Infrared logarithms in Effective Field Theories

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Infrared Logarithms in Effective Field Theories

- Introduction. Perturbative expansion for non-linear σ-model in $D = 4$
- LLogs from unitarity and analyticity
- Connection with the renormalization-group
- Geometric formulation of RG-equations

Conclusions

Introduction. Effective field theories

Non-linear σ-model has the Lagrangian of the form:

$$\mathcal{L}_2 = \frac{1}{2} g_{ab}(\phi)\, \partial_\mu \phi^a \partial_\mu \phi^b = \frac{F^2}{4} \mathrm{tr}\left(\partial_\mu U \partial_\mu U^\dagger\right),$$

where $U = \exp(i\lambda^a \phi^a)$ is an element of some group $\mathcal{G}$.
In $D > 2$ dimensions non-linear σ-models are **non-renormalizable**. However such models are often used as Effective Field Theories (EFTs) for description of Goldstone particles integrations, e.g. Chiral Perturbation theory (ChPT), which is the $\mathcal{G} = SU_L(3) \times SU_R(3)/SU_V(3)$ σ-model and describes interaction of massless pions.

Non-renormalizability of the theory requires introduction higher-order Lagrangians, with new low-energy constants (LEC)

$$\mathcal{L}^{EFT} = \underbrace{\mathcal{L}_2}_{\substack{\text{terms with}\\ \text{2 derivatives}}} + \underbrace{\mathcal{L}_4}_{\substack{\text{terms with}\\ \text{4 derivatives}}} + \underbrace{\mathcal{L}_6}_{\substack{\text{terms with}\\ \text{6 derivatives}}} + \ldots\ .$$

in ChPT 1 constants	in ChPT ~ 4 constants	in ChPT ~ 30 constants

Introduction. Structure of low-energy expansion

Perturbative expansion in $D = 4$ dimensions has the form (4-point function):

$$A(s,t) = \underbrace{c_1 \frac{E^2}{\Lambda^2}}_{\substack{\text{tree order} \\ \text{only } \mathcal{L}_2 \\ \text{parameters}}} + \frac{E^4}{\Lambda^4}\Bigg(\underbrace{c_2 \ln\left(\frac{\mu^2}{E^2}\right)}_{\substack{\text{1-loop} \\ \text{only } \mathcal{L}_2 \\ \text{parameters}}} + \underbrace{c_3}_{\substack{\text{1-loop with } \mathcal{L}_2 \\ \text{parameters } + \\ \text{tree order with} \\ \mathcal{L}_4 \text{ parameters}}} \Bigg) + \frac{E^6}{\Lambda^6}(\ldots) + \ldots$$

E^2 is a generic momentum parameter, s and t are Mandelstam variables.

$$A(s,t) = \underbrace{\sum_{n=1}^{\infty} \omega_n \left(\frac{E^2}{\Lambda^2}\right)^n \ln^{n-1}\left(\frac{\mu^2}{E^2}\right)}_{\substack{\text{Leading} \\ \text{Logarithms (LLogs)} \\ \text{contains only} \\ \mathcal{L}_2 \text{ parameters}}} + \underbrace{\sum_{n=2}^{\infty} \varpi_n \left(\frac{E^2}{\Lambda^2}\right)^n \ln^{n-2}\left(\frac{\mu^2}{E^2}\right)}_{\substack{\text{Next-to-Leading} \\ \text{Logarithms (NLLogs)} \\ \text{contains} \\ \mathcal{L}_2 + \mathcal{L}_4 \text{ parameters}}} + \ldots$$

The usual renormalization group is not applicable here.

Unitarity and LLog coefficients

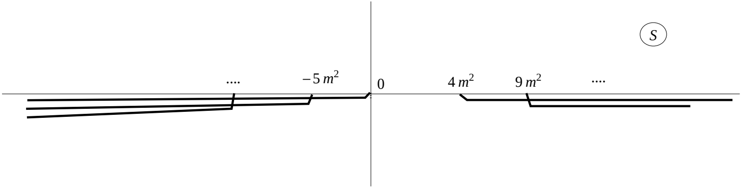
Let us consider non-linear σ-model with the symmetry group $\mathcal{G}$. Expanding the metric above the origin in normal coordinates

$$\mathcal{S}_2 = \int d^4x \; \frac{1}{2} g_{ab}(\phi) \partial_\mu \phi^a \partial_\mu \phi^b = \int d^4x \left(\frac{1}{2} \overset{o}{g}_{ab} \, \partial_\mu \phi^a \partial_\mu \phi^b - \frac{1}{6} \overset{o}{R}_{ac,bd} \, \phi^d \phi^c \partial_\mu \phi^b \partial_\mu \phi^a + \ldots \right) ,$$

g_{ab} is the group metric, $R_{ab,cd}$ is the Riemann tensor.

