Flat bands in nodal topological matter

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1. Gapless & gapped topological media

2. Fermi surface as topological object

3. Fermi points (Weyl, Majorana & Dirac points) & nodal lines

- superfluid 3He-A, topological semimetals , cuprate superconductors , graphene
  vacuum of Standard Model of particle physics in massless phase
- QED, QCD and gravity as emergent phenomena; quantum vacuum as 4D graphene
- exotic fermions: quadratic, cubic & quartic dispersion; dispersionless fermions

4. Flat bands on surface of topological matter

- superfluid 3He-A, semimetals , cuprate superconductors , graphene , graphite
- towards room-temperature superconductivity

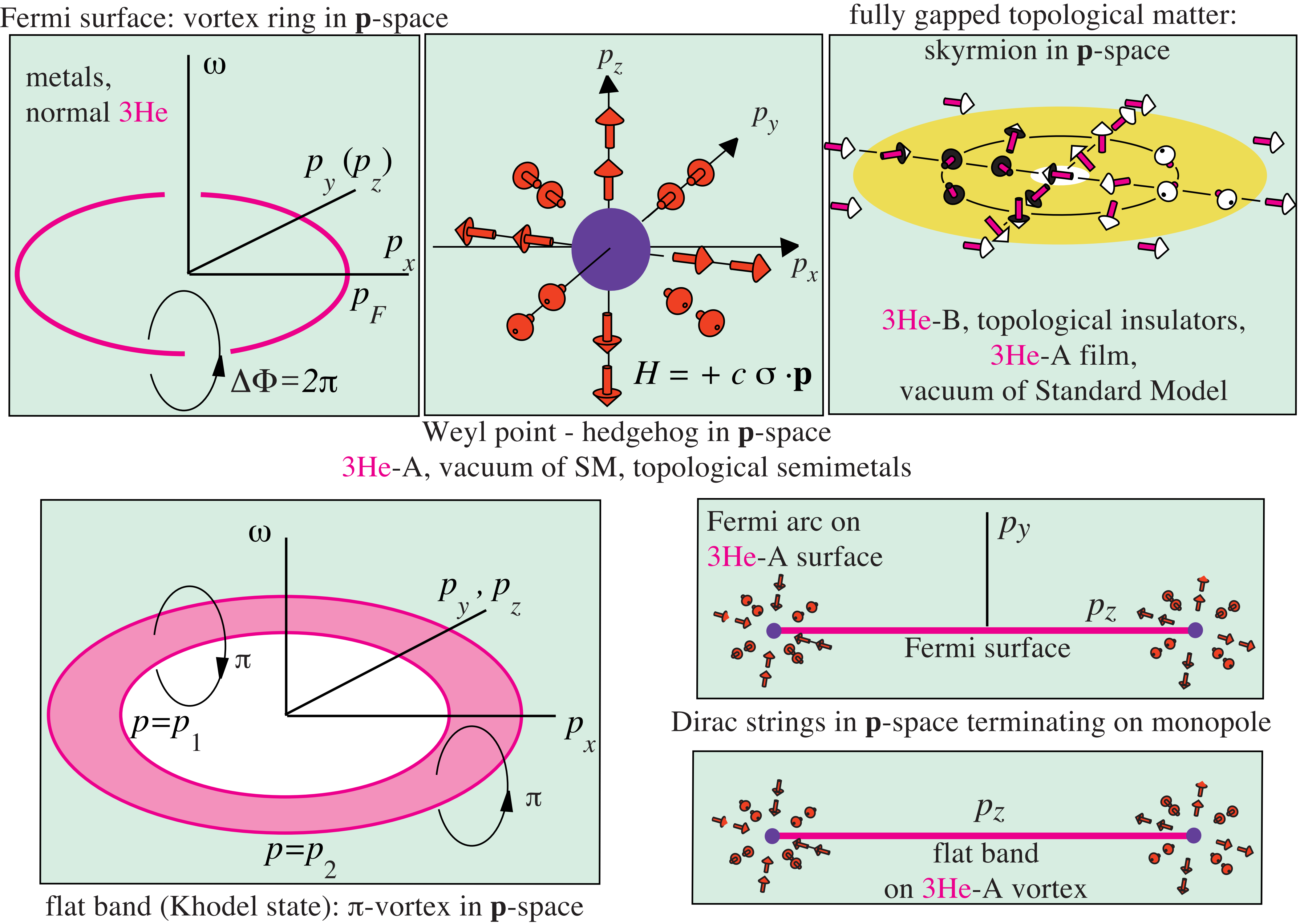
5. 1D flat band in the vortex core and Fermi-arc on the surface
of topological matter with Weyl points

Heikkilä, Kopnin, GV
***arXiv:*1012.0905, 1011.4185, 1011.4665, 1103.2033**

6. Supplemented material: Fully gapped topological media

- superfluid 3He-B, topological insulators , chiral superconductors,
  vacuum of Standard Model of particle physics in present massive phase
- edge states & Majorana fermions (planar phase , topological insulator & 3He-B)

classes of topological matter as momentum-space objects



topological correspondence:

topology in bulk protects gapless fermions on the surface or in vortex core

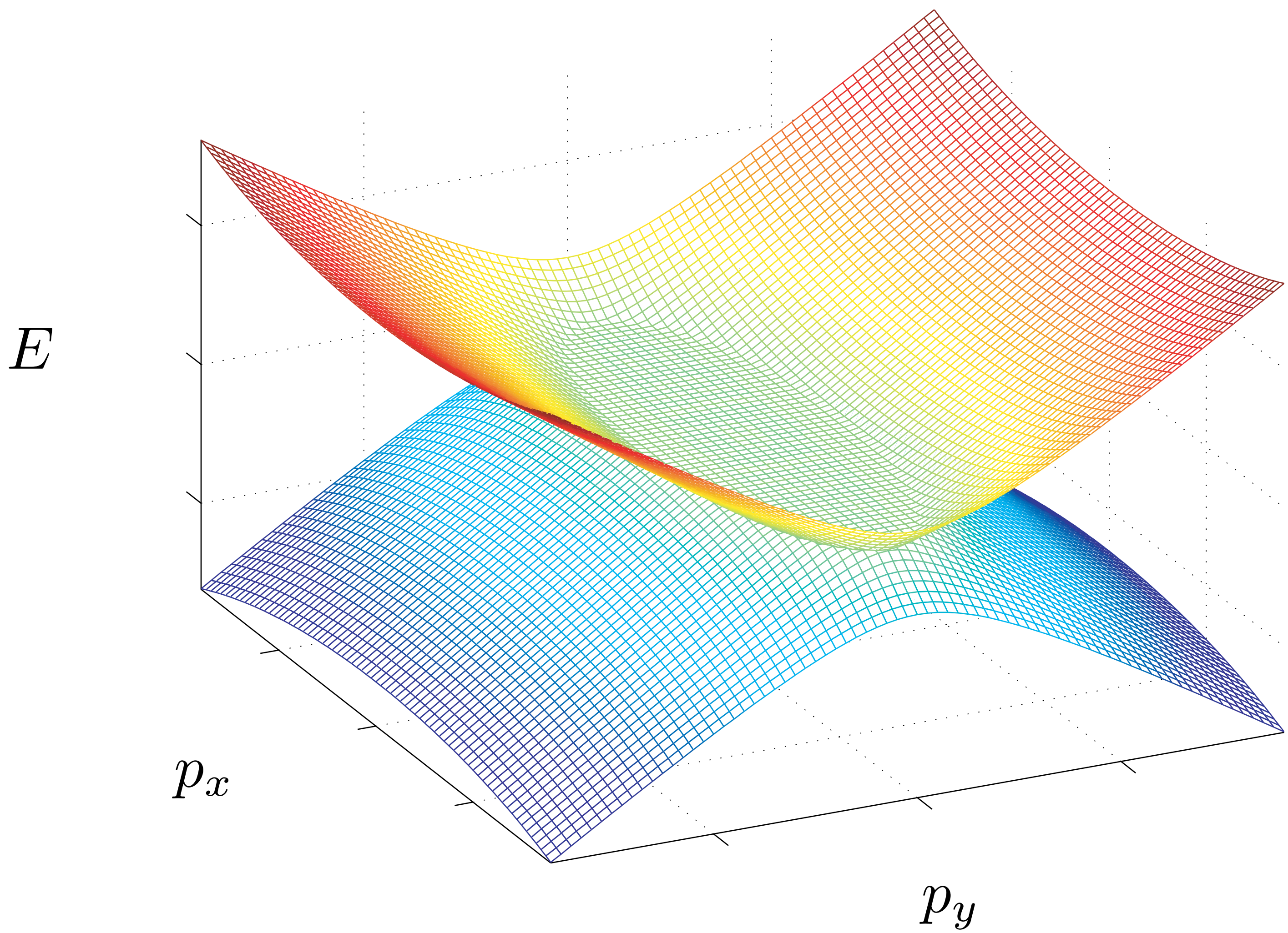
bulk-surface correspondence:

2D Quantum Hall insulator & 3He-A film	chiral edge states
3D topological insulator	Dirac fermions on surface
superfluid 3He-B	Majorana fermions on surface
superfluid 3He-A, Weyl point semimetal	Fermi arc on surface
graphene	dispersionless 1D flat band on surface
semimetal with Fermi lines	2D flat band on the surface

bulk-vortex correspondence:

superfluid 3He-A	1D flat band of zero modes in the core

New topological object in momentum space: flat band with zero energy



Flat band on the surface of topological matter with nodal lines

2. Effective theory of vacuum with Fermi surface

two major universality classes of gapless fermionic vacua

Landau theory of Fermi liquid

vacuum with Fermi surface:
normal ^{3}He

Standard Model + gravity

vacuum with Fermi point:
^{3}He-A, planar phase

gravity emerges from
Fermi point
analog of
Fermi surface

$$g^{\mu\nu}(p_\mu - eA_\mu - e\tau \cdot \mathbf{W}_\mu)(p_\nu - eA_\nu - e\tau \cdot \mathbf{W}_\nu) = 0$$

Theory of topological matter:
Nielsen, Ishikawa, Haldane, Yakovenko, Horava, Kitaev, Ludwig, Schnyder, Ryu, Furusaki,
S-C Zhang, Kane, Liang Fu, ...

Topological stability of Fermi surface

Energy spectrum of
non-interacting gas of fermionic atoms

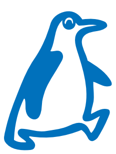
$$\varepsilon(p) = \frac{p^2}{2m} - \mu = \frac{p^2}{2m} - \frac{p_F^2}{2m}$$

$\varepsilon > 0$
empty levels

$\varepsilon < 0$
occupied levels:
Fermi sea

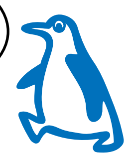
Fermi surface
$\varepsilon = 0$

$p = p_F$



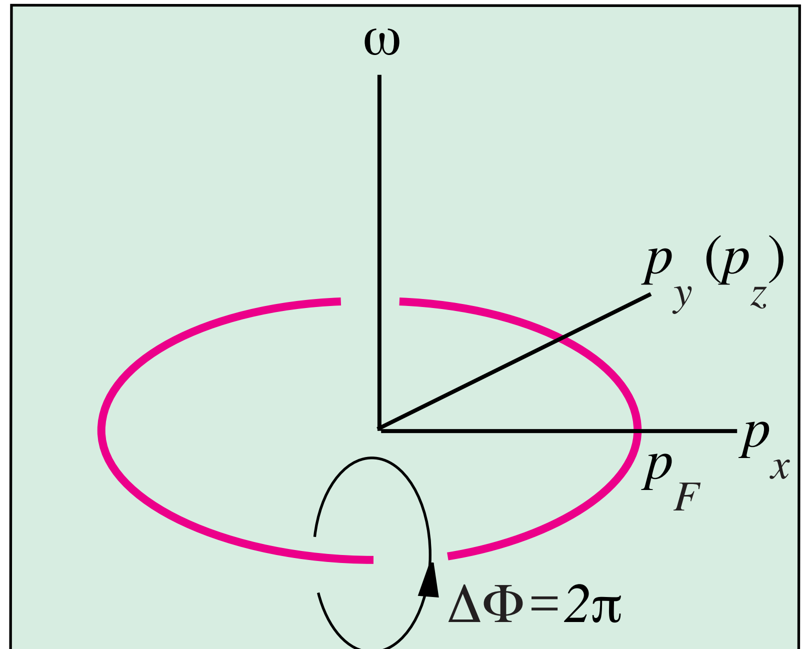
is Fermi surface a domain wall in momentum space?

no!
it is a **vortex ring**



Green's function

$$G^{-1} = i\omega - \varepsilon(p)$$



Fermi surface:
vortex ring in **p**-space

phase of Green's function

$$G(\omega,\mathbf{p}) = |G| e^{i\Phi}$$

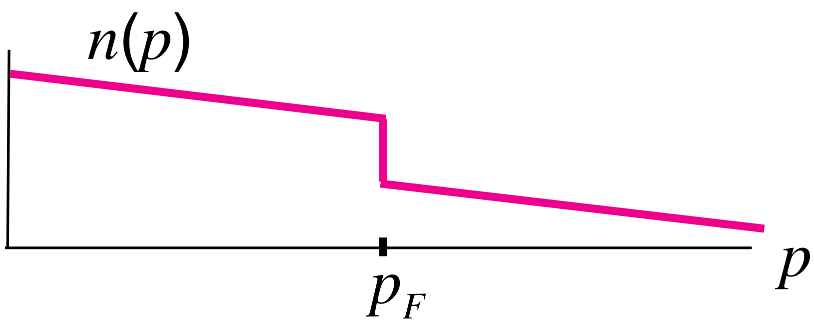
has winding number $N = 1$

Migdal jump & p-space topology

* Singularity at Fermi surface is robust to perturbations:

winding number $N=1$ *cannot change continuously,*
interaction (perturbative) cannot destroy singularity

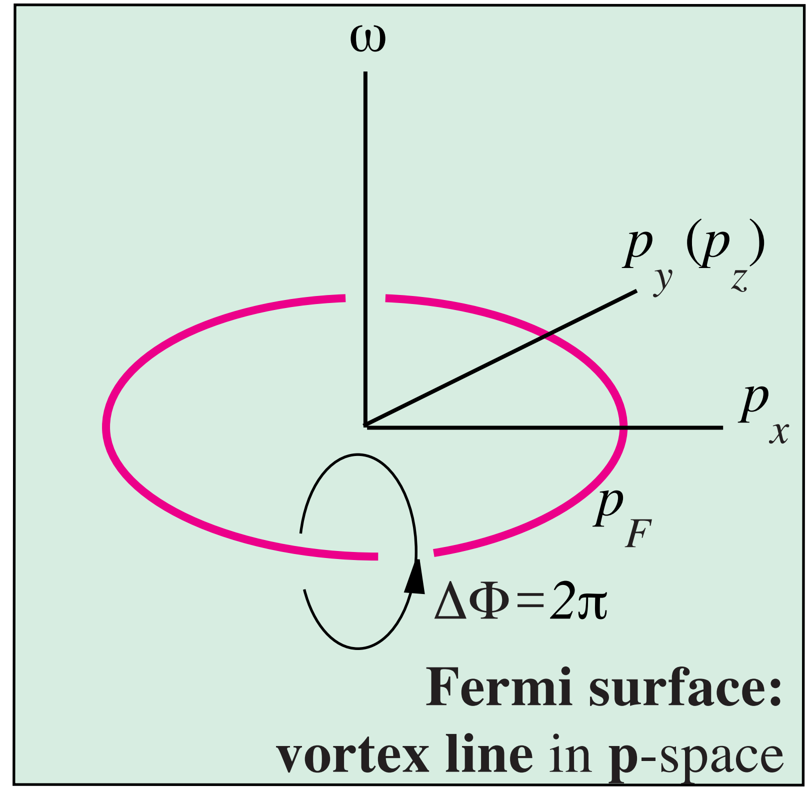
* Typical singularity: Migdal jump



* Other types of singularity: Luttinger Fermi liquid, marginal Fermi liquid, pseudo-gap ...

$$G(\omega,\mathbf{p}) = \frac{Z(p,\omega)}{i\omega - \varepsilon(p)}$$

$$Z(p,\omega) = (\omega^2 + \varepsilon^2(p))^\gamma$$



$$G(\omega,\mathbf{p}) = |G|e^{\,i\Phi}$$

* Zeroes in Green's function instead of poles (for $\gamma > 1/2$) have the same winding number $N=1$

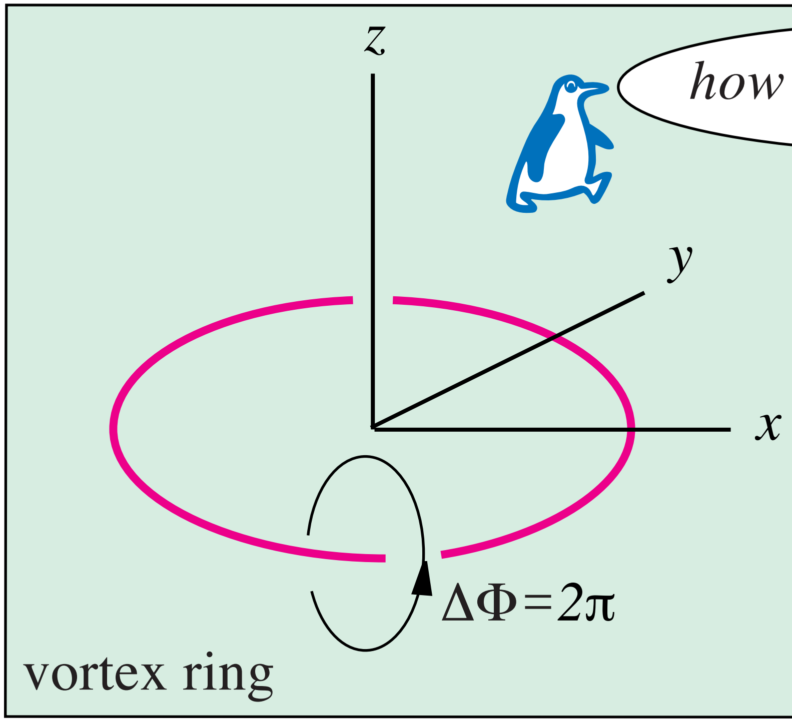
quantized vortex in **r**-space $\equiv$ **Fermi surface** in **p**-space

