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Localization in disordered systems with chiral symmetry

Pavel Ostrovsky (KIT, Landau ITP)

in collaboration with

S. Bera, I. V. Gornyi, E. König, A. D. Mirlin, I. Protopopov, M. Titov

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Outline

1. Introduction
2. From ballistics to diffusion
3. From diffusion to localization
4. Summary

Outline

Scaling theory of localization

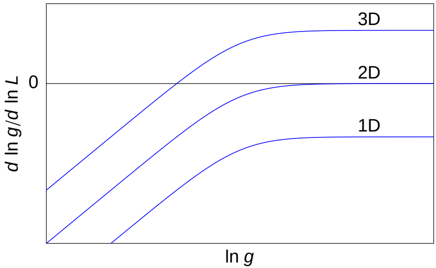
Abrahams, Anderson, Licciardello, Ramakrishnan '79

Dimensionless conductance [in units e^2/h]:

- Metallic sample (Ohm's law): $g \sim L^{d-2}$
- Insulating sample (tunneling): $g \sim e^{-L/\xi}$

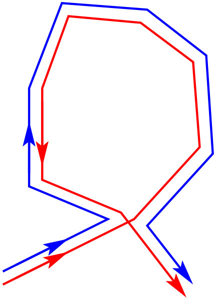
Universal scaling function:

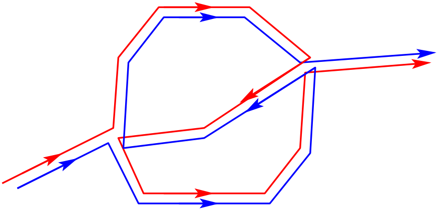
$$\frac{d\ln g}{d\ln L} = \beta(g) = \begin{cases} d-2, & g \gg 1, \quad \text{(metal)}, \\ \ln g, & g \ll 1, \quad \text{(insulator)}. \end{cases}$$



Quantum interference corrections

Quantum corrections are sensitive to time-reversal symmetry





TR symmetry preserved

spin decoupled ⇒ negative correction

strong spin-orbit ⇒ positive correction

TR symmetry broken

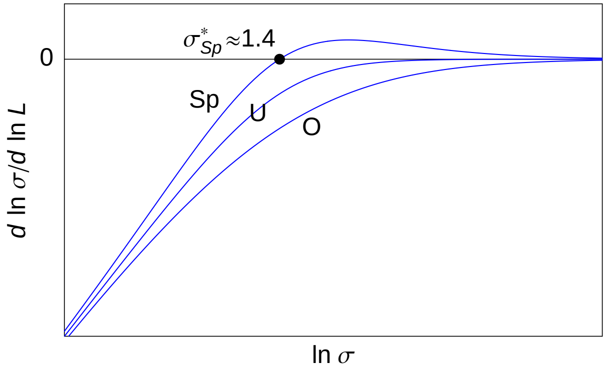
two-loop interference

very weak negative correction

Weak localization correction in 2D

Gor'kov, Larkin, Khmelnitskii '79; Hikami, Larkin, Nagaoka '80

$$\frac{d\ln g}{d\ln L} = \begin{cases} -(\pi g)^{-1}, & \text{orthogonal (TR preserved, spin preserved)}, \\ -(2\pi g^2)^{-1}, & \text{unitary (TR broken)}, \\ +(\pi g)^{-1}, & \text{symplectic (TR preserved, spin broken)} \end{cases}$$



Symmetry Classification

Wigner ’51, Dyson ’62, Altland and Zirnbauer ’97, Schnyder et al ’08

- Time-reversal symmetry (T):

$$H = UH^T U^{-1}, \qquad T^2 = UU^* = \pm \mathbf{1}$$

- Chiral symmetry (C):

$$H = -UHU^{-1}$$

- Particle-hole symmetry (CT):

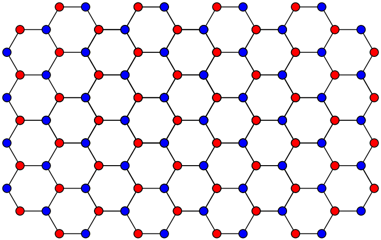
$$H = -UH^T U^{-1}, \qquad CT^2 = UU^* = \pm \mathbf{1}$$

