Multiple Rogue Waves Solutions to the Gross-Pitaevskii Equation.

Vladimir B. Matveev

[1]Université de Bourgogne, Institut de Mathématiques de Bourgogne, Dijon, France
email: matveev@u-bourgogne.fr

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Main Results

Abstract

We propose two different descriptions of the multi-rogue waves solutions to the Gross-Pitaevskii equation depending on $2n$ arbitrary real parameters providing very large family of multiple rogue-wave solutions "appearing from nowhere and disappearing again". The conjecture is that for given n "in general position" the absolute values of these solutions have $n(n+1)/2$ **maxima** and $n(n+1)$ **minima** of the height comparable with that of Peregrine breather. Exceptionally, under appropriate choice of parameters one can observe the "extreme" rogue wave=higher Peregrine breather solution of order n with one highest maximum of amplitude, of the height $2n+1$ surrounded by a greater number, (conjecturally equal to n(n+1) -1), of smaller maxima. These solutions generalize both the famous Peregrine breather (Peregrine 1983) and its higher order versions obtained by Akhmediev et all (1985-2010). We explain the link of these solutions with a subfamily of smooth localized rational solutions of the KP-I equation. This talk is based on the articles P.Gailard,P.Dubard , C.h.Klein, V.B. Matveev Eur.Phys. J,Special topics **185**,247-258, (2010), EDP Sciences , Springer-Verlag 2010,
P.Dubard and V.B. Matveev Nat.Hazards Earth syst.Sci.,11, 667-672,2011,
www.nat-hazards-earth-syst-sci.net/11/2011/
and some new, yet unpublished results, obtained by P.Dubard, P.Gaillard and myself.

Focusing NLS equation reads

$$iv_t + 2|v|^2 v + v_{xx} = 0, \quad x, t \in \mathbb{R}.$$

Multi rogue waves solutions of the NLS equation are its quasi rational solutions:

$$v = e^{2iB^2t} R(x,t), \quad R(x,t) = \frac{N(x,t)}{D(x,t)}, \quad B > 0,$$

Here $N(x,t), D(x,t)$ are polynomials of x and t, and $\deg N(x,t) = \deg R(x,t) = n(n+1)$,

$$|v^2| \to B^2, \quad x^2 + t^2 \to \infty$$

The rational function $R(x,t)$ satisfies the 1D Gross-Gitaevskii equation:

$$iR_t + 2R(|R|^2 - B^2) + R_{xx} = 0, \quad |R| = |v|.$$

$$q_{2n}(k) := \prod_{j=1}^{n}\left(k^2 - \frac{\omega^{2m_j+1}+1}{\omega^{2m_j+1}-1}B^2\right), \quad \omega := \exp\left(\frac{i\pi}{2n+1}\right).$$

Numbers m_j are some positive integers satisfying the condition

$$0 \le m_j \le 2n-1, \quad m_l \ne 2n - m_j, 1 \le l, j \le n.$$

In particular, it is possible to set $m_j = j - 1$.

$$\Phi(k) := i \sum_{l=1}^{2n} \varphi_l (ik)^l, \quad \varphi_j \in \mathbf{R},$$

$$f(k,x,t) := \frac{\exp(kx + ik^2 t + \Phi(k))}{q_{2n}(k)}, \quad D_k := \frac{k^2}{k^2+B^2} \frac{\partial}{\partial k},$$

$$f_j(x,t) := D_k^{2j-1} f(k,x,t)\,|_{k=B} \;,$$

$$f_{n+j}(x,t) := D_k^{2j-1} f(k,x,t)\,|_{k=-B} \;, \quad j = 1\ldots,n.$$

Consider two Wronskians:

$$W_1 := W(f_1, \dots, f_{2n}) \equiv \det A, \quad A_{lj} := \partial_x^{l-1} f_j,$$

$$W_2 := W(f_1, \dots, f_{2n}, f).$$

Multi-rogue solutions to the focusing NLS equation I

Theorem

The function $v(x,t)$ defined by the formula

$$v(x,t) = -q_{2n}(0)B^{1-2n}e^{2iB^2t}\frac{W_2}{W_1}\,|_{k=0}\;, \tag{1}$$

represents a family of nonsingular (quasi)-rational solutions to the focusing NLS equation depending on 2n independent real parameters φ_j.