$\boxed{\text{homotopy group } \pi_1}$

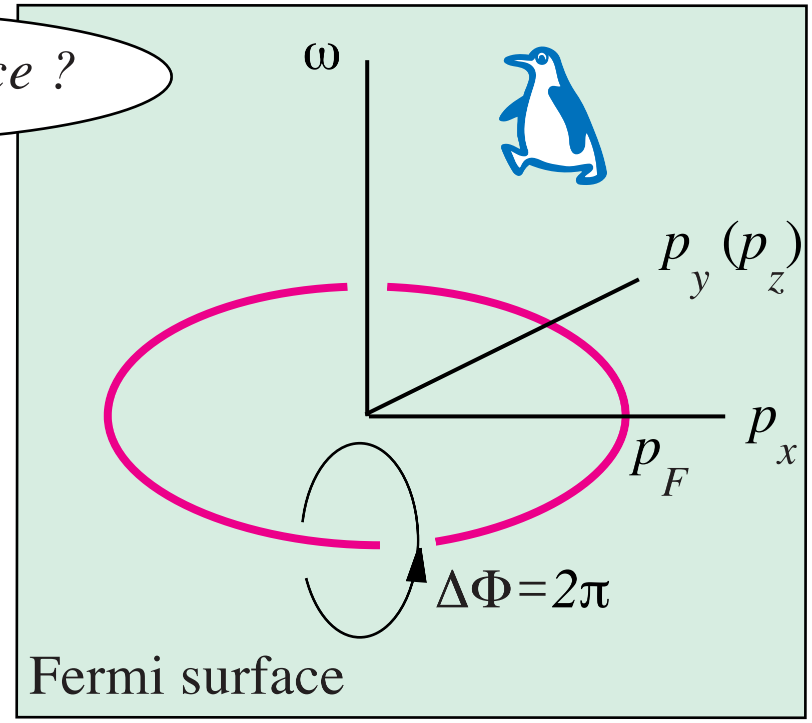
Topology in **r**-space

Topology in **p**-space

*how is it in **p**-space ?*



winding number $N_1 = 1$



$\Psi(\mathbf{r})=|\Psi|\, e^{i\Phi}$

scalar order parameter
of superfluid & superconductor

$G(\omega,\mathbf{p})=|G| e^{i\Phi}$

Green's function (propagator)

classes of mapping $S^1 \rightarrow U(1)$
manifold of
broken symmetry vacuum states

classes of mapping $S^1 \rightarrow GL(\mathrm{n,C})$
space of
non-degenerate complex matrices

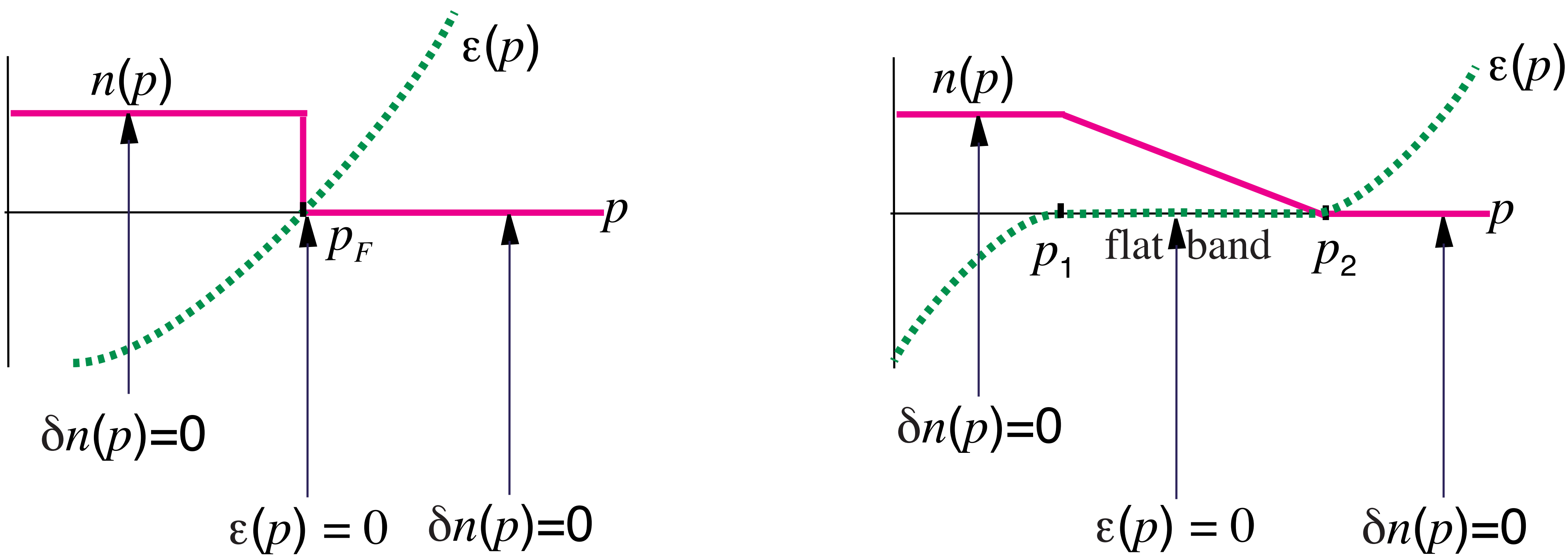
non-topological flat bands due to interaction

Khodel-Shaginyan fermion condensate

JETP Lett. **51**, 553 (1990)
GV, JETP Lett. **53**, 222 (1991)
Nozieres, J. Phys. (Fr.) **2**, 443 (1992)

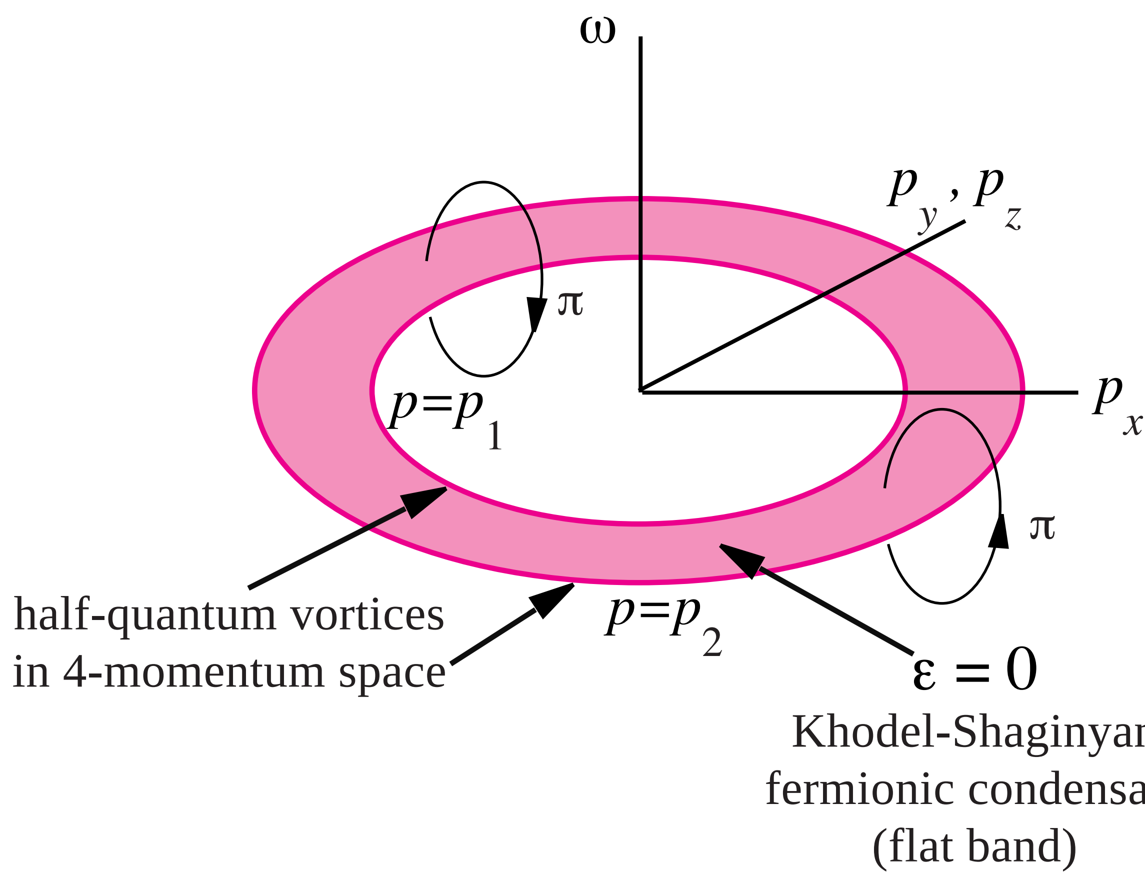
$$E\{n(p)\} \qquad \delta E\{n(p)\} = \int \varepsilon(p)\,\delta n(p)\, d^{\mathrm{d}}p = 0$$

solutions: $\varepsilon(p) = 0$ or $\delta n(p)=0$



splitting of Fermi surface to flat band

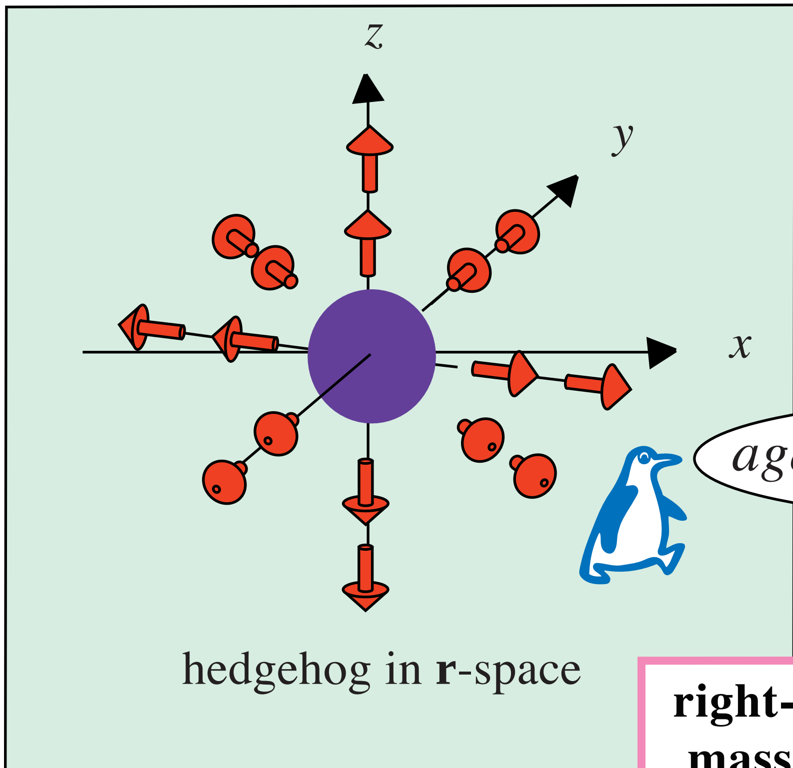
Flat band as momentum-space dark soliton terminated by half-quantum vortices



phase of Green's function changes by π across the "dark soliton"

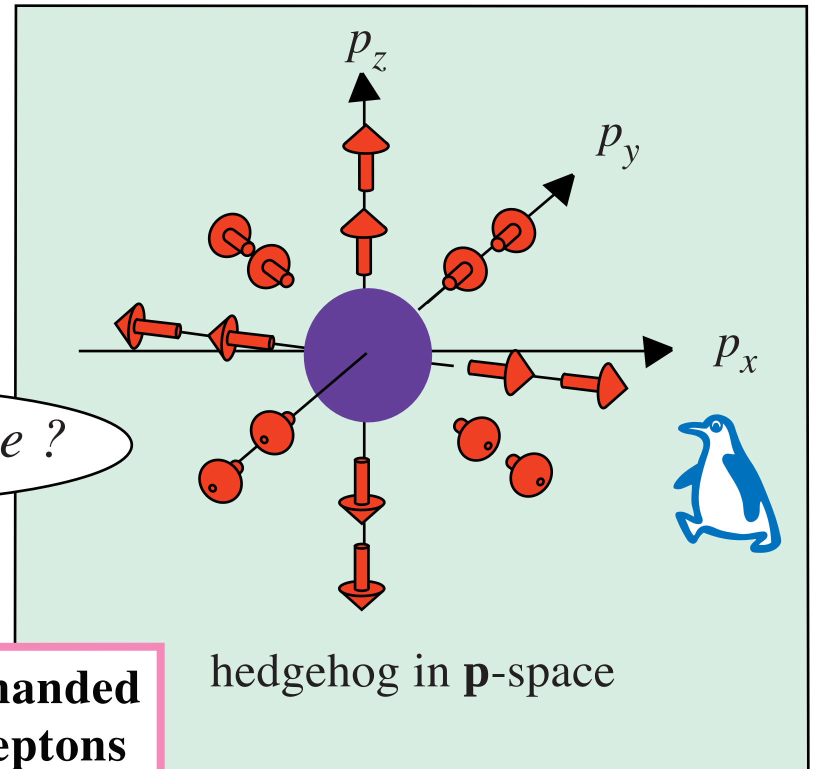
3. Classes of Fermi points & nodal lines:
superfluid ^{3}He-A, Standard Model, semimetals, graphene, cuprate SC, ...
surface of ^{3}He-B & topological insulators

magnetic hedgehog vs Weyl point



$\sigma(\mathbf{r}) = \hat{\mathbf{r}}$

again no difference ?



$\sigma(\mathbf{p}) = \hat{\mathbf{p}}$

right-handed and left-handed massless quarks and leptons are elementary particles in Standard Model

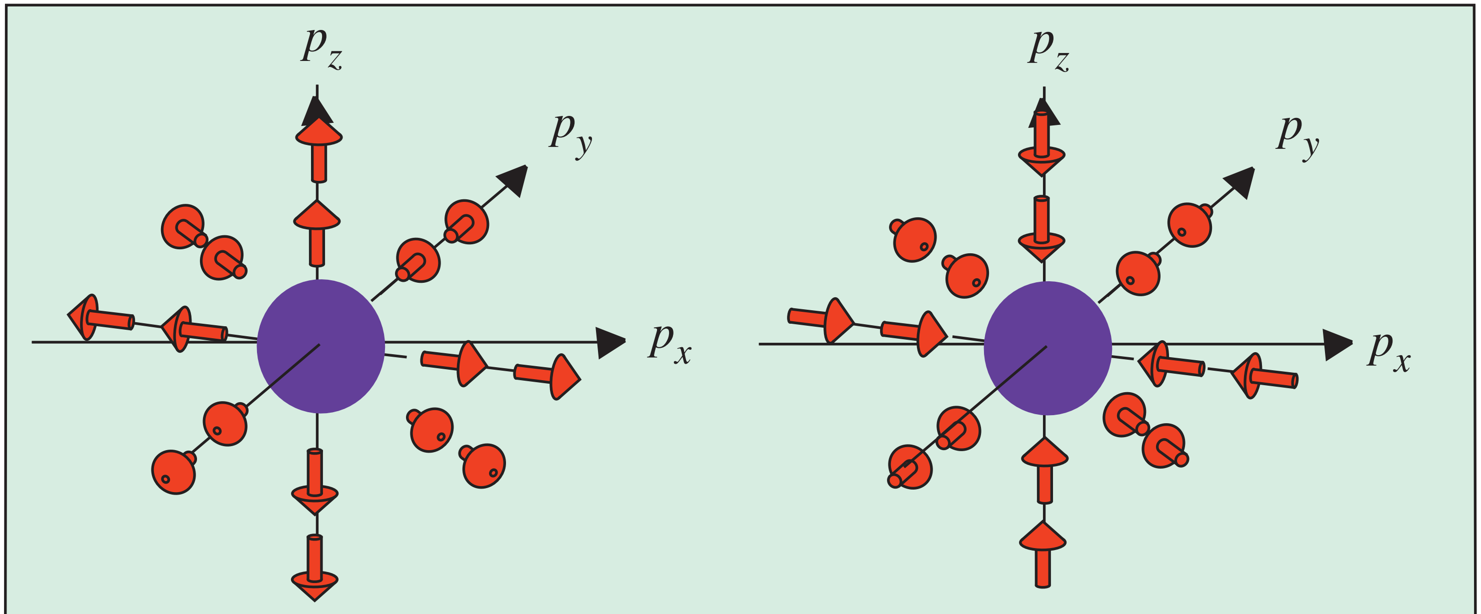
Landau CP symmetry is emergent

close to Fermi point

$$H = + c\, \sigma \cdot \mathbf{p}$$

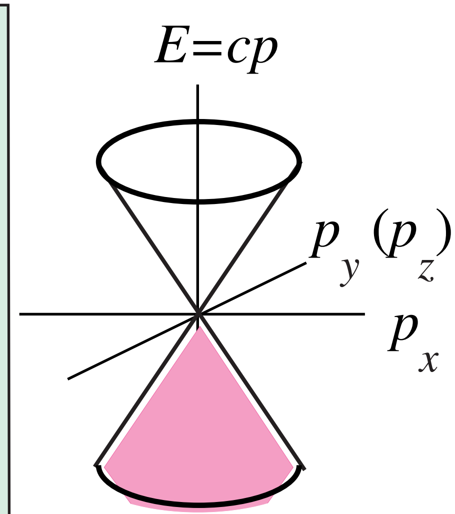
right-handed Weyl electron =
hedgehog in **p**-space with spines = spins