$$\langle \phi^d \phi^c | S | \phi^b \phi^a \rangle = I + 2\pi i (4\pi)^4 \delta(\sum_{i=1}^{4} p_i) \sum_I P_I^{abcd} \sum_l (2l+1) P_l \left(1 + \frac{2t}{s} \right) t_l^I(s),$$

P_I^{abcd} is the projector to representation I, s and t are Mandelstam variables (built from Clebsch-Gordan coefficients).



$$t_l^I = \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{s}{(4\pi F)^2} \right)^n \omega_{nl}^I \ln^{n-1} \left(\frac{\Lambda^2}{|s|} \right) + \mathcal{O}\left(s^n \ln^{n-2} \right)$$

- The 2-particle cuts result only from LLogs
- The coefficient ω_{nl}^I (the "general" LLog coefficient) can be found by summing discontinuities over left- and right- cuts.
- The right-cut discontinuity is given by the unitarity relation

$$\text{Disc } t_l^I(s) = |t_l^I(s)|^2 + \mathcal{O}\left(\text{Inelastic part} \sim \text{NLLog}\right), \qquad s > 0.$$

- The left-cut discontinuity can be found by the analytical continuation of the unitarity relation (like imaginary part of the Roy equation)

$$\text{Disc } t_l^I(s) = \sum_{l',I'} C_{su}^{II'} \frac{2(2l'+1)}{s} \int_0^{-s} ds' P_l\left(\frac{s+2s'}{-s} \right) P_{l'}\left(\frac{2s+s'}{-s'} \right) \text{Disc } t_{l'}^{I'}(s'), \quad s < 0,$$

where $C_{su}^{II'} = \frac{1}{d_I} P_I^{ab,cd} P_{I'}^{bd,ac}$ is the crossing matrix between $s \leftrightarrow u$ channels.

The LLog coefficients are given by the recursive equation [J.Koschinski,M.Polyakov, AV, 1004.2197]

$$\omega^I_{nl} = \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{l=0}^{n}\frac{1}{2}\left(\delta^{II'}\delta^{ll'} + C_{st}\Omega^{l'l}_n + C_{su}(-1)^{l+l'}\Omega^{l'l}_n\right)\frac{\omega^{I'}_{i,l'}\omega^{I'}_{n-i,l'}}{2l'+1}$$

$\Omega^{l'l}_n$ is the crossing matrix in the partial wave space at LLog approximation, and C_{st} are crossing matrices in representation space.

It is universal form of equation on LLog coefficients

$$\vec{\omega}_n = \frac{1}{2(n-1)}\sum_{i=1}^{n-1}(\mathcal{I}+\mathcal{C}_{st}+\mathcal{C}_{su})\cdot(\vec{\omega}_i\vec{\omega}_{n-i})\,.$$

One needs to calculate only the crossing matrices and the boundary conditions $\vec{\omega}_1$.

The boundary conditions are given by the tree order of the amplitude.

Example calculation: $O(N)$ σ-model

The Lagrangian of the Weinberg model:

$$\mathcal{L}_2 = \left(\delta_{ij} + \frac{\phi_i \phi_j}{F^2 - \phi^2}\right) \partial_\mu \phi^i \partial_\mu \phi^j .$$

There are 3 isospin spaces ($I = 0, 1, 2$) with projectors.

$$P_0^{abcd} = \frac{1}{N} \delta^{ab} \delta^{cd} \; , \quad P_1^{abcd} = \frac{1}{2}\left(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}\right) \; , \quad P_2^{abcd} = \frac{1}{2}\left(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right) - \frac{1}{N}\delta^{ab}\delta^{cd}$$

Boundary conditions $\{\omega_{10}^0, \omega_{11}^1, \omega_{10}^2\} \; = \; \{N-1, \; 1, \; -1\}$. Then the simple calculation gives

$$\frac{\omega_{n0}^0}{N-1} = \{1, \; \frac{N}{2} - \frac{1}{9}, \; \frac{N^2}{4} - \frac{61N}{144} + \frac{59}{144}, \; \frac{N^3}{8} - \frac{631N^2}{2700} + \frac{46279N}{194400} - \frac{13309}{194400}, \; \ldots\}$$

- The results agrees with known 1 and 2 -loop calculations
- The large-N approximation and its correction are correct.
- The equation is fast evaluated by computer, e.g. calculation of ω_{100} ($N = 3$) takes ~ 10 min.