Plots of quasi-rational solutions for n=1,2

We choose $m_j = j - 1$ and we limit ourselves to the case $B = 1$. The whole set of solutions with any B can be obtained by the scaling transformation:

$$x \to Bx, \quad t \to B^2 t, v \to B^{-1} v.$$

Peregrine solution

The case n=1 is well known:

$$v(x,t) = \frac{(x-\varphi_1)^2 + 4(t-\varphi_2)^2 - (2\sqrt{3}+4i)(t-\varphi_2) + i\sqrt{3}}{(x-\varphi_1)^2 + 4(t-\varphi_2)^2 - 2\sqrt{3}(t-\varphi_2) + 1} e^{2it}. \tag{2}$$

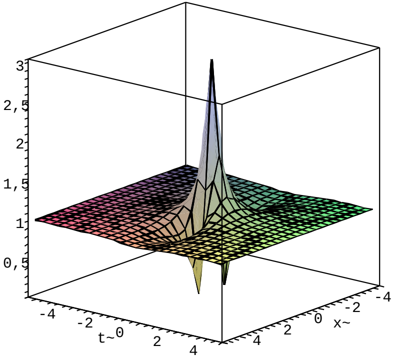


Figure: n=1 solution for $\varphi_1 = 0$ and $\varphi_2 = 0$.

For $n = 2$ the nominator and denominator of the rational part of the solution are a 6-th order polynomials with respect to x and t. With out less of generality it can be written in a following form:

Let

$$\varphi_1 = 3\varphi_3, \quad \varphi_2 = 2\varphi_4 + \frac{3+\sqrt{5}}{16} \cdot \sqrt{10 - 2\sqrt{5}},$$

$$\alpha := (5+\sqrt{5})\sqrt{10 - 2\sqrt{5}} - 96\varphi_4,$$

$$\beta := 96\varphi_3$$

. Then the solution to the case n=2 is given by the formula:

"general" solution for n=2

$$v(x,t) = \frac{N(x,t)}{D(x,t)} \exp(2it),$$

$$\begin{aligned}
N(x,t) &= 64x^6 + (768t^2 - 144 - (768i)t)x^4 - 16\beta x^3 \\
&+ (3072t^4 + 192\alpha t - (96i)\alpha - 5760t^2 - (6144i)t^3 \\
&+ (1152i)t - 180)x^2 + (192\beta t^2 - 36\beta - (192i)\beta t)x \\
&+ 45 - (1536i)t^3 + \beta^2 + 4\alpha^2 - 1872t^2 \\
&- 8448t^4 - 256\alpha t^3 + 4096t^6 + 48\alpha t + (720i)t \\
&- (12288i)t^5 + (384i)\alpha t^2 - (24i)\alpha,
\end{aligned}$$

$$\begin{aligned}
D(x,t) &= 64x^6 + (768t^2 + 48)x^4 - 16\beta x^3 \\
&+ (-1152 + t^2 + 192\alpha t + 108 + 3072t^4)x^2 \\
&+ (12\beta + 192\beta t^2)x + 4096t^6 + 6912t^4 - 256\alpha t^3 \\
&+ 1584t^2 - 144\alpha t + \beta^2 + 4\alpha^2 + 9.
\end{aligned}$$

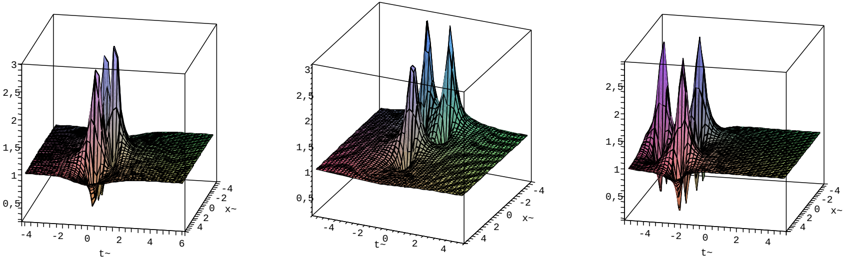


Figure: Amplitude of the solution to the NLS equation for $n = 2$ with $\varphi_2 = 1$ and $\varphi_1 = \varphi_3 = \varphi_4 = 0$ on the left, $\varphi_3 = 1$ and $\varphi_1 = \varphi_2 = \varphi_4 = 0$ in the middle and $\varphi_4 = 1$ and $\varphi_1 = \varphi_2 = \varphi_3 = 0$ on the right.