Topological invariant for right-handed and left-handed elementary particles



hedgehog with spines (spins) outward ($N_3 = +1$)

hedgehog with spines (spins) inward ($N_3 = -1$)



right neutrino

$$H = +c\,\sigma\cdot\mathbf{p}$$
$$\mathbf{g}(\mathbf{p}) = +c\mathbf{p}$$

$$\boxed{H = \sigma\cdot\mathbf{g}(\mathbf{p})}$$

$$H = -c\,\sigma\cdot\mathbf{p}$$
$$\mathbf{g}(\mathbf{p}) = -c\mathbf{p}$$

left neutrino

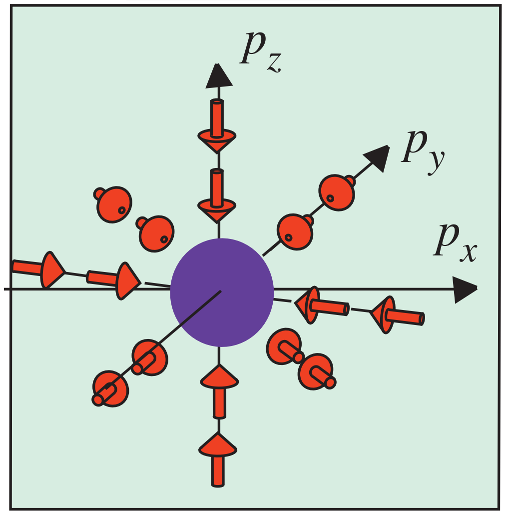
$$N_3 = \frac{1}{8\pi}\, e_{ijk} \int dS^i\, \hat{\mathbf{g}}\cdot(\partial^j \hat{\mathbf{g}} \times \partial^k \hat{\mathbf{g}})$$

over 2D surface around Fermi point

Chiral Weyl fermions in Standard Model

Family #1 of quarks and leptons

left particles



hedgehog with spines (spins) inward ($N_3 = -1$)

quarks

+2/3 u_L +1/6	-1/3 d_L +1/6
+2/3 u_L +1/6	-1/3 d_L +1/6
+2/3 u_L +1/6	-1/3 d_L +1/6

$SU(3)_C$

$SU(2)_L$

leptons

0 ν_L -1/2	-1 e_L -1/2

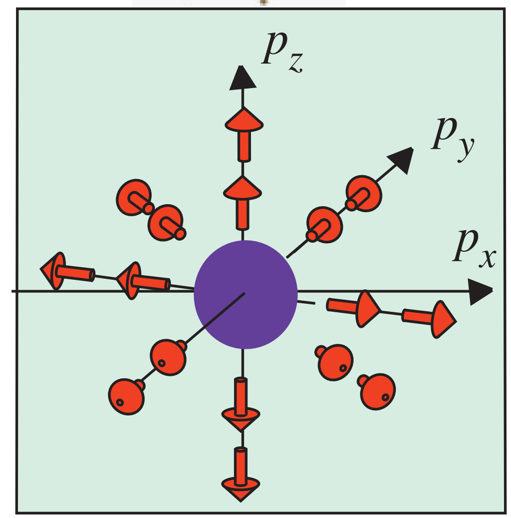
$$H = -c\,\sigma\cdot\mathbf{p}$$
$$N_3 = -1$$

+2/3 u_R +2/3	-1/3 d_R -1/3
+2/3 u_R +2/3	-1/3 d_R -1/3
+2/3 u_R +2/3	-1/3 d_R -1/3

0 ν_R 0	-1 e_R -1

$$H = +c\,\sigma\cdot\mathbf{p}$$
$$N_3 = +1$$

right particles



hedgehog with spines (spins) outward ($N_3 = +1$)

$$N_3 = \frac{1}{24\pi^2}\, e_{\mu\nu\lambda\gamma}\, \mathbf{tr} \int dS^\gamma\, \mathbf{G}^\mu\, \mathbf{G}^{-1}\, \mathbf{G}^\nu\, \mathbf{G}^{-1} \mathbf{G}^\lambda\, \mathbf{G}^{-1}$$

over 3D surface S in 4D momentum space

general topological invariant
in terms of Green's function
for interacting systems

Standard Model topological invariant

Topological invariant protected by symmetry

$$N_K = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \, \mathbf{tr} \int_{\text{over } S^3} dV \, \mathbf{K} \, \mathbf{G} \, \partial^\mu \mathbf{G}^{-1} \, \mathbf{G} \, \partial^\nu \mathbf{G}^{-1} \mathbf{G} \, \partial^\lambda \mathbf{G}^{-1}$$

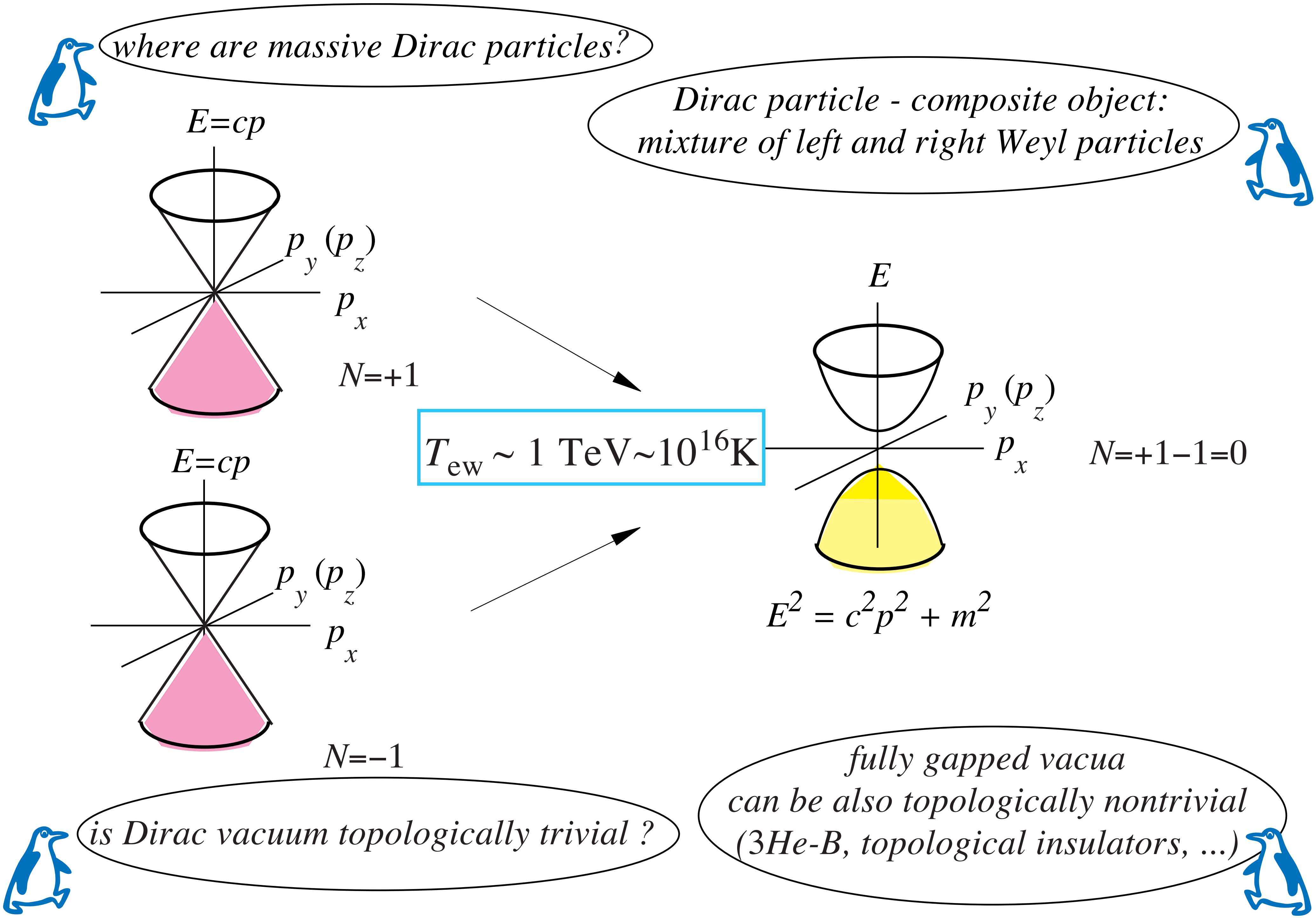
G is Green's function, K is symmetry operator **GK =+/− KG**

for Standard Model vacuum $K = \exp 2\pi i \tau_3$
weak isotopic spin

$$N_K = 16\, n_g$$

16 massless Weyl particles in one generation are protected by combined symmetry and topology

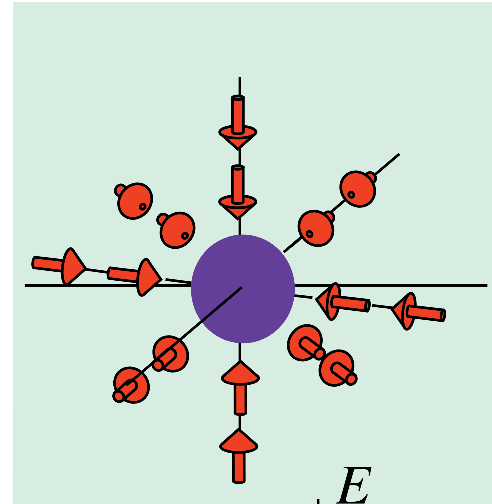
From massless Weyl particles to massive Dirac particles



Weyl fermions in 3+1 gapless topological cond-mat

topologically protected Weyl points in:

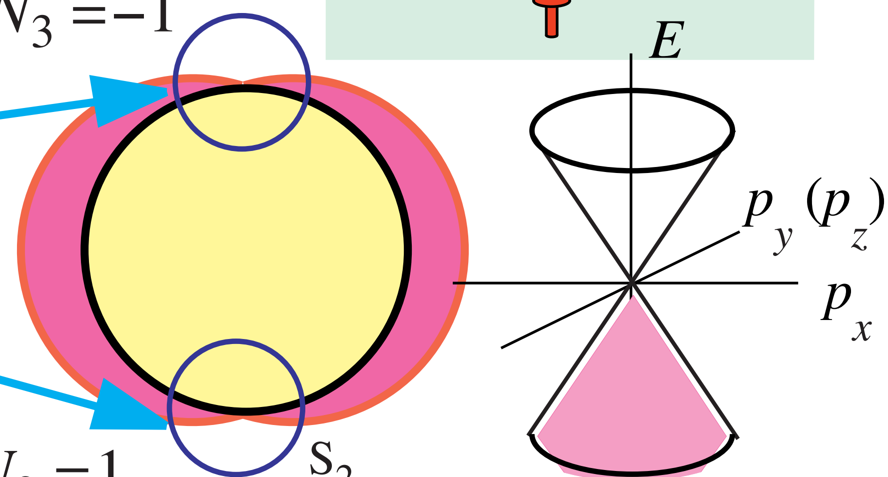
topological semi-metal (Abrikosov-Beneslavskii 1971),
^{3}He-A (1982), triplet Fermi gases



Gap node - Weyl point
(anti-hedgehog)

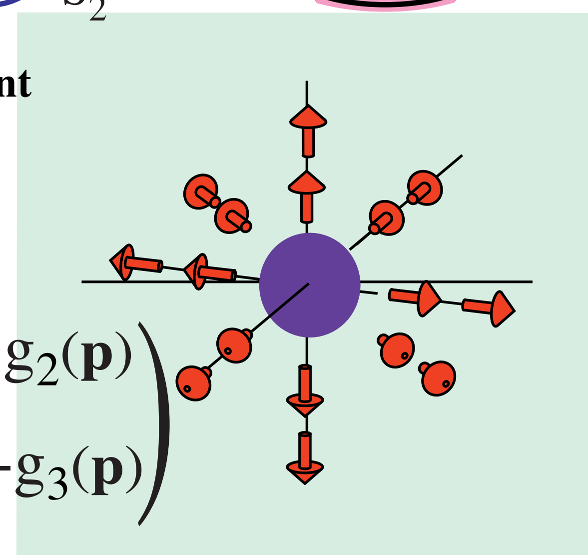
$$N_3 = \frac{1}{8\pi} e_{ijk} \int dS^k \; \hat{\mathbf{g}} \cdot (\boldsymbol{\partial}_{p_i} \hat{\mathbf{g}} \times \boldsymbol{\partial}_{p_j} \hat{\mathbf{g}})$$

over 2D surface S
in 3D p-space



$$p^2 = p_x{}^2 + p_y{}^2 + p_z{}^2$$

Gap node - Weyl point
(hedgehog)



$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix}$$

emergence of relativistic QFT near Fermi (Dirac) point

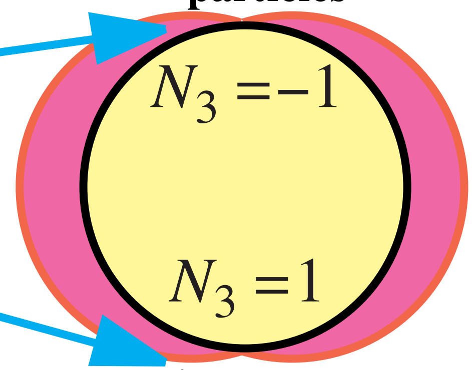
original non-relativistic Hamiltonian

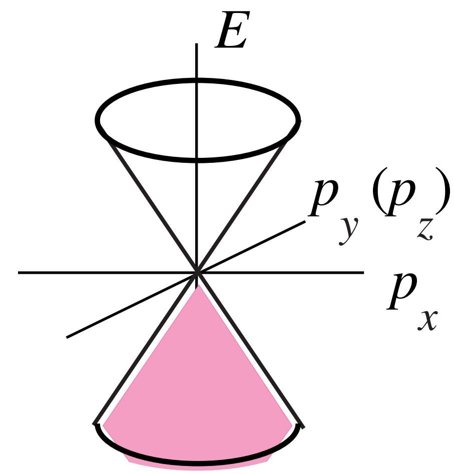
$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \boldsymbol{\tau} \cdot \mathbf{g}(\mathbf{p})$$

close to nodes, i.e. in low-energy corner
relativistic chiral fermions emerge

$$H = N_3\, c\, \boldsymbol{\tau} \cdot \mathbf{p}$$

$$E = \pm\, cp$$





chirality is emergent ??

top. invariant determines chirality in low-energy corner

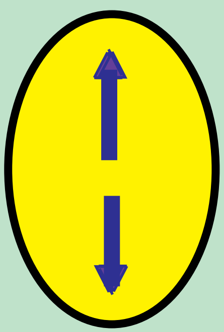
what else is emergent ?

relativistic invariance as well

bosonic collective modes in two generic fermionic vacua

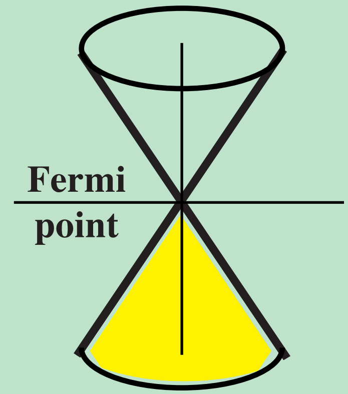
Landau theory of Fermi liquid

collective Bose modes of fermionic vacuum:
propagating
oscillation of **shape**
of Fermi surface



Landau, ZhETF **32**, 59 (1957)