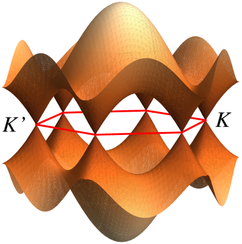
Symmetry Classification

Wigner '51, Dyson '62, Altland and Zirnbauer '97, Schnyder et al '08

	T^2	C	CT^2	NLσM	π_1	π_2	π_3	WL
A	0	0	0	$U(2N)/U(N) \times U(N)$	0	$\mathbb{Z}$	0	0
AI	1	0	0	$Sp(4N)/Sp(2N) \times Sp(2N)$	0	0	0	-
AII	-1	0	0	$O(2N)/O(N) \times O(N)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	+
AIII	0	1	0	$U(N)$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\equiv 0$
BDI	1	1	1	$U(2N)/Sp(2N)$	$\mathbb{Z}$	0	0	$\equiv 0$
CII	-1	1	-1	$U(N)/O(N)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\equiv 0$
D	0	0	1	$O(2N)/U(N)$	0	$\mathbb{Z}$	0	+
C	0	0	-1	$Sp(2N)/U(N)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	-
DIII	-1	1	1	$O(N)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	+
CI	1	1	-1	$Sp(2N)$	0	0	$\mathbb{Z}$	-

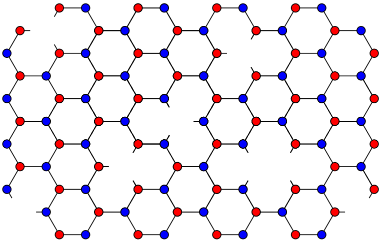
Graphene

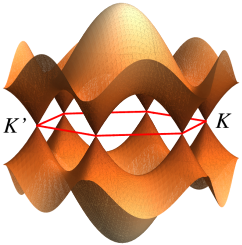




- Chiral structure: two sublattices: A, B
- Two valleys of the spectrum: K, K′
- linear dispersion: $\varepsilon = v_0|\mathbf{p}|$
- Massless Dirac Hamiltonian in each valley: $H = v_0\boldsymbol{\sigma}\mathbf{p}$, $\boldsymbol{\sigma} = \{\sigma_x, \sigma_y\}$
- Vacancies preserve chiral symmetry (class BDI)

Graphene





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- **Vacancies preserve chiral symmetry (class BDI)**

Outline

Generating function

- Matrix Green function [Nazarov '94]

$$\check{G} = \begin{pmatrix} \epsilon + i0 - H & -\delta(x) v_x \sin\frac{\phi}{2} \\ -\delta(x-L) v_x \sin\frac{\phi}{2} & \epsilon - i0 - H \end{pmatrix}^{-1}$$

- Generating function (free energy): $\mathcal{F}(\phi) = \mathbf{Tr} \log \check{G}^{-1}(\phi)$

$$\Longrightarrow \quad \text{Conductance: } G = -\frac{2e^2}{h} \frac{\partial^2 \mathcal{F}}{\partial \phi^2}\bigg|_{\phi=0}$$

$$\Longrightarrow \quad \text{Fano factor: } F = \frac{1}{3} - \frac{2}{3} \frac{\partial^4 \mathcal{F}/\partial \phi^4}{\partial^2 \mathcal{F}/\partial \phi^2}\bigg|_{\phi=0}$$

- Clean graphene

$$\mathcal{F}_0(\phi) = -\frac{W\phi^2}{\pi L}, \qquad G = \frac{4e^2}{\pi h}\frac{W}{L}, \qquad F = \frac{1}{3}$$

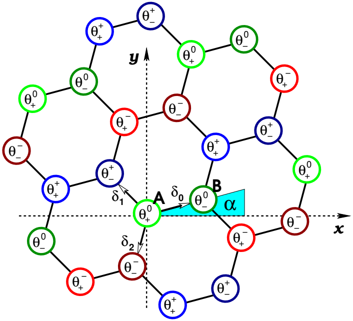
On-site potential

- From lattice to Dirac: $\Psi_i = \langle u_i | \Phi(\mathbf{r}) \rangle$

  $|\Phi(\mathbf{r})\rangle$ – smooth envelope function (Dirac Hamiltonian)

  Bloch function $\langle u_i| = \begin{cases} (e^{i\theta_+/2}, 0, 0, e^{-i\theta_+/2}), & \mathbf{r}_i \in \mathrm{A}, \\ (0, ie^{i\theta_-/2}, ie^{-i\theta_-/2}, 0), & \mathbf{r}_i \in \mathrm{B}. \end{cases}$