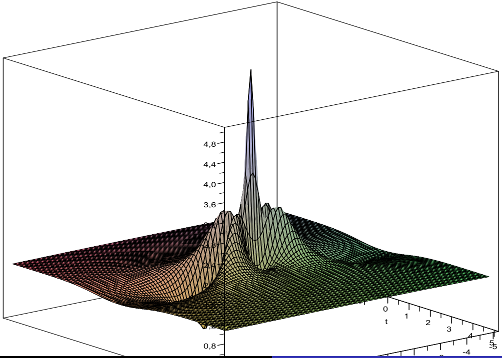
How to get 2-d Akhmediev-Peregrine breather

We can see the solutions above as a natural generalization of the higher Peregrine breather found in 1985 by Akhmediev, Eleonski and Kulagin. The later corresponds to the following choice of the phases:

$$\varphi_1 = \varphi_3 = 0,$$

$$\varphi_2 = \frac{7+2\sqrt{5}}{6} \sin\frac{\pi}{5}, \varphi_4 = \frac{5+\sqrt{5}}{24} \sin\frac{\pi}{5}.$$

Plot of the Amplitude of Peregrine-Akhmediev breather of order 2.



When the parameters α, β are small enough the related deformation of the higher Peregrine breather keeps its extreme rogue wave character i.e. the maximum of its magnitude is very close to 5 and a plot of the solution is quite similar to what we have when $\alpha = \beta = 0$.

Solutions of the NLS equation above provide $2n$-parametric family of the smooth rational solutions to the KP-I equation:

$$\partial_x(4u_t + 6uu_x + u_{xxx}) = 3u_{yy}.$$

Replace t by y and φ_3 by t. Obviously the function

$$f(k,x,y,t) := \exp(kx + ik^2y + k^3t + \phi(k)),$$

where

$$\phi(k) := \Phi(k) - \varphi_3 k^3,$$

satisfies the system

$$f_t = f_{xxx}, \quad if_y + f_{xx} = 0.$$

The same is true for the functions f_j, defined above if we denote t by y and φ_3 by t. Now from (Matveev LMP 1979 p.214-216) we get following result:

Theorem

$$u(x, y, t) = 2\partial_x^2 \log W(f_1, \dots, f_{2n}) = 2(|v|^2 - B^2)$$

gives a family of smooth rational solutions to the KP-I equation. It is obvious that

$$\int_{-\infty}^{\infty} u(x, y, t) dx = 0.$$

These solutions depend on $2n$ real parameters $\varphi_j, B, j \neq 3$, representing the action of the KP-I hierarchy flows. The phases φ_1, φ_2 correspond respectively to space and time translations.

Above we presented the evidence of existence for any given n of the $2n$-parametric family of the multi rogue waves solutions to the focusing NLS solutions absorbing the previously known (isolated) solutions with similar properties. We explained also their relevance to some particular $2N$ parametric family of smooth, localized real valued solutions to the KP-I equation. This approach allows also to isolate the higher Peregrine solution of order 2 and also to study its vicinity in a space of parameters but it is not clear how to isolate in a frame of this approach higher order extreme rogue waves.

The weakness of the described approach is that it makes difficult to isolate the extremal wave solutions corresponding to the higher Peregrine breathers except the case n=2. For higher values of *n* it seems to be hopeless. It is also difficult enough from the point of view of visualization and numerical evaluation of the solutions. I suggested another systematic approach to description of multi rogue waves solutions.It contains to take an appropriate limit in the formulas for elementary solutions describing the N-phase modulation of the plane wave solutions, when all periods tend to infinity and the related "phases" tend to zero in appropriate way.

Some notations

Let $\lambda_m, \quad 1 \leq m \leq 2n$ are any real numbers satisfying the conditions

$$0 < \lambda_j < 1, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq n;$$

and e_ν, are complex numbers $1 \leq \nu \leq 2n$ satisfying the relations

$$e_j = iA_j - B_j, \quad e_{n+j} = iA_j + B_j, \quad 1 \leq j \leq n.$$

further definitions

κ_ν, δ_ν, γ_ν are the following functions of λ_ν:

$$\kappa_\nu := 2\sqrt{1-\lambda_\nu^2}, \quad \delta_\nu := \kappa_\nu \lambda_\nu, \quad \gamma_\nu := \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}},$$

$$\kappa_{n+j} := \kappa_j, \quad \delta_{n+j} := -\delta_j,$$
$$\gamma_{n+j} := 1/\gamma_j, \quad j = 1 \dots n.$$

$$\epsilon_j := j, \quad 1 \le j \le n$$
$$\epsilon_j := j+1, \quad n+1 \le j \le 2n.$$

$$x_j(0) := 0, \quad 1 \leq j \leq 2n,$$
$$x_j(3) := 2\ln\frac{\gamma_j - i}{\gamma_j + i}, \quad 1 \leq j \leq n,$$
$$x_{n+j}(3) := -2\ln\frac{\gamma_j - i}{\gamma_j + i} - 2i\pi, \quad 1 \leq j \leq n.$$