Standard Model + gravity

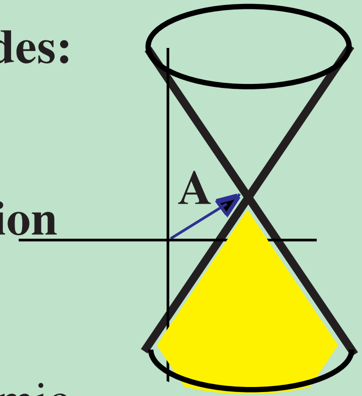


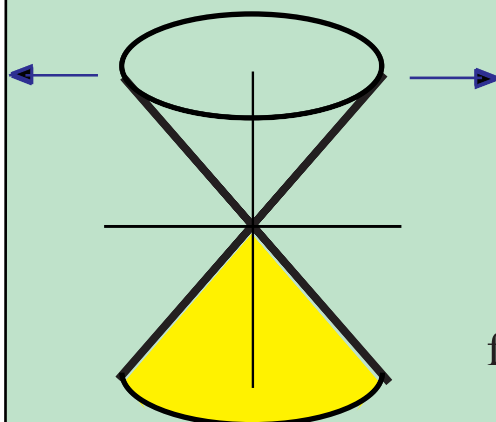
collective Bose modes:

propagating
oscillation of **position**
of Fermi point

$$\mathbf{p} \rightarrow \mathbf{p} - e\mathbf{A}$$

form effective dynamic
electromagnetic field

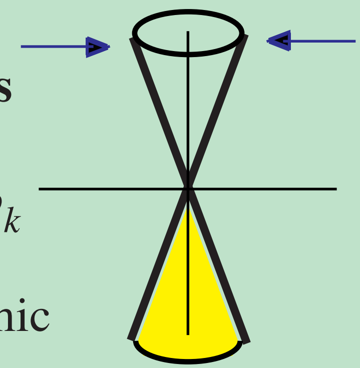




propagating
oscillation of **slopes**

$$E^2 = c^2p^2 \rightarrow g^{ik}p_i p_k$$

form effective dynamic
gravity field



two generic quantum field theories of interacting bosonic & fermionic fields

relativistic quantum fields & gravity emerging near Weyl point

Atiyah-Bott-Shapiro construction:
linear expansion of Hamiltonian near the nodes in terms of Dirac Γ-matrices

$$\boxed{E = v_F\,(p - p_F)}$$

emergent relativity

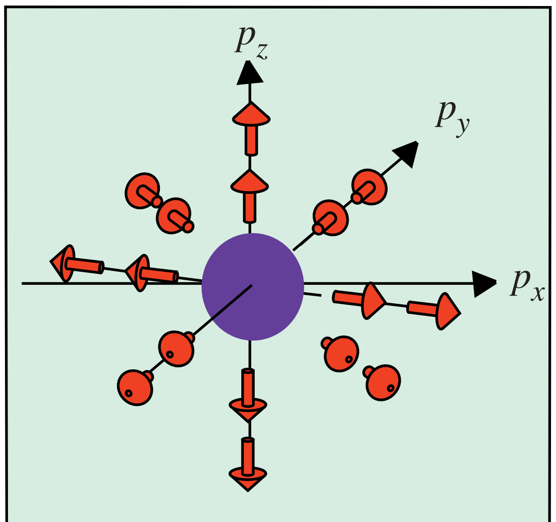
$$\boxed{H = e_a{}^k\,\Gamma^a \cdot (p_k - p_k^0)}$$

linear expansion near Fermi surface

linear expansion near Weyl point

primary object: tetrad $e_a{}^\mu$

secondary object: metric $g^{\mu\nu} = \eta^{ab}\, e_a{}^\mu\, e_b{}^\nu$



hedgehog in **p**-space

$$\boxed{g^{\mu\nu}(p_\mu - eA_\mu - e\tau \cdot \mathbf{W}_\mu)(p_\nu - eA_\nu - e\tau \cdot \mathbf{W}_\nu) = 0}$$

effective metric: emergent gravity

effective electromagnetic field

effective $SU(2)$ gauge field

effective electric charge $e = +1$ or -1

effective isotopic spin

all ingredients of Standard Model :
chiral fermions & gauge fields
emerge in low-energy corner
together with spin, Dirac Γ−matrices, gravity & physical laws:
Lorentz & gauge invariance, equivalence principle, etc

gravity & gauge fields are collective modes of vacua with Weyl point

crossover from Landau 2-fluid hydrodynamics to Einstein general relativity

they represent two different limits of hydrodynamic type equations

equations for $g^{\mu\nu}$ depend on hierarchy of ultraviolet cut-off's:
Planck energy scale E_{Planck} vs Lorentz violating scale E_{Lorentz}

$E_{\mathrm{Planck}} >> E_{\mathrm{Lorentz}}$

emergent Landau two-fluid hydrodynamics

$E_{\mathrm{Planck}} << E_{\mathrm{Lorentz}}$

emergent general covariance & general relativity

^{3}He-A with Fermi point

$E_{\mathrm{Lorentz}} << E_{\mathrm{Planck}}$

$E_{\mathrm{Lorentz}} \sim 10^{-3} E_{\mathrm{Planck}}$

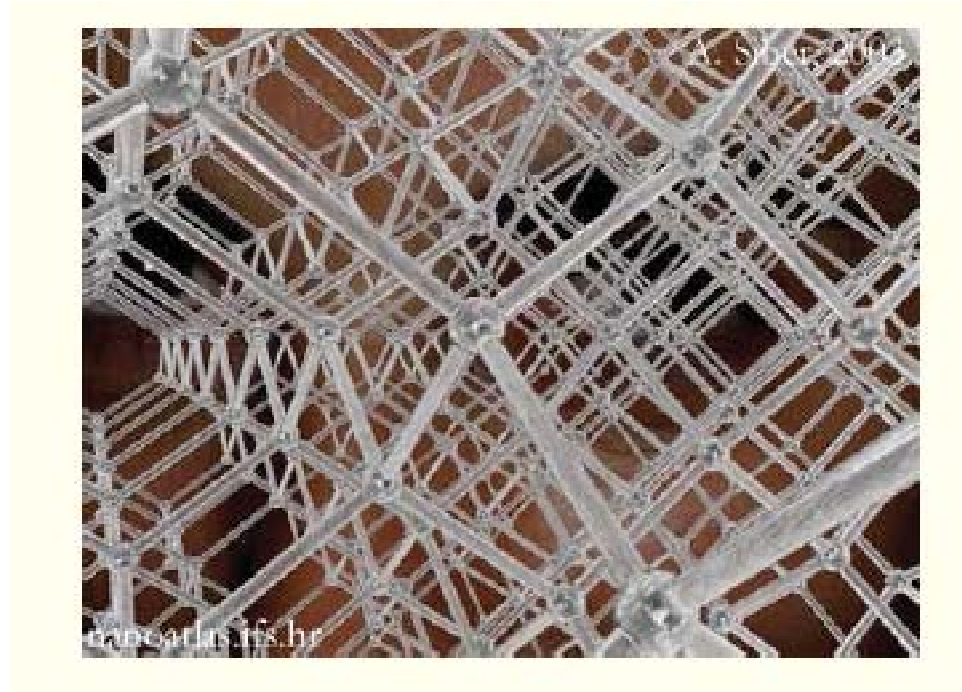
Universe

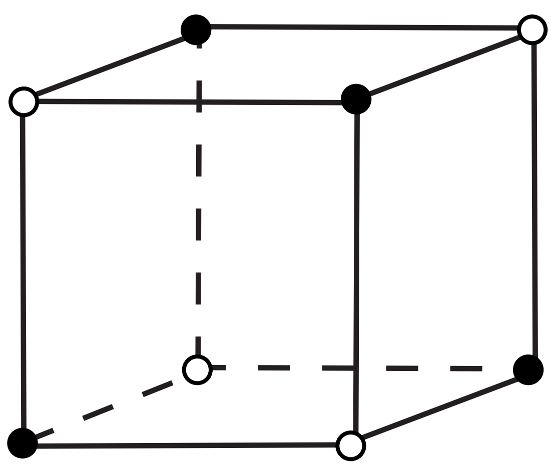
$E_{\mathrm{Lorentz}} >> E_{\mathrm{Planck}}$

$E_{\mathrm{Lorentz}} > 10^{9} E_{\mathrm{Planck}}$

quantum vacuum as crystal

4D graphene
Michael Creutz JHEP 04 (2008) 017





● Fermi (Dirac) points with $N_3 = +1$

○ Fermi (Dirac) points with $N_3 = -1$

topology of graphene nodes

$$\boxed{N = \frac{1}{4\pi i}\,\mathbf{tr}\,[\mathbf{K}\oint dl\,\mathbf{H}^{-1}\,\boldsymbol{\partial}_l\,\mathbf{H}]}$$

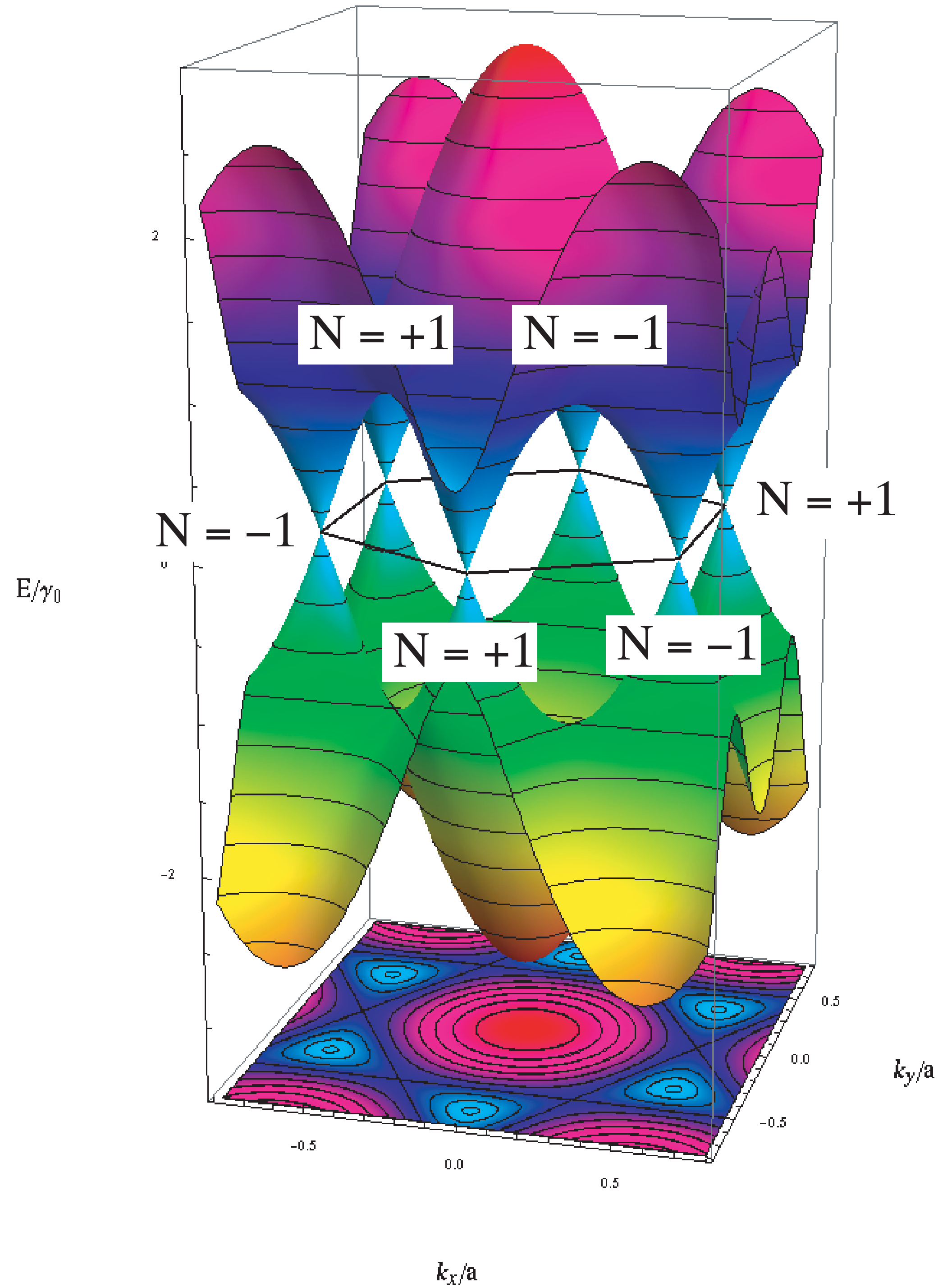
K - symmetry operator,
commuting or anti-commuting with **H**

close to nodes:

$$\mathbf{H}_{\mathrm{N}\,=\,+1} = \tau_x p_x + \tau_y p_y$$

$$\mathbf{H}_{\mathrm{N}\,=\,-1} = \tau_x p_x - \tau_y p_y$$

$$\mathbf{K} = \tau_z$$

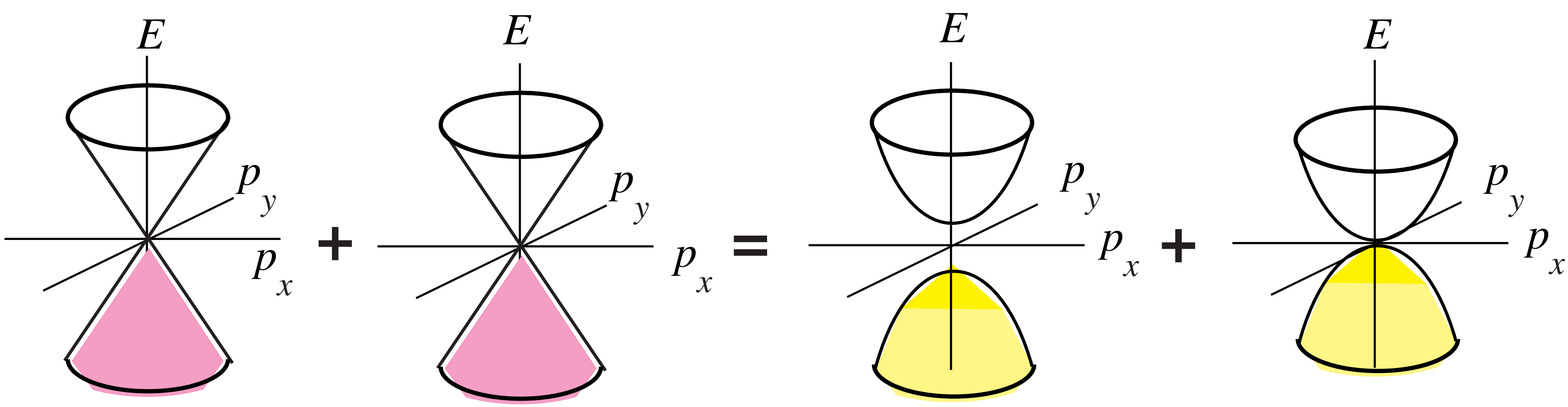


exotic fermions:
massless fermions with quardatic dispersion
semi-Dirac fermions
fermions with cubic and quartic dispersion

bilayer graphene
double cuprate layer
surface of top. insulator
neutrino physics

$$\boxed{N = \frac{1}{4\pi i}\,\mathbf{tr}\,[\,\mathbf{K}\oint dl\,\mathbf{H}^{-1}\,\partial_l\,\mathbf{H}]}$$

N=+1 N=+1 N=0 N=+2



$E=cp$ $E=cp$ $E^2 = 2c^2p^2 + 4m^2$ $E^2 = (p^2/\,2m)^2$

Dirac fermions

massive fermions

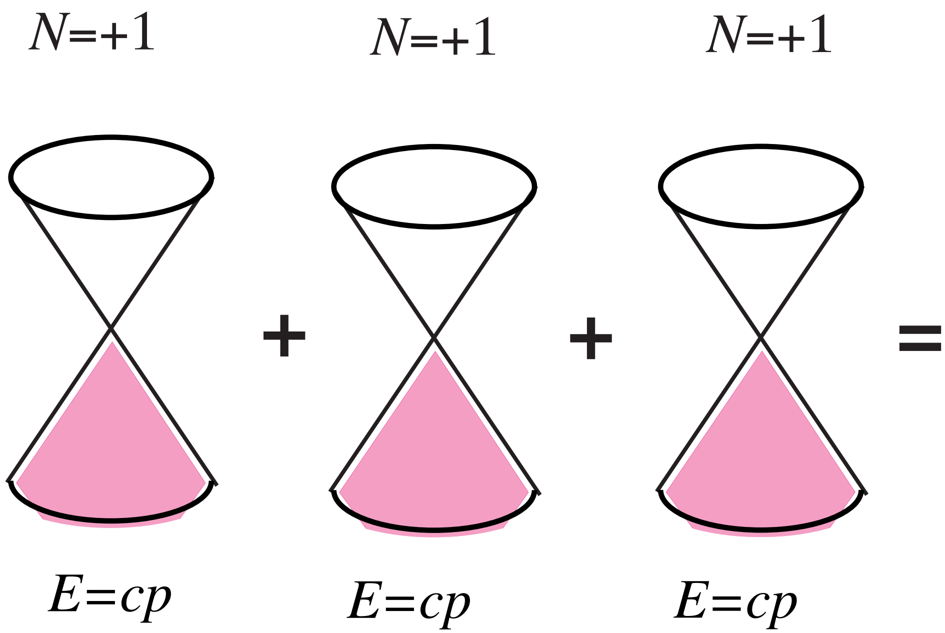
massless fermions
with quadratic dispersion