- On-site potential in the Dirac language: $|u_i\rangle V_i \langle u_i|$



Phases $\theta_\pm$ depend on sublattice and "color" of the site:

$\theta_\pm = \pm\alpha + 2\mathbf{K}\cdot\mathbf{r}_i = \pm\alpha + 2\pi i c/3$

Color index: $c = 0, \pm 1$

Unfolded representation

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- Generating function

$$\mathcal{F}(\phi) = \mathcal{F}_0 + \mathbf{Tr}\log(1 - \check{G}_0 V) \quad \text{with} \quad V = \sum_m |u_m\rangle V(\mathbf{r}_m)\langle u_m|$$

- Unfolding: $\mathcal{F}(\phi) = \mathcal{F}_0 + \log\det\left[\delta_{nm} - V_n\langle u_n|\check{G}_0(\mathbf{r_n}, \mathbf{r_m})|u_m\rangle\right]$
- Vacancies $V_n \to \infty$: $\mathcal{F}(\phi) = \mathcal{F}_0 + \log\det\langle u_n|\check{G}_0(\mathbf{r_n}, \mathbf{r_m})|u_m\rangle$
- Conductance

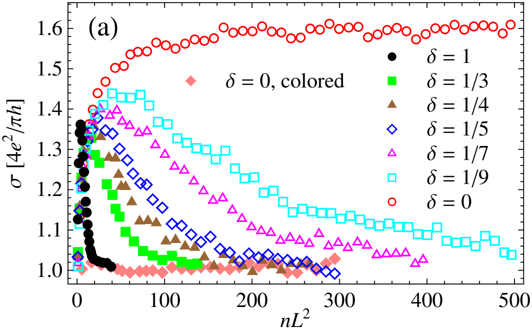
$$G = \frac{4e^2}{\pi h}\left\{\frac{W}{L} + \pi\,\mathrm{Tr}[K, Y](K + K^T)^{-1}[K^T, Y](K + K^T)^{-1}\right\}$$

$$K_{mn} = \frac{e^{\frac{i}{2}(\theta_m - \theta_n)}}{\sin\frac{\pi}{2L}\left[\zeta_m x_m + \zeta_n x_n + i(y_m - y_n)\right]}, \qquad Y = L^{-1}\,\mathrm{diag}\{y_n\}$$

$\zeta_i = \pm 1$ and θ_i are sublattice and color of ith vacancy

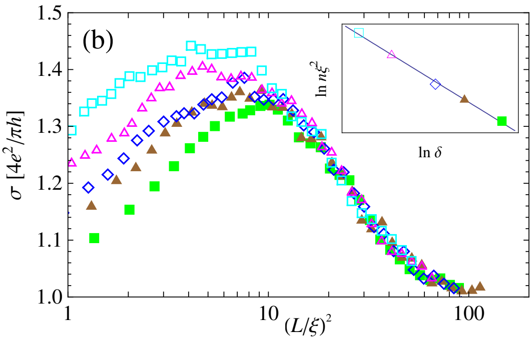
Inversion of an $N \times N$ matrix $\Longrightarrow$ extremely efficient numerics!

Vacancies: numerics



- Single color, armchair boundary ($\alpha = 0$)
- Sublattice imbalance $\delta = (n_A - n_B)/n$
- Unstable fixed point for $\delta = 0$ (conductivity saturates at $\sigma \approx 2e^2/h$)
- Stable fixed point for $n_B \neq n_A$ with $\sigma \approx \dfrac{4e^2}{\pi h}$

Vacancies: scaling

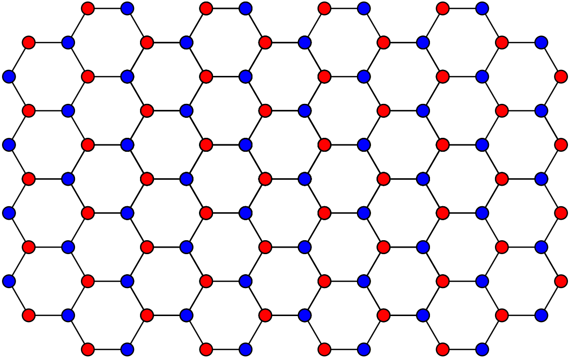


- Crossover curves collapse in units of L/ξ
- Power law scaling $n\xi^2 \sim \delta^{0.72}$