We denote by $\mathbf{G}(r)$, $r = 1, 3$ the $2n \times 2n$ matrices :

$$G_{\nu\mu} := \frac{2(-1)^{\epsilon_\nu}\gamma_\nu}{\gamma_\nu + \gamma_\mu} \exp(i\kappa_\nu x - 2\delta_\nu t + x_\nu(r) + e_\nu), \quad r = 1, 3$$

n-phase multi-periodic solution of the focusing NLS equation

$$v(x,t) := \frac{\det(I + \mathbf{G}(3))}{\det(I + \mathbf{G}(1))} \exp(2it).$$

Novel construction of quasi-rational solutions to NLS equation

Suppose λ_j depends on ϵ in a following way:

$$\lambda_j = 1 - 2\epsilon^2 j^2, \quad 1 \leq j \leq n, \quad \epsilon > 0,$$

and

$$A_j = a_j \epsilon^{2n-1}, \quad B_j = b_j \epsilon^{2n-1},$$

where a_j, b_j do not depend on ϵ.

Crucial observation

Suppose the previous requirements concerning ϵ dependence are satisfied . than there exists the limit :

$$\lim_{\epsilon\to 0} \frac{\det\left(I+\mathbf{G}(3)\right)}{\det\left(I+\mathbf{G}(3)\right)} e^{2it} = \frac{N(x,t)}{D(x,t)} e^{2it},$$

where D(x,t) and N(x,t) are polynomials of x and t of order $n(n+1)$. These polynomials depend on $2n$ parameters a_i, b_i when all this parameters are equal to zero we obtain the n-th AP breather n, of amplitude $2n+1$. Until recently it was a single example of what we propose to call "extreme rogue wave solution" Varying the parameters we can see that for small enough values of a_i, b_i the extreme wave character of the solution is well preserved. Therefore, the extreme rogue waves

For $n = 1$ we get again the Peregrine breather. For $n = 2$ as before the solution depends on 4 parameters. It can be identified with the solution written above via simple transformation :

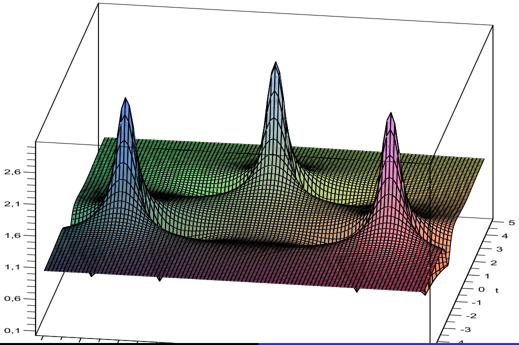
$$\alpha = -\frac{b_2 - 2b_1}{4}, \quad \beta = -\frac{a_2 - 2a_1}{2},$$

To get Peregrine breather of order 2 it is enough to take $a_1 = a_2 = b_1 = b_2 = 0$.

taking

$$a_1 = a_2 = 0, = b_1 = b_2 = 1000,$$

we get again the picture corresponding to the "three sisters" wave solution:



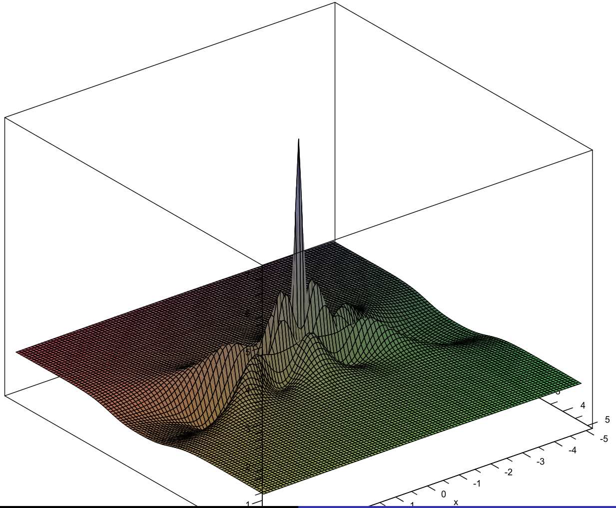
n=3 quasi-rational solutions

Higher Peregrine breather of order 3 is obtained simply by setting $a_j = b_j = 0, \quad j = 1, 2, 3$. Changing a_j and b_j we can observe the deformation of higher Peregrine breather to a 6-peaks rogue-wave solution.