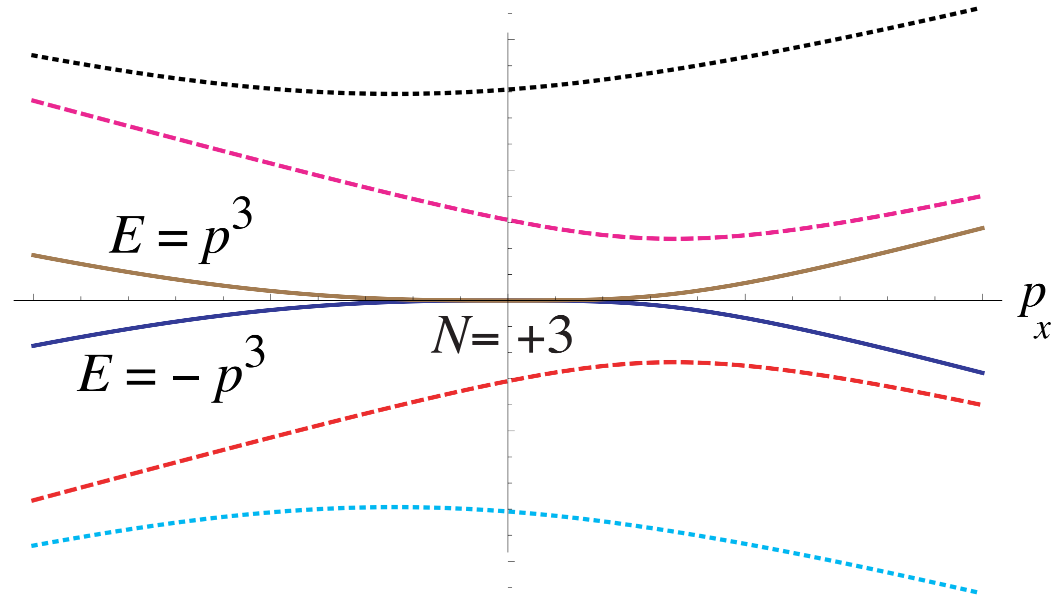
multiple Fermi point

T. Heikkilä & GV arXiv:1010.0393

cubic spectrum in trilayer graphene

$$N = 1 + 1 + 1 = 3$$





multilayered graphene

$$N = 1 + 1 + 1 + \ldots$$

spectrum in the outer layers

$$E = p^N$$

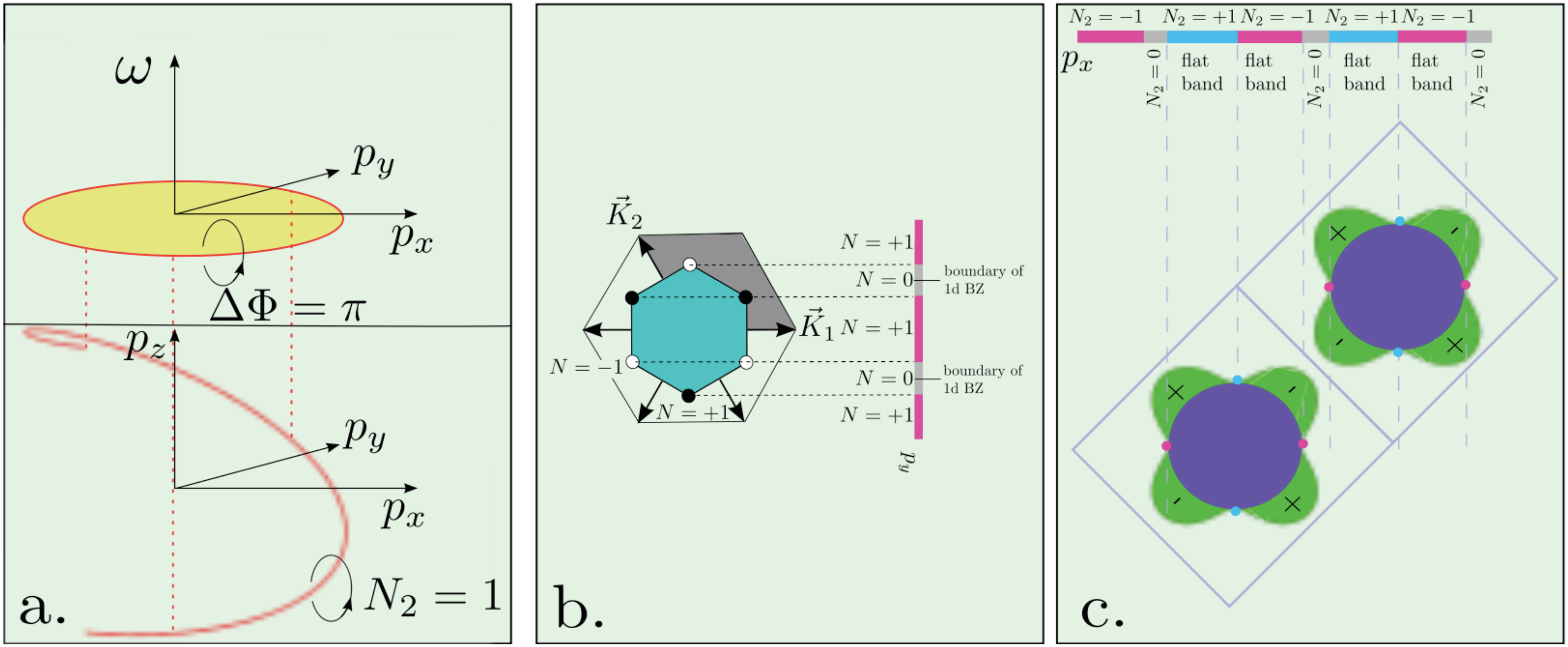
$$E = -p^N$$

what kind of induced gravity emerges near degenerate Fermi point?

route to topological flat band on the surface of 3D material

Flat bands in topological matter

flat band: half-quantum vortex in **p**-space



nodal spiral in multilayered graphene
generates flat band with zero energy
in the top and bottom layers

Hekilla, Kopnin, GV

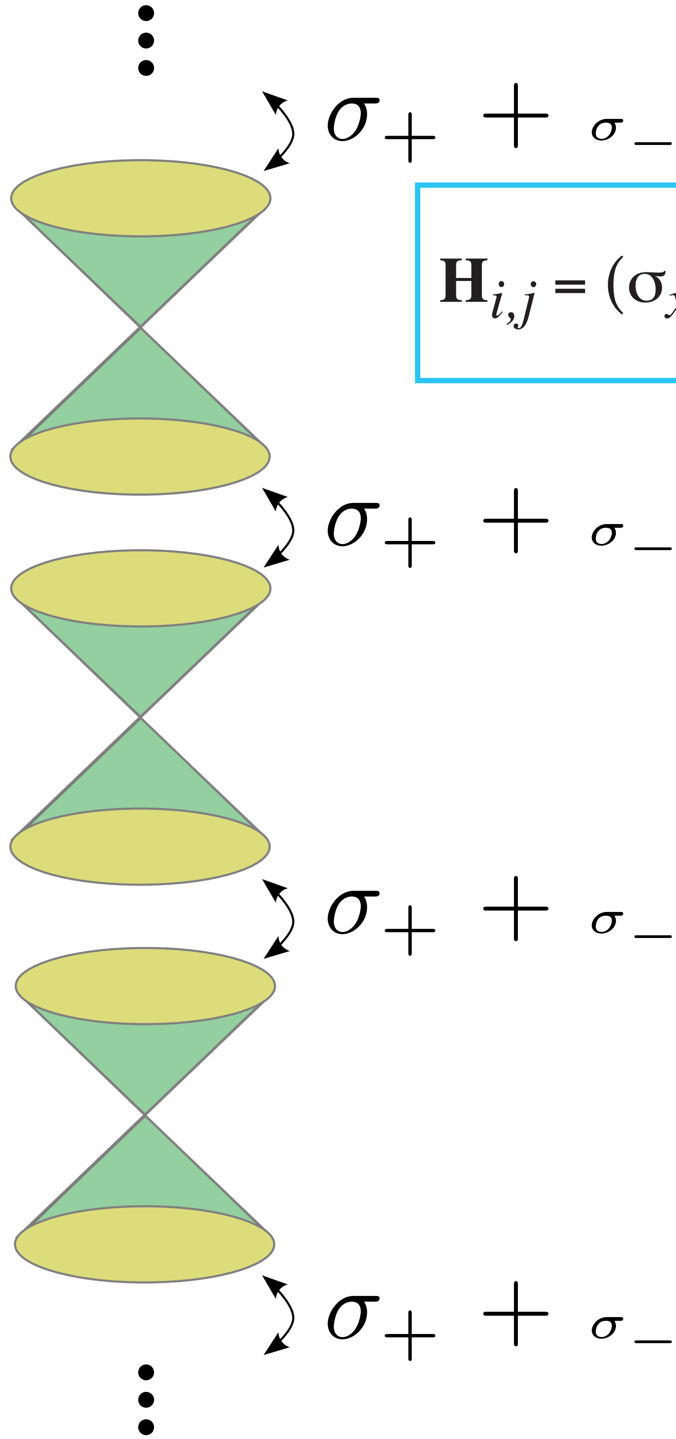
nodes in graphene
generate flat band on zigzag edge

nodal lines
in cuprate superconductors
generate flat band on side surface

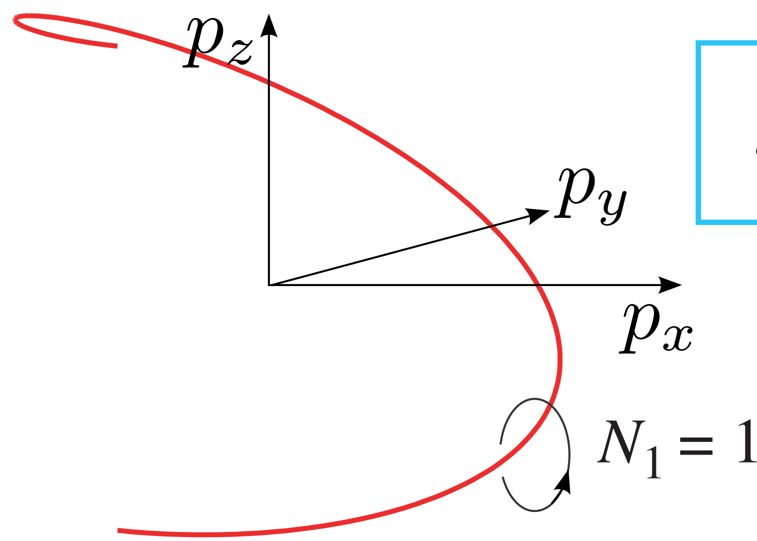
Shinsei Ryu

approximate flat band on side surface
of graphite

formation of nodal spiral in bulk (together with flat band on the surface)
by stacking of graphene layers



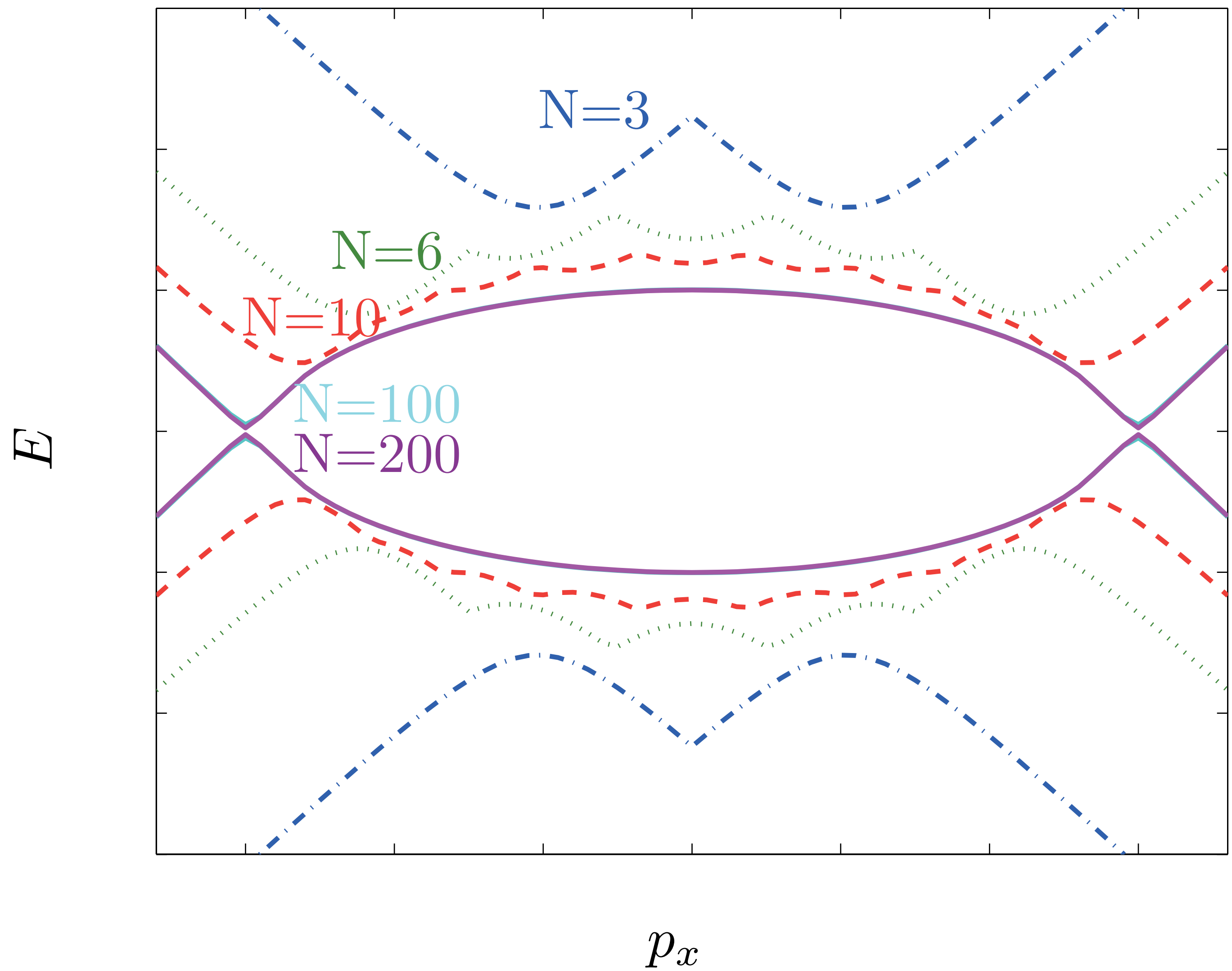
$$\mathbf{H}_{i,j} = (\sigma_x p_x + \sigma_y p_y)\delta_{i,j} + (\sigma_+ t_+ + \sigma_- t_-)\delta_{i,j+1}$$



$$t_+ > t_-$$

$$N_1 = \frac{1}{4\pi i}\,\mathbf{tr}\left[\,\mathbf{K}\oint dl\;\mathbf{H}^{-1}\,\partial_l\,\mathbf{H}\right]$$

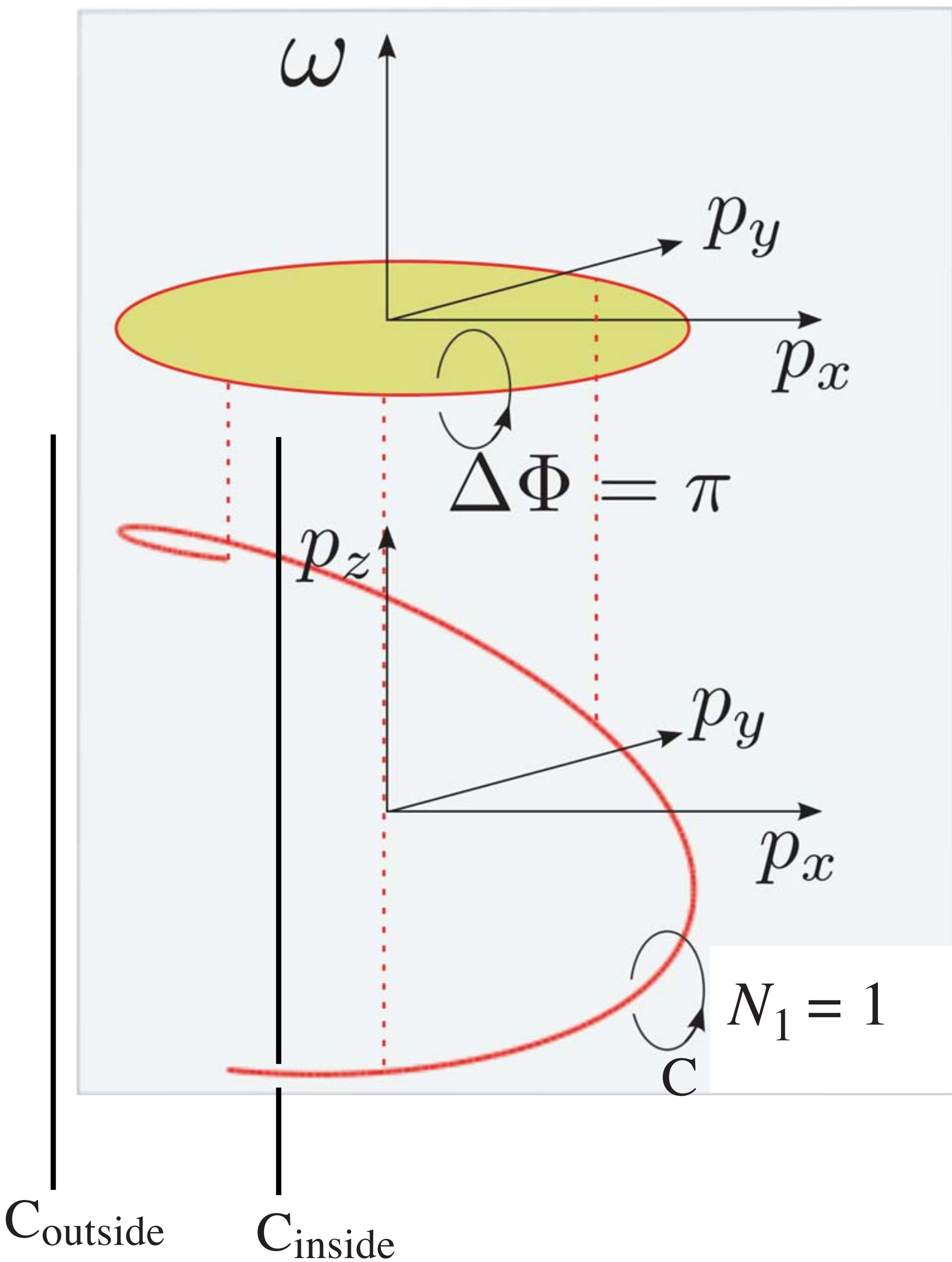
Emergence of nodal line from gapped branches by stacking graphene layers