Novel strong-coupling criticality in class BDI beyond sigma model

Outline

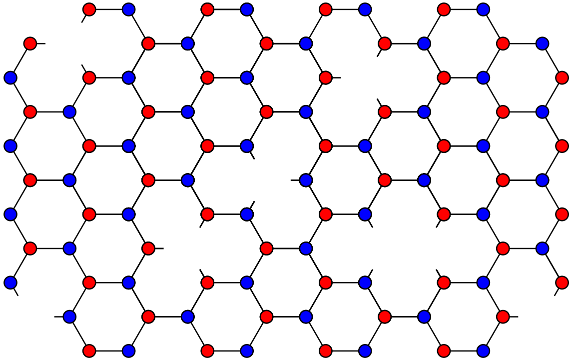
From metal to insulator



Gradually remove sites from graphene

Metal-insulator transition expected!

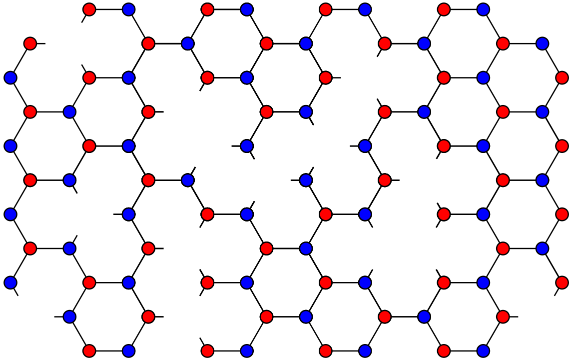
From metal to insulator



Gradually remove sites from graphene

Metal-insulator transition expected!

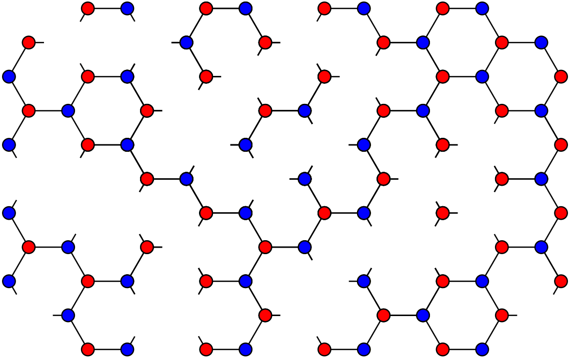
From metal to insulator



Gradually remove sites from graphene

Metal-insulator transition expected!

From metal to insulator

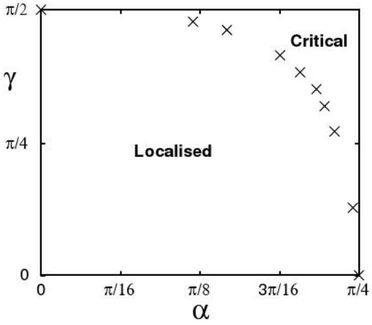


Gradually remove sites from graphene

Metal-insulator transition expected!

Numerics: Chiral Network Models

Bocquet and Chalker '03



- Chiral unitary (AIII) network model
- Both critical (Gade) and localized phase observed
- Similar results for dimerized lattice model [Motrunich *et al* '02]

Localization from metallic perspective

Gade and Wegner '91, Gade '93

- 2D nonlinear sigma model for a chiral system

$$S[Q] = \int d^2x \left\{ \frac{\sigma}{8\pi} \mathrm{tr} \left[\nabla Q^{-1} \nabla Q \right] - \frac{c}{8\pi} \left[\mathrm{tr}\, Q^{-1} \nabla Q \right]^2 \right\}$$

- Matrix field

$$Q \in \begin{cases} U(N), & \text{unitary (AIII)}, \\ U(N)/Sp(N), & \text{orthogonal (BDI)}, \\ U(N)/O(N), & \text{symplectic (CII)}. \end{cases}$$

- Replica limit $N \to 0$ is assumed
- σ – conductivity per square

Localization from metallic perspective

Gade and Wegner '91, Gade '93

- Rewrite $Q = e^{i\phi} U$ (det $U = 1$)

$$S[U, \phi] = \int d^2x \left\{ \frac{\sigma}{8\pi} \mathrm{tr} \left[\nabla U^{-1} \nabla U \right] + \left(\frac{\sigma + Nc}{8\pi} \right) (\nabla \phi)^2 \right\}$$

- Decoupled Gaussian theory in ϕ:

$$\frac{d}{d\ln L} (\sigma + Nc) = 0$$

- Replica limit

$$\frac{d\sigma}{d\ln L} = -N \frac{dc}{d\ln L} \overset{N \to 0}{\to} 0$$

Absence of localization to all orders in perturbation theory!

