac{p^A_{\mu'} p^B_{\mu} + p^{A'}_{\mu} p^B_{\mu'}}{p^A p^B} + \frac{q_1^2}{2} \frac{p^B_{\mu} p^B_{\mu'}}{(p^A p^B)^2}$$

Reggeon–reggeon-graviton vertex

$$\Gamma_{\rho\sigma} = \frac{\kappa}{4} \left(C_\rho C_\sigma - N_\rho N_\sigma \right) \,, \; N = \sqrt{q_1^2 q_2^2} \left(\frac{p^A}{kp^A} - \frac{p^B}{kp^B} \right)$$

10 Graviton trajectory and amplitudes

Graviton Regge trajectory (L. (1982))

$$\omega(q^2) = \frac{\kappa^2}{(2\pi)^2} \int \frac{q^2\, d^2k}{k^2(q-k)^2} f(k,q)\,,\ q^2 = -\overrightarrow{q}^{\,2}$$

$$f(k,q) = (k, q-k)^2 \left(\frac{1}{k^2} + \frac{1}{(q-k)^2} \right) - q^2 + \frac{N}{2}(k, q-k)$$

Gravitino action

$$S_{3/2} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \int d^4x \sum_{r=1}^{N} \bar{\psi}_\mu^r \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma^r$$

Amplitudes in the DL (ladder) approximation

$$A = A_{Born} s^{-a \ln \frac{q^2}{\lambda^2}} \frac{1}{b\xi} I_1(2b\xi)\,,\ b = \frac{\sqrt{4-N}}{4\pi}\kappa\,,\ \xi = \ln\frac{s}{t}$$

11 Effective action for gravity

Locality in the rapidity space

$$y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|}\,, \;\; |y - y_0| < \eta\,, \;\; \eta \ll \ln\, s$$

Reggeized graviton fields

$$\delta A^{++}(x) = \delta A^{--}(x) = 0\,, \;\; \partial_+ A^{++}(x) = \partial_- A^{--}(x) = 0$$

Effective action for the high energy gravity (L. 2011)

$$S = -\frac{1}{2\kappa} \int d^4x \left(\sqrt{-g}\, R + \frac{1}{2} \left(\partial_+ j^- \, \partial_\mu^2 A^{++} + \partial_- j_+ \, \partial_\mu^2 A^{--} \right) \right)$$

Effective currents in two first orders

$$\partial_\pm j^\mp = h_{\pm\pm} - X_{\rho\pm}^2 + \frac{\kappa^2}{2} \, \bar{\psi}_\pm \frac{\gamma_\pm}{\partial_\pm} \psi_\pm + \ldots\,, \;\; X_{\rho\pm} = h_{\rho\pm} - \frac{1}{2} \frac{\partial_\rho}{\partial_\pm} h_{\pm\pm}$$

12 General covariance

Perturbative expansion

$$\partial_\pm j^\mp = h_{\pm\pm} + P^{(2)}_{\pm\pm}(h) + P^{(3)}_{\pm\pm}(h) + \ldots$$

Recurrent relations

$$\frac{\delta P^{(n)}_{++}}{\delta h_{\rho\sigma}}\, 2\,\partial_\sigma \chi_\rho + \frac{\delta P^{(n)}_{++}}{\delta h_{\rho+}}\, \partial_+ \chi_\rho = \sum_{k=1}^{n-1} \frac{\delta P^{(k)}_{++}}{\delta h_{\rho\sigma}}\, 2\Gamma^\nu_{\rho\sigma}\chi_\nu\,,$$

$$\frac{\delta P^{(n)}_{++}}{\delta h_{++}}\, 2\,\partial_+ \chi_+ + \frac{\delta P^{(n)}_{++}}{\delta h_{\rho+}}\, \partial_\rho \chi_+ = 0\,.$$

Perturbative solution in pure gravity

$$\partial_\pm j^\mp = h_{\pm\pm} - X^2_{\rho\pm} + X_{\sigma\pm}X_{\rho\pm}h_{\rho\sigma} - X_{\sigma\pm}\frac{\partial_\sigma}{\partial_\pm}\, X^2_{\rho\pm} + \ldots$$

13 Hamilton-Jacobi equation

"Fan" equation

$$g^{\sigma\mp}\,\partial_\sigma j^{\mp} = g^{\mp\mp} + \frac{g^{\sigma\rho}}{4}\,(\partial_\sigma j^{\mp})(\partial_\rho j^{\mp})$$