Nodal spiral generates flat band on the surface

projection of spiral on the surface determines boundary of flat band

$$N_1 = \frac{1}{4\pi i} \mathbf{tr} \left[\mathbf{K} \oint_C dl\, \mathbf{H}^{-1} \partial_l \mathbf{H} \right]$$



at each (p_x,p_y) except the boundary of circle one has 1D gapped state (insulator)

$N_{outside} = 0$ trivial 1D insulator

$N_{inside} = 1$ topological 1D insulator

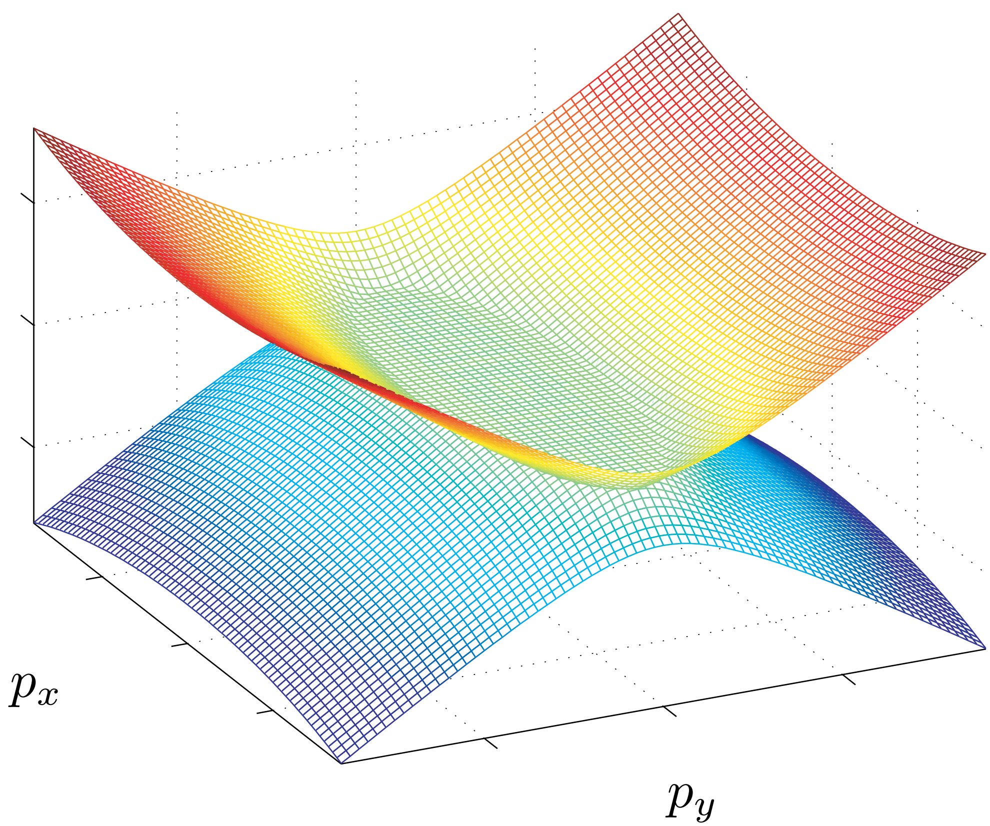
at each (p_x,p_y) inside the circle
one has 1D gapless edge state
this is flat band

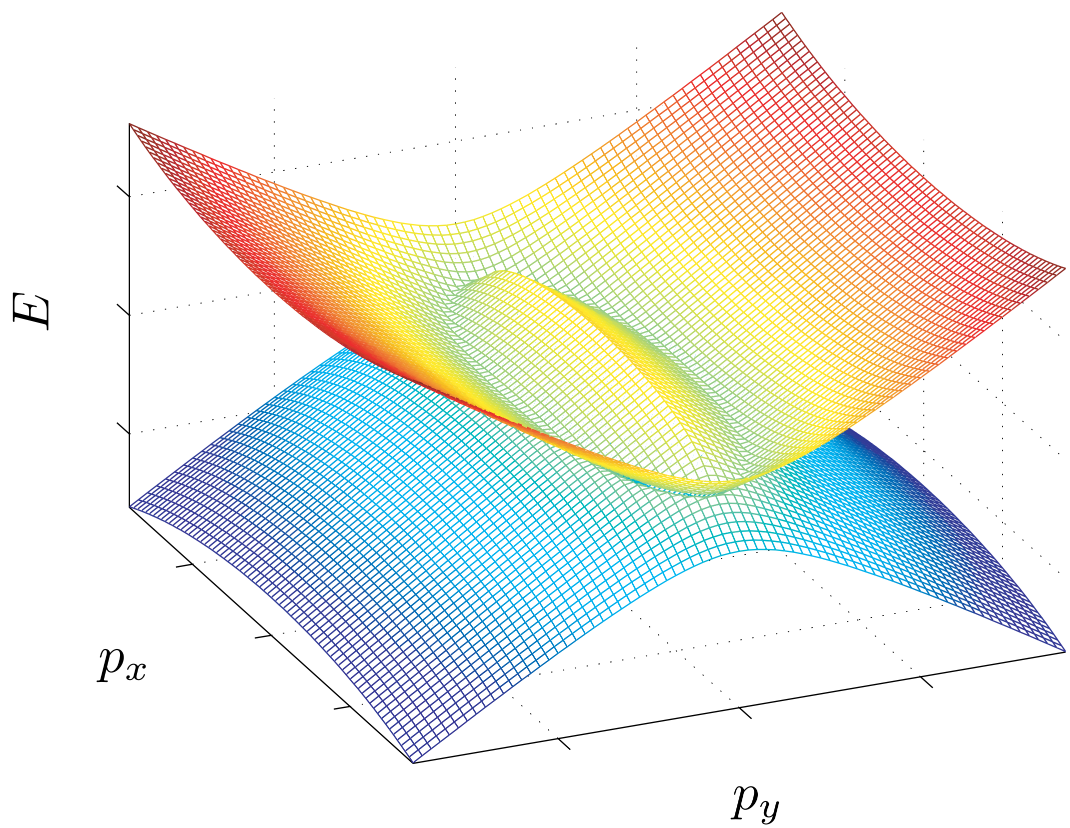
Nodal spiral generates flat band on the surface

projection of nodal spiral on the surface determines boundary of flat band

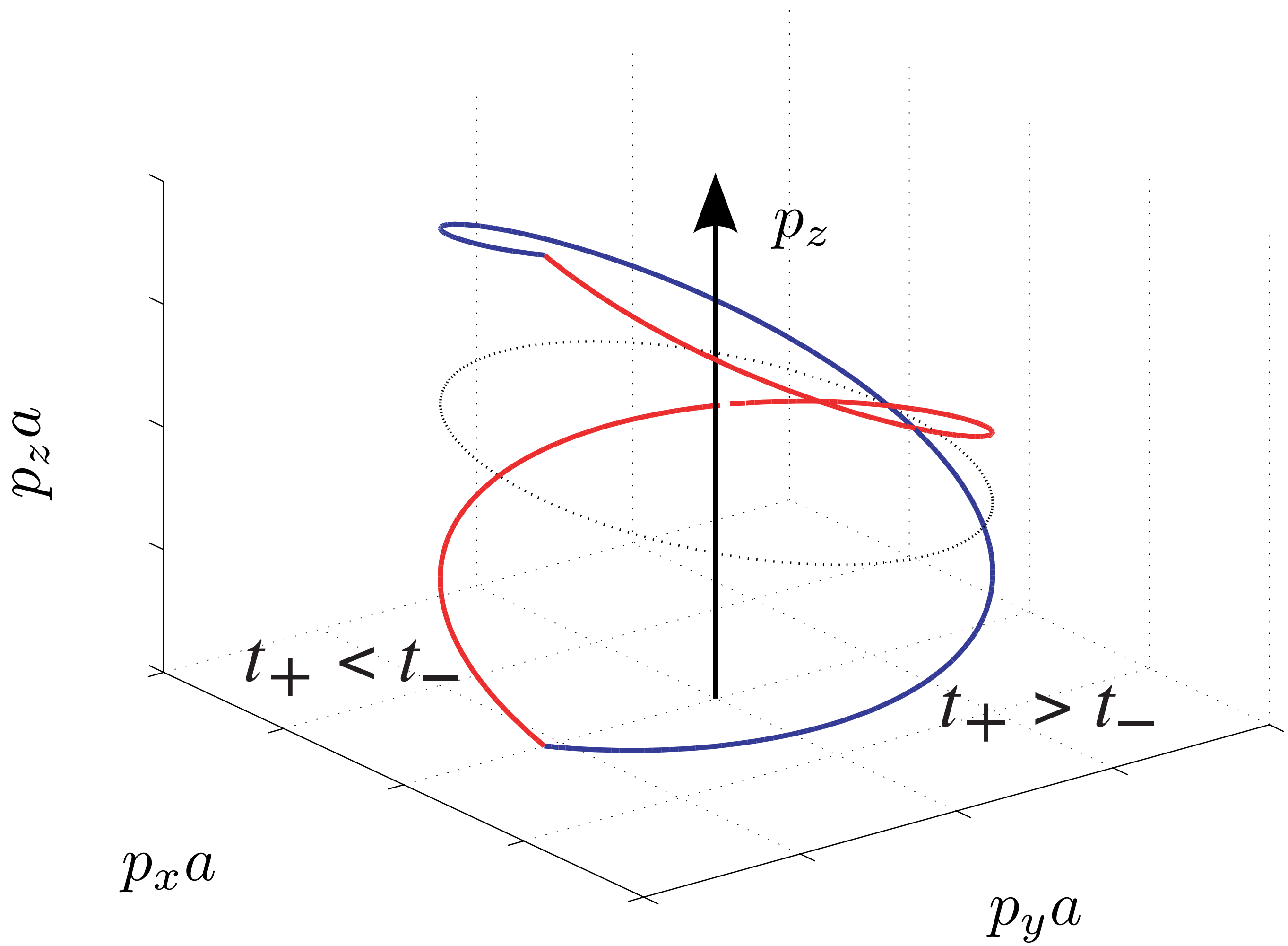
lowest energy states:
surface states form the flat band

energy spectrum in bulk
(projection to p_x , p_y plane)





Helicity of nodal spiral



Modified nodal spiral in rhombohedral graphite: spiral of Fermi surfaces (McClure 1969)

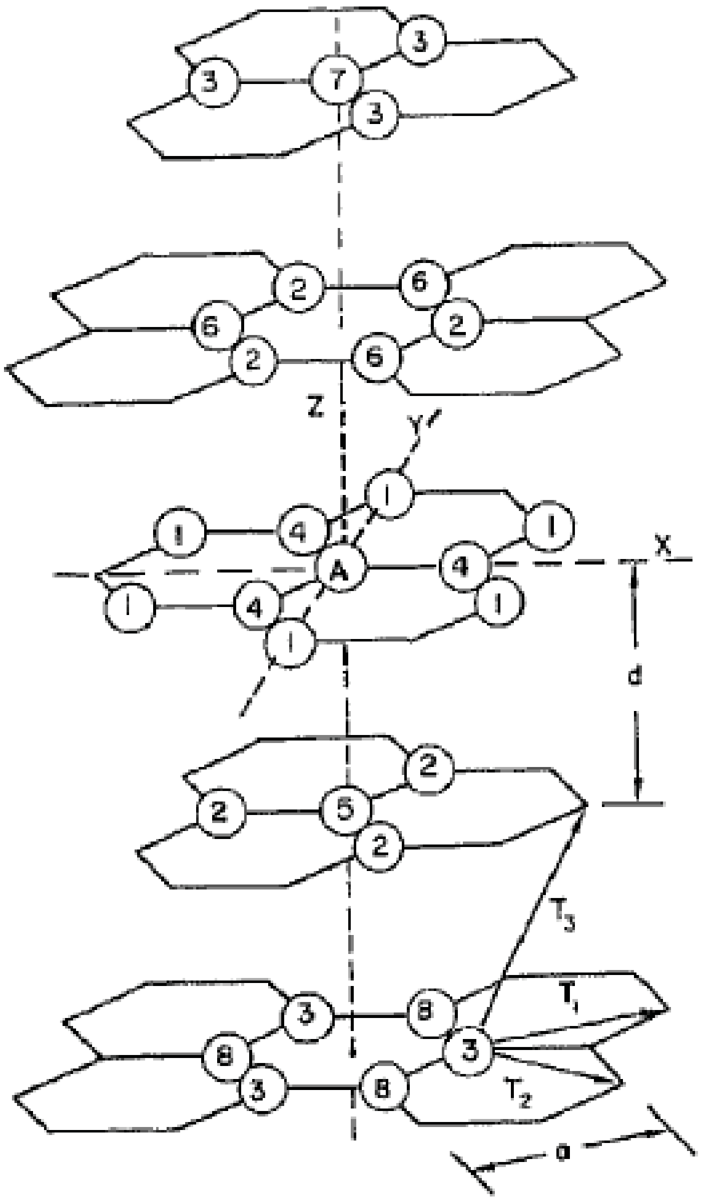


Fig. 1. The crystal lattice of rhombohedral graphite. The numbering of the groups of neighbors of the central *A* atom is explained in the text.

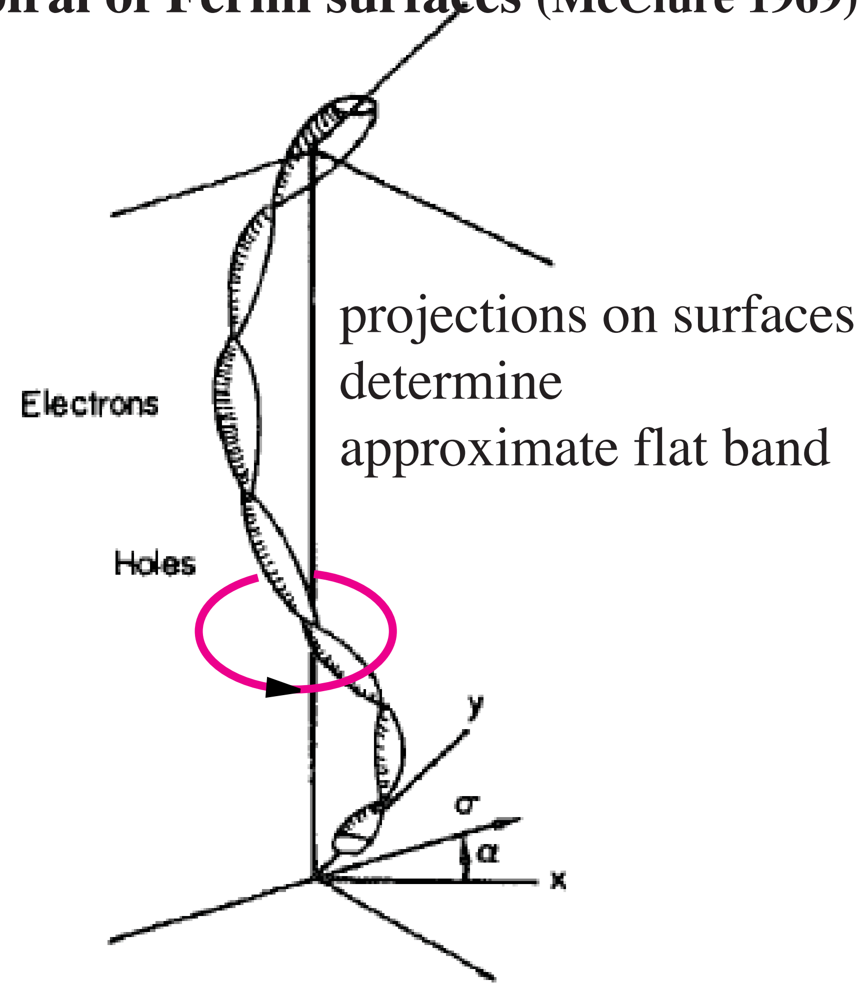
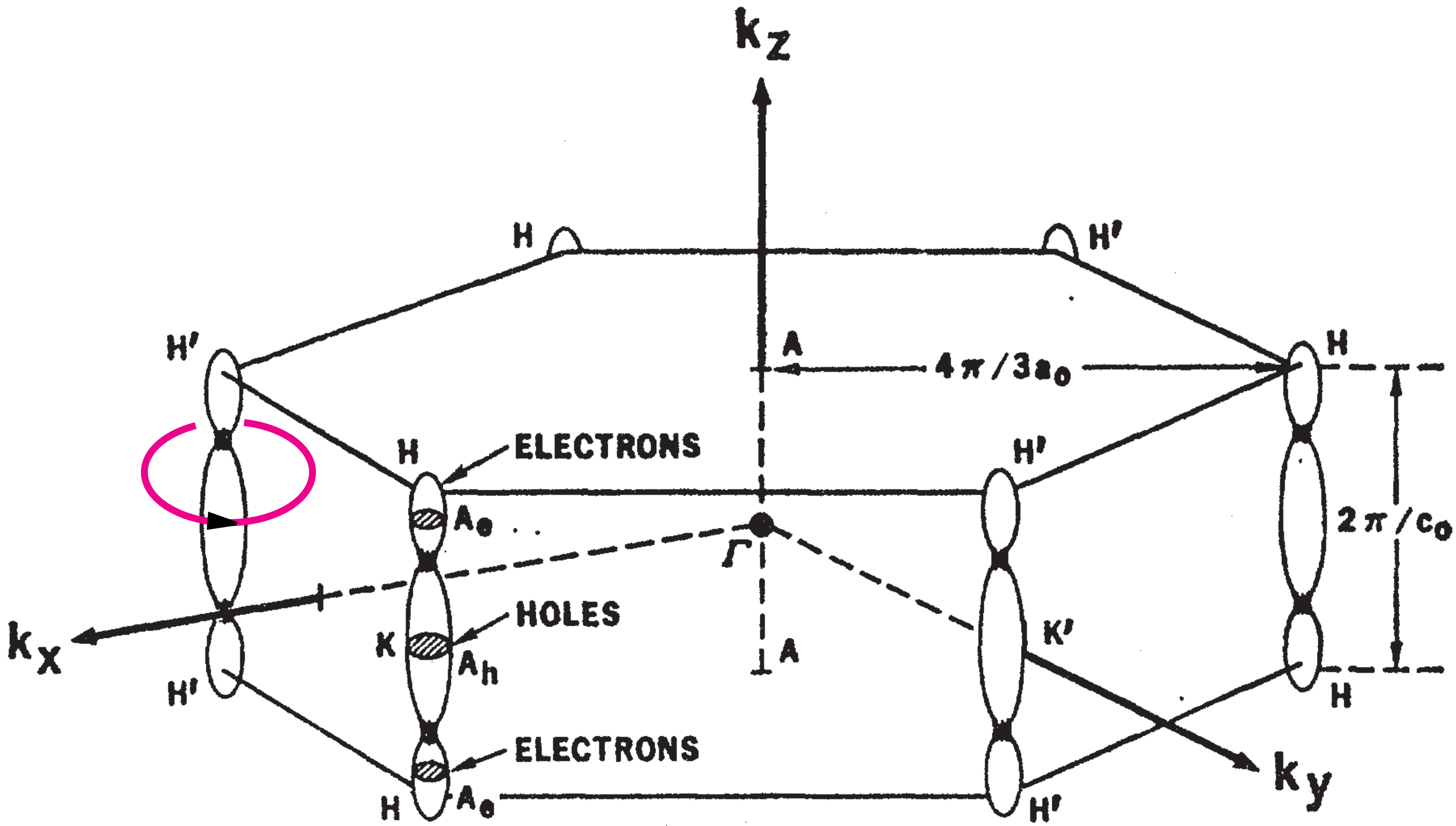