Status quo

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Apparent controversy

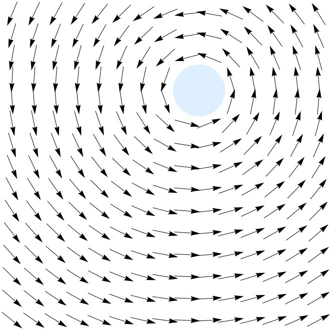
- Strong disorder induces localization in a chiral system (intuition + numerics)
- No traces of localization in the perturbation theory in the metallic limit (Gade and Wegner)

How to resolve?

Take into account non-perturbative effects

Bypass Gade and Wegner argument

- Loophole to escape Gade and Wegner argument:
  $\det Q = e^{i\phi} \in U(1) \simeq \mathbb{S}^1$
  $\Rightarrow$ **vortex excitations allowed!**



- Recalls Berezinskii-Kosterlitz-Thouless transition!

BKT Transition

Berezinskii ’70, Kosterlitz and Thouless ’73

- Continuum limit of xy-model:

$$S[\phi] = \int d^2x \, \frac{J}{2} (\nabla \phi)^2$$

- Vortex excitation (γ – inverse core size, n – vorticity)

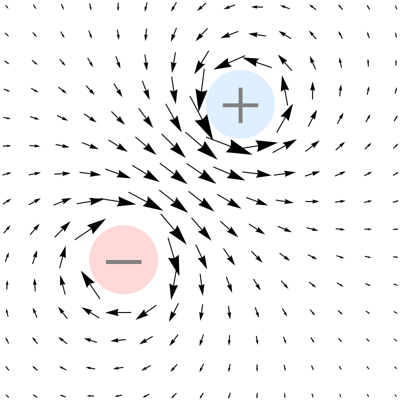
$$\partial_\mu \phi = n \, \mathtt{rot}_\mu \ln \gamma ||\mathbf{x} - \mathbf{x}_0||$$

- Within the core, gradient expansion breaks down: regularize by S_{core}
  $\Rightarrow$ statistical weight $y_0 = e^{-S_{\text{core}}}$
- 2D Coulomb gas of vortices

$$S_{\text{Vortices}} = -2\pi J \sum_{i<j} n_i n_j \ln \gamma ||\mathbf{x}_i - \mathbf{x}_j||$$

BKT Transition

Berezinskii ’70, Kosterlitz and Thouless ’73



The gradient field $\nabla\phi$ of a vortex-antivortex dipole

BKT Transition

Berezinskii '70, Kosterlitz and Thouless '73

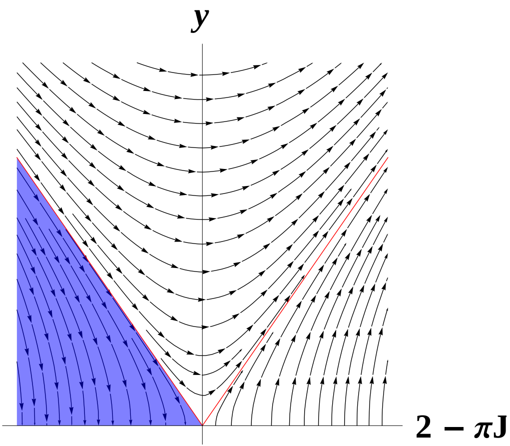
- Large J (low temperature):
  vortices strongly bound in tiny dipoles
  ordered phase (quasi long-range order)
- Small J (high temperature):
  vortex plasma, disordered phase
- Real space RG (integrate out small dipoles and rescale)

$$\frac{dJ}{d\ln L} = -y^2 J^2$$

$$\frac{dy}{d\ln L} = (2 - \pi J)\, y$$

BKT Transition

Berezinskii ’70, Kosterlitz and Thouless ’73



RG-flow in the vicinity of the critical “end” point.