Formal solution

$$j^{\mp} = 2x^{\mp} - \omega^{\mp}$$

Light front (Hamilton-Jacobi) equation

$$g^{\sigma\rho}\partial_\sigma\omega^{\mp}\partial_\rho\omega^{\mp} = 0$$

Hamilton dynamics for massless particle

$$\delta J = 0\,,\;\; J = \int_{-\infty}^{\infty} d\tau \left(p_\mu\partial_\tau x^\mu - \frac{g^{\mu\nu}}{2}p_\mu p_\nu\right)\,,\;\; x^\mu_{\tau\to-\infty} = 2\delta^\mu_\pm x^\pm + \rho^\mu_\perp$$

14 Effective currents for shock waves

Aichelburg - Sexl metric

$$(ds)^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu + a\, \ln|\overrightarrow{x}|\delta(x^-)\,(dx^-)^2\,,\ \ z = a\,\frac{x^-}{|\overrightarrow{x}|^2}\,,\ \ a = \frac{8}{\sqrt{2}}\,G\,\mu$$

Effective current for the shock wave

$$j^+ = -a\,\mu\left(\ln|\overrightarrow{x}| + \ln f(z) - \frac{1}{4}\,\frac{z}{f^2(z)}\right)\,,\ \ f(z) = \frac{1}{2} + \frac{\sqrt{1+2z}}{2}$$

Perturbative expansion

$$j^+ = -a\ln|\overrightarrow{x}| + \frac{a^2}{\partial_-}\left(\frac{x_\sigma}{2|\overrightarrow{x}|}\right)^2) - \frac{a^3}{\partial_-}\,\frac{x_\mu}{2|\overrightarrow{x}|}\,\frac{\partial_\mu}{\partial_-}\left(\frac{x_\sigma}{2|\overrightarrow{x}|}\right)^2 + \ldots$$

Variational principle for j^+

$$j^+ = \min_{\overrightarrow{\rho}} \int_{-\infty}^{x^-} dy^-\,\left(g^{++}(y^-, \overrightarrow{\rho}(y^-)) + (\partial_-\overrightarrow{\rho})^2\right)\,,\ \ \frac{\delta j^+}{\delta\,\overrightarrow{\rho}\,(y^+)} = 0$$

15 Effective reggeon vertices

Hilbert-Einstein Lagrangian

$$\sqrt{-g}\,R = \frac{\partial_\sigma h_{\mu\sigma}}{2}\left(\partial_\mu h_{\rho\rho} - \partial_\rho h_{\mu\rho}\right) + \frac{1}{4}\left((\partial_\sigma h_{\mu\nu})^2 - (\partial_\sigma h_{\mu\mu})^2\right) +$$

$$h_{\rho\sigma}\left(-\frac{\partial_\rho h_{\mu\nu}}{4}\,\partial_\sigma h_{\mu\nu} - \frac{\partial_\mu h_{\nu\sigma}}{2}(\partial_\nu h_{\mu\rho} + \partial_\mu h_{\rho\nu}) + \frac{\partial_\mu h_{\mu\nu}}{2}\left(2\partial_\rho h_{\nu\sigma} - \partial_\nu h_{\rho\sigma}\right)\right)$$

$$+h_{\rho\rho}\left(h_{\mu\nu}(\partial_\mu\partial_\sigma h_{\nu\sigma} - \frac{1}{2}\,\partial_\sigma^2 h_{\mu\nu}) + \frac{1}{2}\,(\partial_\nu h_{\nu\sigma})^2 - \frac{3}{8}\,(\partial_\sigma h_{\mu\nu})^2 + \frac{1}{4}\,(\partial_\sigma h_{\mu\nu})\,\partial_\mu h_{\sigma\nu}\right)$$

$$+h_{\rho\sigma}\left((\partial_\mu h_{\mu\sigma})\partial_\nu h_{\nu\rho}\right) - \frac{h_{\rho\rho}}{8}\,(\partial_\nu h_{\sigma\sigma})^2 - \frac{h_{\rho\rho}}{4}\,h_{\mu\nu}\,\partial_\mu\partial_\nu h_{\sigma\sigma} + \ldots\,.$$

Triple reggeon interaction

$$S^{1\to 2} = -\frac{1}{2\kappa}\int\frac{d^4x}{8}\left(\left(\frac{\partial_\rho}{\partial_+}A_{++}\right)^2\partial_\sigma^2 A_{--} + \left(\frac{\partial_\rho}{\partial_-}A_{--}\right)^2\partial_\sigma^2 A_{++}\right)$$

16 Discussion

1. Locality of reggeon interactions in the rapidity space.
2. BFKL equation for Pomeron wave function
3. High energy effective action for gluons in QCD.
4. Pomeron-graviton duality in $N = 4$ SUSY.
5. Multi-regge processes in gravity.
6. The graviton trajectory and double logarithms.
7. Effective action for the high energy gravity.
8. Hamilton-Jacobi equation for effective currents.