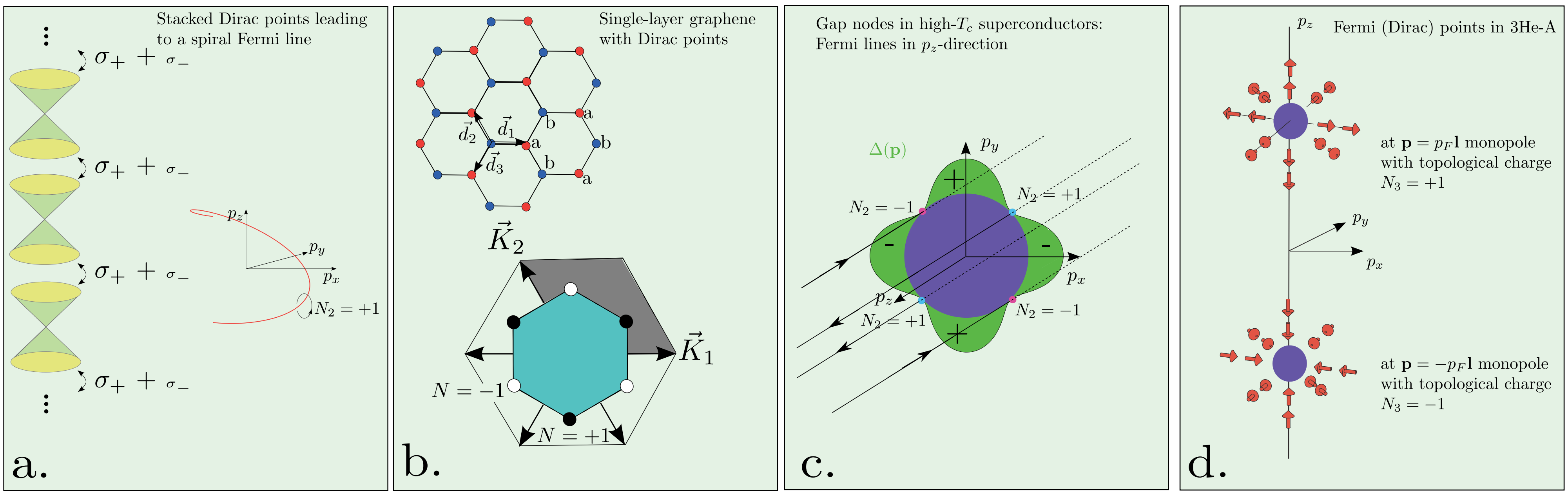
Fig. 2. The Fermi surface of rhombohedral graphite. The surface is centered on one of the six vertical zone edges. The widths of the surfaces have been magnified by more than an order of magnitude.

Nodal lines in graphite tranformed to chain of electron and hole FS



for conventional graphite:
approximate flat band
on the lateral surface

Gapless topological matter with protected flat band on surface or in vortex core



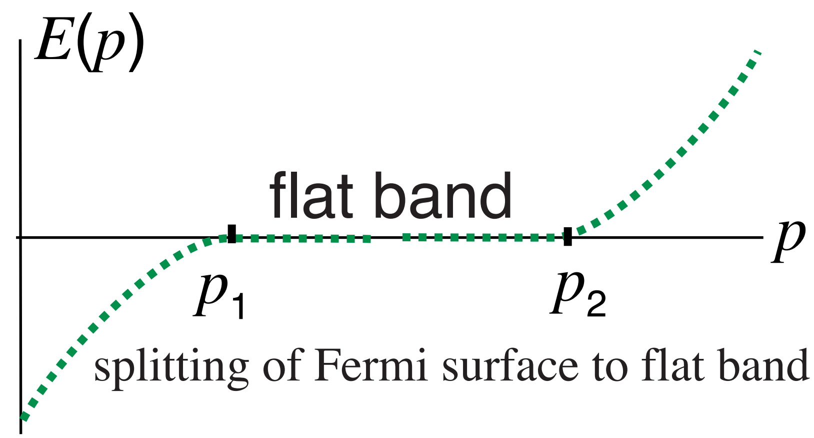
non-topological flat bands due to interaction

Khodel-Shaginyan fermion condensate

JETP Lett. **51**, 553 (1990)

GV, JETP Lett. **53**, 222 (1991)

Nozieres, J. Phys. (Fr.) **2**, 443 (1992)



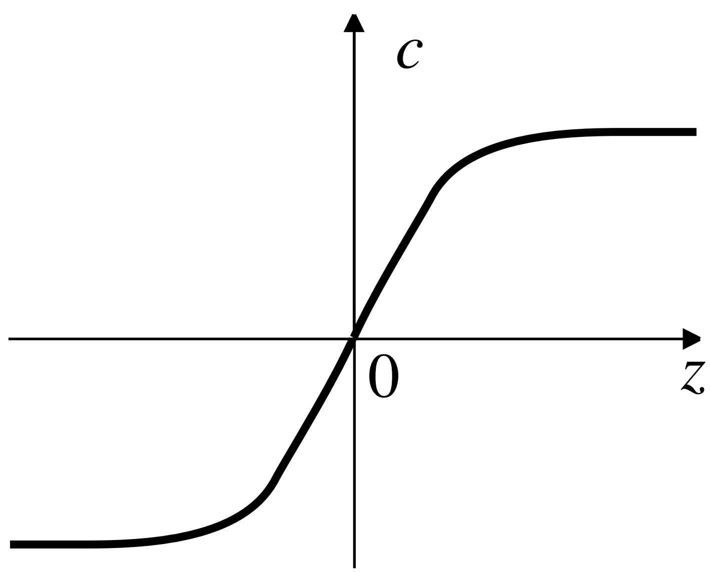
flat band in soliton

$$H = \tau_3 (p_x^2 + p_z^2 - p_F^2)/2m + \tau_1 c(z) p_z$$

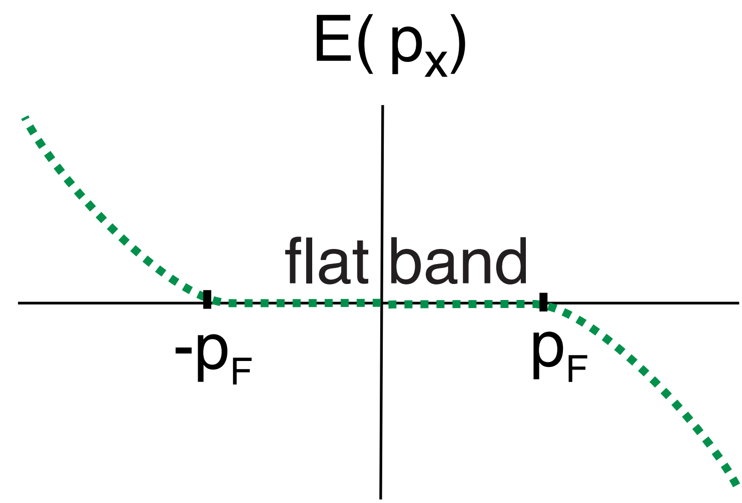
nodes at $p_z = 0$ and $p_x^2 = p_F^2$

$$\boxed{N = \frac{1}{4\pi i} \mathbf{tr} \left[\mathbf{K} \oint dl \, \mathbf{H}^{-1} \partial_l \mathbf{H} \right]}$$

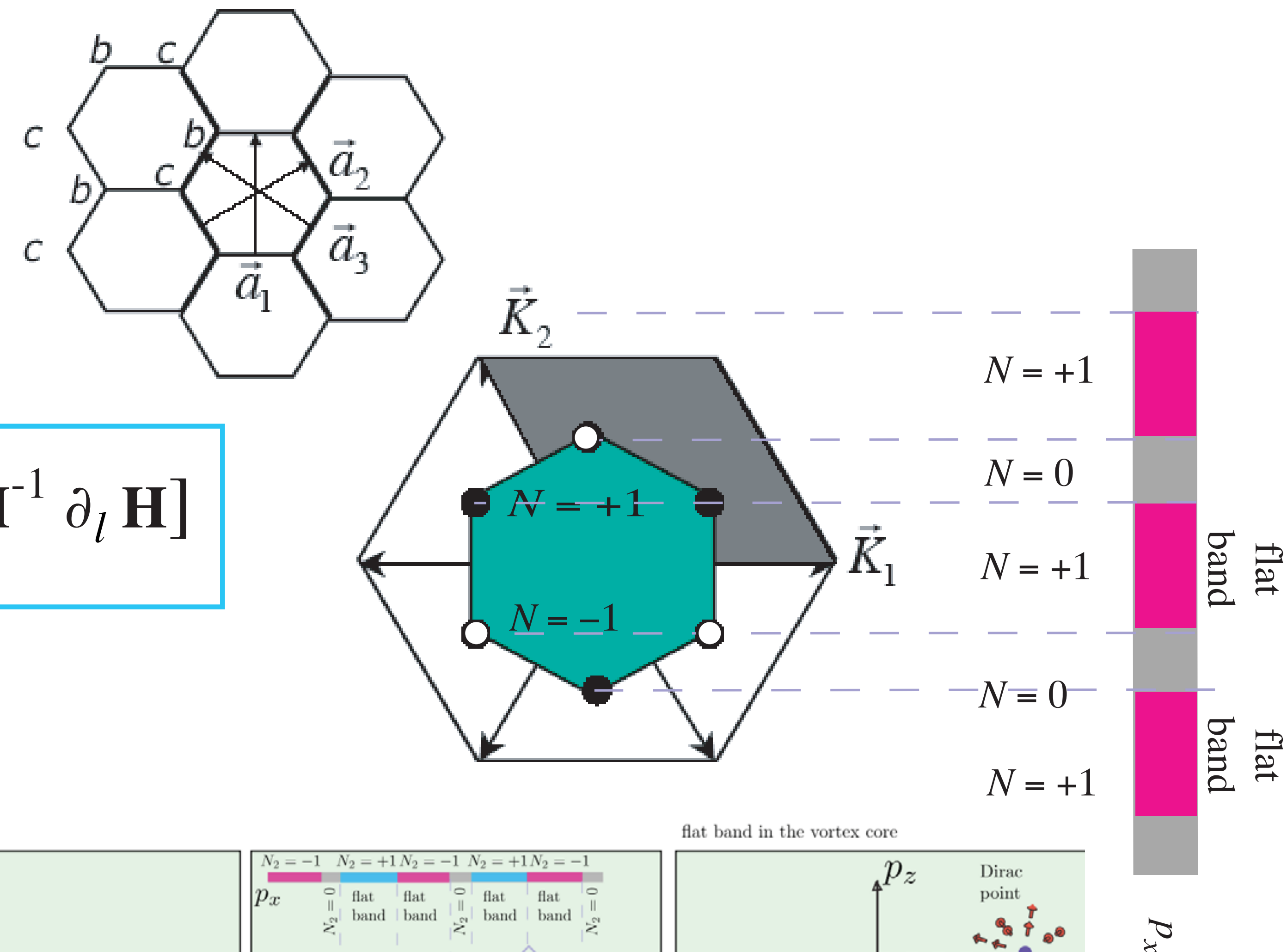
soliton



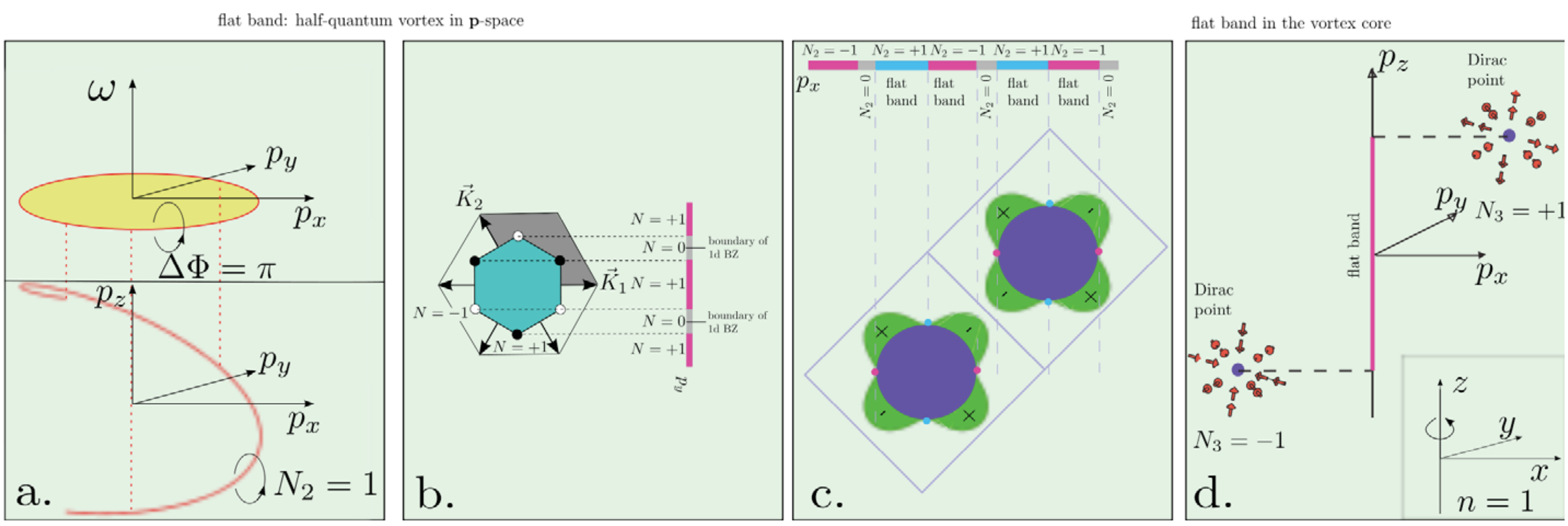
spectrum in soliton



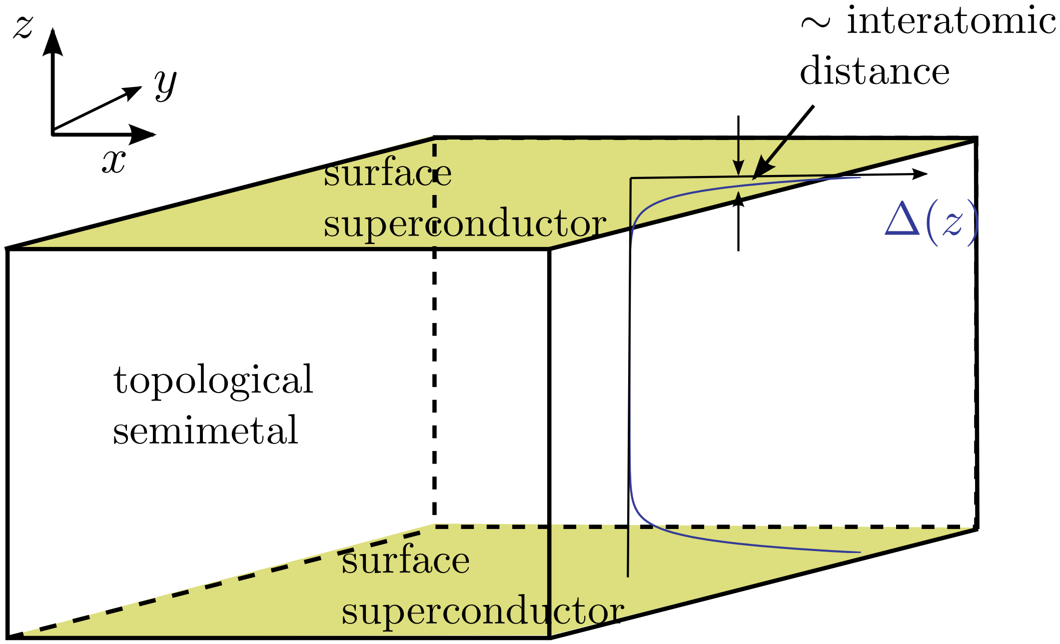
Flat band on the graphene edge

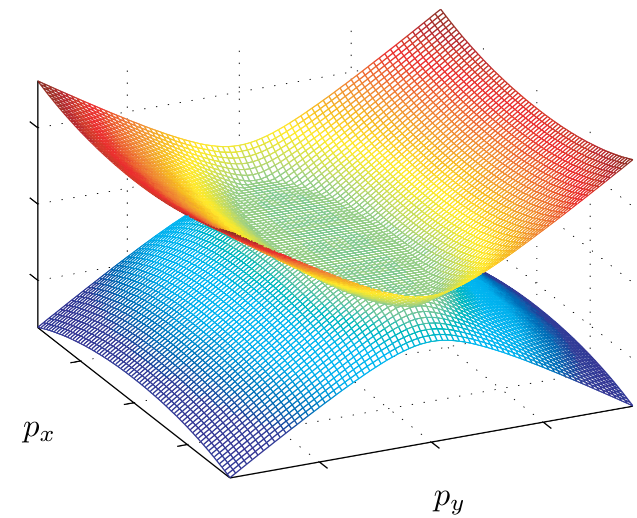


$$N = \frac{1}{4\pi i}\,\mathbf{tr}\left[\,\mathbf{K}\oint dl\ \mathbf{H}^{-1}\,\partial_l\,\mathbf{H}\right]$$