An example

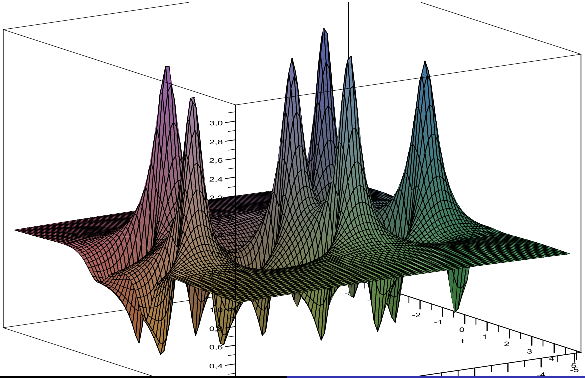
$$a_j = b_j = 1, \quad j = 1, 2, 3$$

illustrates a small deformation of the Peregrine's breather of order 3 showing that it is stable with respect to small variation of parameters.


4
5
-5
-4
-3
-2
-1
0
x

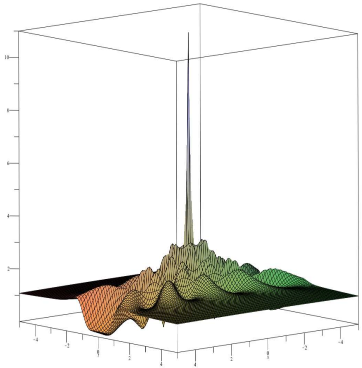
6-rogue wave solution to FNLS equation, n=3

$$a_1 = 10000,\ a_2 = a_3 = 0,\ b_1 = 10000,\ b_2 = b_3 = 0.$$



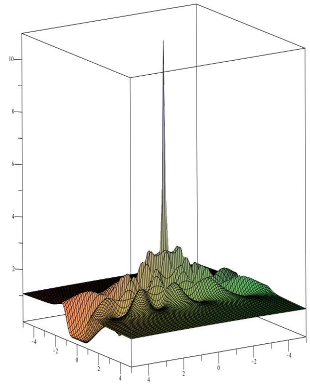
n=6 solutions

To get the 6-th AP breather we take as before $a_j = b_j = 0 \quad j = 1, \ldots 6$, The maximum of amplitude of the solution is equals $2 \times 6 + 1 = 13$. For the related solution of the KP-I equation the maximum of amplitude equals $13^2 - 1 = 168$ 168



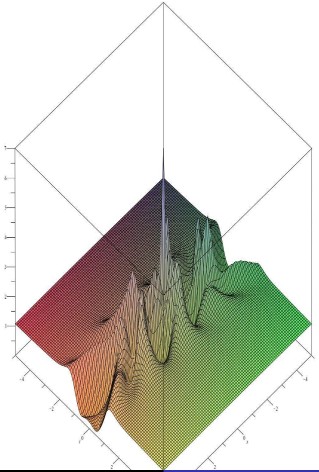
Small deformation of the 6-th AP breather: extreme rogue wave

For $a_j = b_j = 1, \forall j$ we get following plot of $|v|$:


10
8
6
4
2
-4
-2
0
2
4
4
2
0
-2
-4

large deformation of 6-th AP breather

$$a_j = 0, j = 1 \ldots 6, b_j = 1000000, \quad j = 1 \ldots 6$$



Conclusions

Our major new observation is that the n-th (AP breather), which before was considered as very isolated special solution to the FNLS equation, in fact, can be considered as a special reduction of more general 2n-parametric families of families of quasi-rational solutions of the same equation. Moreover, we found the particularly comfortable parametrization for which the afore-mention reduction becomes trivial: we just have to set all the parameters to be equal to zero. Small variation of the aforementioned parameters leads to almost the same plots of the amplitude as for the n-th AP breather, preserving the character of "extreme rogue wave" solution with a height close to 2n+1. Therefore, "extreme rogue waves" are not so rare as a single Peregrine breather: they form a 2n parametric family containing 2n-2 "nontrivial" parameters. For the "intermediate" values of parameters the picture becomes more rich and complicated when the parameters or a part of them become larger. For "much larger" values of parameters we get a multi rogue waves solutions attending n(n+1)/2 peaks of the amplitude of the height comparable with that of Peregrine breather: this can be considered as a generic picture.

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THANK YOU FOR YOUR ATTENTION !