RG: background field formalism

Polyakov ’75, Pruisken ’87

- Bare action

$$S_0[Q] = \int d^2x \, \left\{ \frac{\sigma_0}{8\pi} \mathrm{tr} \left[\nabla Q^{-1} \nabla Q \right] - \frac{c_0}{8\pi} \left[\mathrm{tr} \, Q^{-1} \nabla Q \right]^2 \right\}$$

- Separate fast and slow variables $Q = U^{-1} \tilde{Q} V$

  $\tilde{Q}$ – fast; U, V – slow

- Integrate out fast variables

- Sigma-model action for slow $Q' = U^{-1} V$ with corrected constants

Perturbative RG

- Expand the fast field $\tilde{Q}$ near 1
- One-loop perturbative RG (chiral unitary system):

$$\frac{d\sigma}{d\ln L} = -N + N^2\, O(1/\sigma), \qquad \frac{dc}{d\ln L} = 1 + N\, O(1/\sigma)$$

- Replica limit $N \to 0$:

$$\frac{d\sigma}{d\ln L} = 0, \qquad \frac{dc}{d\ln L} = 1$$

- Only one loop survives in replica limit [Guruswamy *et al* '00]

Vortex contribution

- Include single vortex-antivortex dipole **d** into fast field:

$$\tilde{Q}_v = 1 + |p\rangle(e^{i\phi} - 1)\langle p|$$

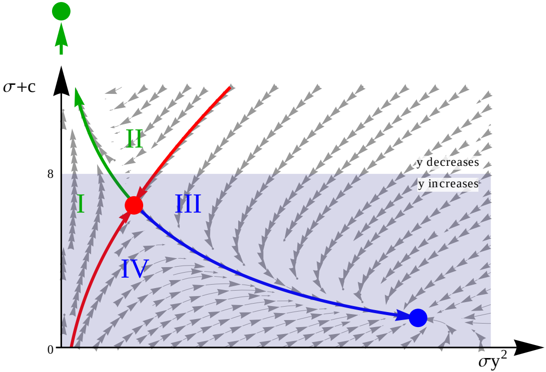
  $|p\rangle$ is a unit vector in replica space
- Dipole action $S[\tilde{Q}_v] = 2S_{\text{core}} + \frac{\sigma_0 + c_0}{2} \ln \gamma \|\mathbf{d}\|$
- Average (integrate over positions of vortices and $|p\rangle$)
- RG equations in replica limit (lowest order in $y = L^2 e^{-S_{\text{core}}}$)

$$\frac{d\sigma}{d\ln L} = -\sigma^2 y^2,$$
$$\frac{dc}{d\ln L} = 1 - (\sigma^2 + 2\sigma c)y^2,$$
$$\frac{dy}{d\ln L} = \left(2 - \frac{\sigma + c}{4}\right) y$$

Flow diagram

Transform RG equations ($K = \sigma + c$, $\Sigma = \sigma y^2$)

$$\frac{d\Sigma}{d \ln L} = -\Sigma^2 + \left(4 - \frac{K}{2}\right)\Sigma, \quad \frac{dK}{d \ln L} = 1 - 2\Sigma K, \quad \frac{dy}{d \ln L} = \left(2 - \frac{K}{4}\right) y$$



Fixed points:

- metal
- insulator
- critical

No minimal metallic conductivity

Vortices vs. topology

- Chiral symplectic class CII admits $\mathbb{Z}_2$ θ-term
  - $\Rightarrow$ Vortices attract instantons
    - $\Rightarrow$ Vortex-instanton fusion changes $S_{\text{core}} \mapsto -S_{\text{core}}$
      - $\Rightarrow$ Internal $\mathbb{Z}_2$ degree of freedom in each vortex
- Chiral unitary class AIII admits Wess-Zumino term
  - $\Rightarrow$ Vortices break gauge symmetry
    - $\Rightarrow$ Internal $U(1)$ degree of freedom in each vortex
      - $\Rightarrow$ Random $\mathrm{Im}\, S_{\text{core}}$

Presence of a topological terms in sigma-model action prevents the theory from vortices!

Summary

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Ballistics ⇔ Diffusion

1. Efficient approach to studying transport in strongly disordered systems is developed
2. The theory is applied to graphene with vacancies
3. Various novel strong-coupling critical regimes are identified

Diffusion ⇔ Localization

1. Renormalization of sigma model due to vortices
2. Non-perturbative weak localization correction in chiral systems
3. Flow diagram near metal-insulator transition

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