The equation in renomalizable theory

In renormalizable theories the LLog approximation can be obtained by solving the 1-loop RG equation.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial_\mu\phi^i - \frac{\lambda_0}{4!}(\phi^2)^2$$

$$\mu^2\frac{\partial}{\partial\mu^2}\lambda = \beta(\lambda) = \frac{N+2}{8}\lambda^2 + \mathcal{O}(\lambda^2) \quad \Longrightarrow \quad A(s,t) = \lambda(\mu^2) = \frac{\lambda_0}{1-\frac{N+2}{8}\lambda_0\ln(\mu^2/s)} \qquad (1)$$

The same solution comes from the recursive equations (the β-function is the sum of crossing matrices, $\Omega_n = 1$):

$$\omega_n = \frac{1}{2(n-1)}\sum_{i=1}^{n-1}\frac{(N+2)}{4}\omega_i\omega_{n-i} \quad \Longrightarrow \quad \omega_n = \left(\frac{N+2}{8}\right)^{n-1} \quad \Longrightarrow \quad A(s,t) = (1)$$

The recursive equations are the particular form of renomalization group equations. For the EFT (i.e. non-renormolizable QFT) it takes the form:

$$\omega_{nl} = \frac{1}{n-1}\sum_{i=1}^{n-1}\beta_n^{ll'}\omega_{il'}\omega_{n-i,l'} \quad \Longrightarrow \quad \mu^2\frac{\partial}{\partial\mu^2}A(s,t) = \int_{-s}^{0} dt_1 dt_2 A(s,t_1)A(s,t_2)K(s,t;t_1,t_2)$$

RG equations in EFTs

In EFTs one has infinite number of coupling constants and β-functions [Buchler,Colangelo,0309049]

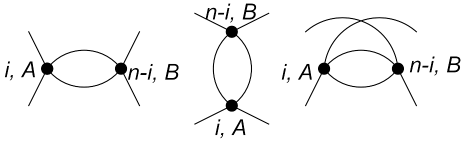
$\mu^2 \frac{\partial}{\partial \mu^2} g_1 = 0,$ the lowest order constant does not run

$\mu^2 \frac{\partial}{\partial \mu^2} g_{2C} = \beta(1,1,C) g_1 g_1,$ the second order constant does run through g_1

...

$\mu^2 \frac{\partial}{\partial \mu^2} g_{nC} = \sum_{i=1}^{n-1} \beta(i,A;n-i,B/C) g_{iA} g_{n-i,B} + \mathcal{O}(g^3),$ the equation for 1-loop running of g_{nA}.

The β function is the pole-part of diagrams:



$$\mu^2 \frac{d}{d\mu^2} A(s,t,\mu^2,g) = \left(\mu^2 \frac{\partial}{\partial \mu^2} + \sum_{n=1}^{\infty} \sum_A \beta(g_{nA}) \frac{\partial}{\partial g_{nA}} \right) A(s,t,\mu^2,g)$$

$$\mu^2 \frac{d}{d\mu^2} A(s,t,\mu^2,g) = \left(\mu^2 \frac{\partial}{\partial \mu^2} + \sum_{n=1}^{\infty} \sum_{A} \beta(g_{nA}) \frac{\partial}{\partial g_{nA}} \right) A(s,t,\mu^2,g)$$

$$A(s,t,\mu_0^2,g_0) = \sum_{n,A} g_{0,nA} V_{nA}(s,t) \qquad \text{– boundary condition}$$

The key point that one has the finite number of β-function at given order. Then the system can be reformulated in terms of recursive equation:

$$A(s,t,\mu^2,g) = \sum_{n,A} \omega_{nA} V_{nA}(s,t) g_1^n \ln^{n-1}\left(\frac{\mu^2}{\mu_0^2}\right)$$

$$\omega_{nC} = \frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{AB} \beta(i,A;n-i,B/C) \omega_{i,A} \omega_{n-i,B}$$

[Kivel,Polyakov,AV,0809.3236][Kivel,Polyakov,AV,0904.3008]

Such analysis can be applied only for massless fields.

The presence of mass breaks the hierarchy of equations and the system can not be solved (the tadpole diagrams generates the infinite number of additional RG-functions, see [Bijnens,Carloni,1008.3499]). The presence of mass also breaks the unitarity derivation, since $\ln(m^2)$ are "invisible" in complex s-plane.

1-loop β-functions for σ-model

In $D=2$ the running of the σ-model is given by Friedan (Ricci-flow) equation:

$$\mu^2 \frac{\partial}{\partial \mu^2} g_{ij}(\phi) = \frac{1}{8\pi} R_{ij} + \ldots \qquad (D=2)$$

In higher dimensions the base manifold metric does not run. But the higher order tensors run:

$$\mathcal{S} = \int d^4x \ \ \frac{1}{2} g_{ij}(\phi)\partial_\mu \phi^i \partial_\mu \phi^j + h_{ij}\partial^2\phi^i\partial^2\phi^j + T^{(1)}_{ijkl}(\phi)\partial_\mu\phi^i\partial_\mu\phi^j\partial_\nu\phi^k\partial_\nu\phi^l + ..$$

$$= \int d^4x \ \ \frac{1}{2} g_{ij}(\phi)\partial_\mu \phi^i \partial_\mu \phi^j + \sum_{n,l} T^{(nl)}_{ijkl}(\phi)(\phi^i \nabla_{\mu_1}..\nabla_{\mu_l}\phi^j)\partial^{2(n-l)}(\phi^k\nabla_{\mu_1}..\nabla_{\mu_l}\phi^l) + T_{ijklmn} \ldots.$$

Their 1-loop running is given by the recursive equation in the form:

$$\mu^2 \frac{\partial}{\partial \mu^2} T^{(nl)}_{abcd} \simeq (\omega_{nl})_{abcd} + \ldots$$

$$(\omega_{nl})_{abcd} = \frac{1}{2(n-1)} \Bigg[\sum_{i=1}^{n-1} \frac{(\omega_{il})_{ab\alpha\beta}\,(\omega_{n-i,l})^{\beta\alpha}{}_{cd}}{2l+1} + \sum_{i=1}^{n-1}\sum_{l'=0}^{n} \frac{(\omega_{il'})_{ad\alpha\beta}\,(\omega_{n-i,l'})^{\beta\alpha}{}_{cb}\,\Omega_n^{l'l} + (\omega_{il'})_{ac\alpha\beta}\,(\omega_{n-i,l'})^{\beta\alpha}{}_{bd}\,(-1)^{l+l'}\Omega_n^{l'l}}{2l'+1} \Bigg].$$

Boundary conditions:

$$(\omega_{10})_{abcd} = -\frac{1}{2}(R_{acbd} + R_{adbc}), \quad (\omega_{11})_{abcd} = \frac{1}{2}R_{abcd}.$$

The $n=2$ evolution is well-known (e.g. [Percacci,Zanusso,0910.0851])

$$\mu^2 \frac{\partial T^{(21)}_{abcd}}{\partial \mu^2} = \frac{1}{(4\pi)^2}\left(\frac{1}{2}\left(R_{a\beta_1 c\beta_2}R_b{}^{\beta_1}{}_d{}^{\beta_2} + R_{a\beta_1 c\beta_2}R_b{}^{\beta_2}{}_d{}^{\beta_1}\right) + \frac{1}{12}\left(R_{ad\beta_1\beta_2}R_{bc}{}^{\beta_1\beta_2} - R_{ab\beta_1\beta_2}R_{cd}{}^{\beta_1\beta_2}\right)\right)$$

$n=3$ evolution is proportional to R^3, etc., for details see [Polyakov,AV,1012.4205].

Conclusions

◯ The method to find the leading infrared logarithm behavior for the Goldstone particle system is found.

- Easily generalized to higher D (even)
- Can be applied for different types of amplitudes, see e.g. form-factors [Kivel,Polyakov,AV,0904.3008]
- The recursive equations can be reformulated in geometrical, "Friedan"-like form.
- This methods are useful for calculations of leading behaviour of non-local matrix elements (e.g. parton distribution in effective field theory approach). The large-non-locality effectively cancels the powers of energy expansion (the resulting low energy expansion is similar to usual renormalizable perturbative expansion).