Surface superconductivity in topological semimetals: route to room temperature superconductivity





Extremely high DOS of flat band gives high transition temperature:

normal superconductors: exponentially suppressed transition temperature

$$T_c = T_F \exp(-1/g\nu)$$

interaction ***DOS***

$$1 = g \int \frac{d^2p}{2h^2} \frac{1}{E(p)}$$

"Recent studies of the correlations between the internal microstructure of the samples and the transport properties suggest that superconductivity might be localized at the interfaces between crystalline graphite regions of different orientations, running parallel to the graphene planes." PRB. 78, 134516 (2008)

flat band superconductivity: linear dependence of T_c on coupling g

$$T_c \sim g S_{FB}$$

interaction ***flat band area***

PHYSICAL REVIEW B **81**, 184513 (2010)

Stripes of increased diamagnetic susceptibility in underdoped superconducting $Ba(Fe_{1-x}Co_x)_2As_2$ single crystals: Evidence for an enhanced superfluid density at twin boundaries

B. Kalisky,[1,2,*,†] J. R. Kirtley,[1,2,3] J. G. Analytis,[1,2,4] Jiun-Haw Chu,[1,2,4] A. Vailionis,[1,4]
I. R. Fisher,[1,2,4] and K. A. Moler[1,2,4,5,*,‡]

Kathryn Moler:
possible 2D superconductivityof twin boundaries

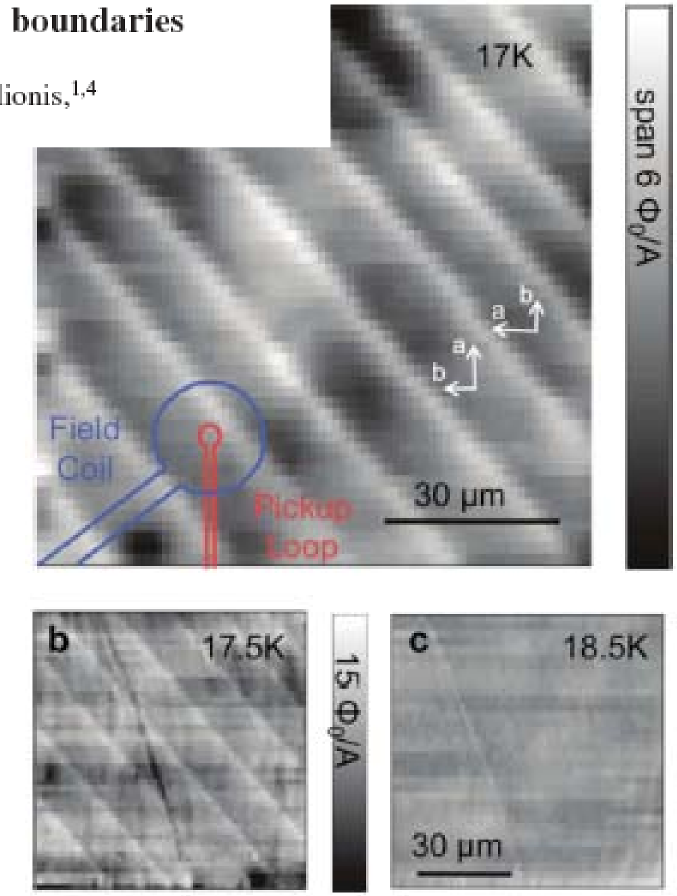


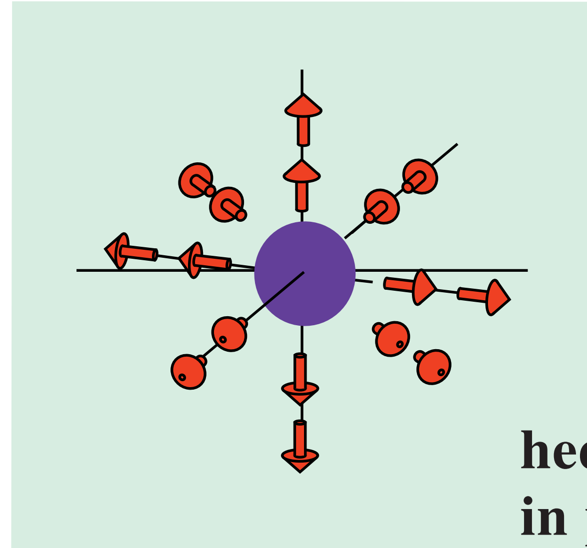
FIG. 1. (Color online) Local susceptibility image in underdoped $Ba(Fe_{1-x}Co_x)_2As_2$, indicating increased diamagnetic shielding on twin boundaries. (a) Local diamagnetic susceptibility, at $T=17$ K, of the *ab* face of sample UD1 ($x=0.051$ and $T_c=18.25$ K), showing stripes of enhanced diamagnetic response (white). In addition there is a mottled background associated with local T_c variations that becomes more pronounced as $T\rightarrow T_c$. Overlay: sketch of the scanning SQUID's sensor. The size of the pickup loop sets the spatial resolution of the susceptibility images. [(b) and (c)] Images of the same region at (b) $T=17.5$ K and (c) at $T=18.5$ K show that the stripes disappear above T_c. A topographic feature (scratch) appears in (b) and (c).

From Weyl point to quantum Hall topological insulators

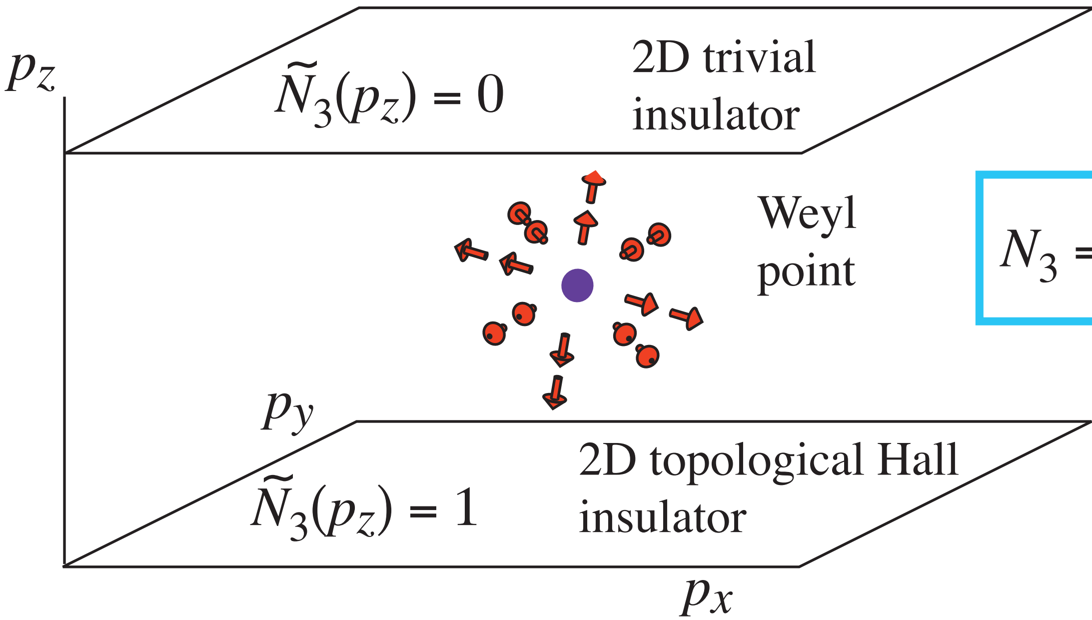
$$N_3 = \frac{1}{8\pi} e_{ijk} \int dS^k \; \hat{\mathbf{g}} \cdot (\partial_{p_i} \hat{\mathbf{g}} \times \partial_{p_j} \hat{\mathbf{g}})$$

over 2D surface S in 3D momentum space

top. invariant for Weyl point in 3+1 system



hedgehog in p-space



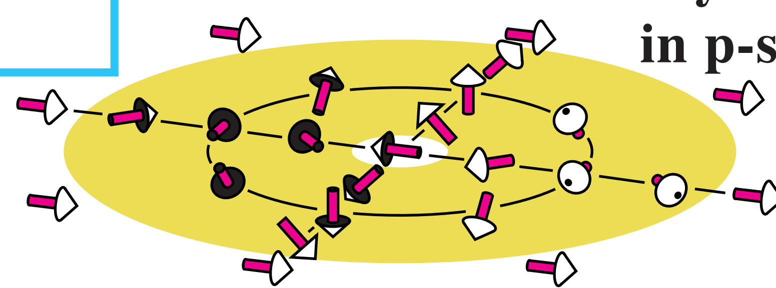
$$N_3 = \tilde{N}_3(p_z < p_0) - \tilde{N}_3(p_z > p_0)$$

at each p_z one has 2D insulator or fully gapped 2D superfluid

$$\tilde{N}_3(p_z) = \frac{1}{4\pi} \int dp_x dp_y \; \hat{\mathbf{g}} \cdot (\partial_{p_x} \hat{\mathbf{g}} \times \partial_{p_y} \hat{\mathbf{g}})$$

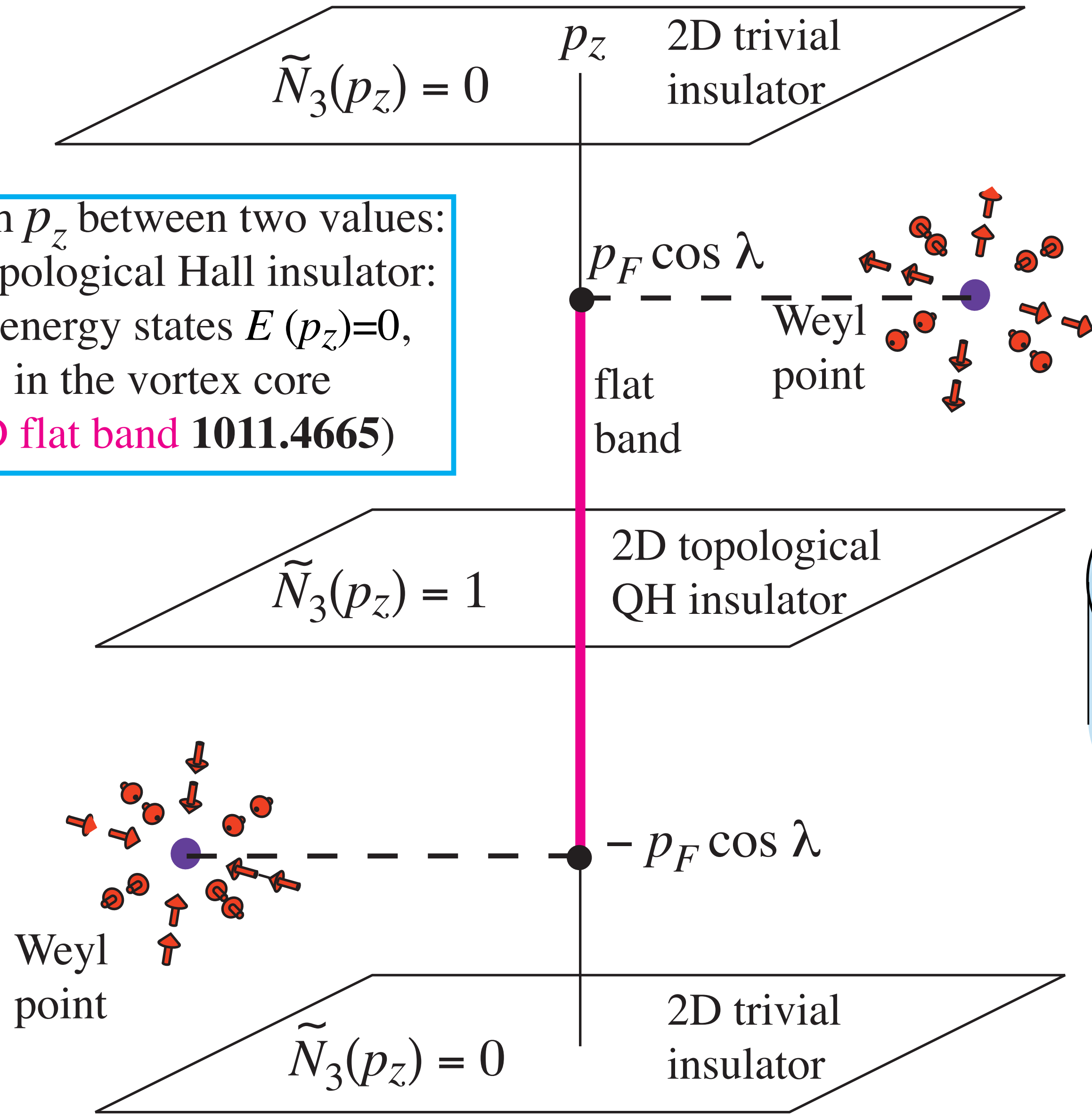
over the whole 2D momentum space or over 2D Brillouin zone

top. invariant for fully gapped 2+1 system

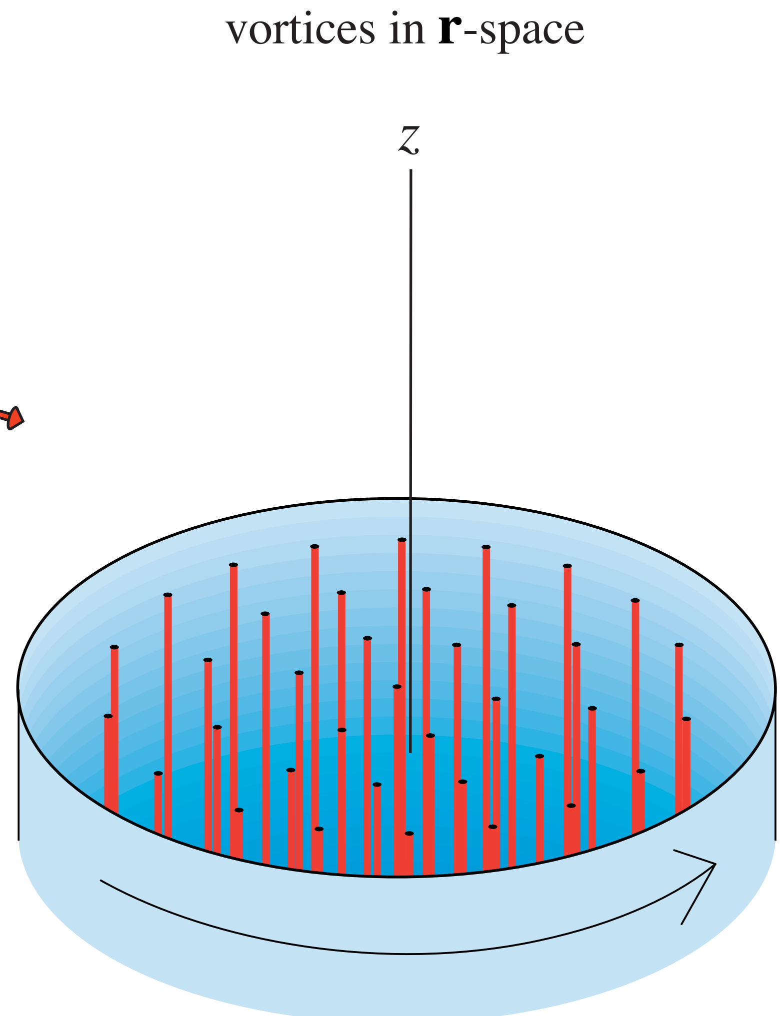


skyrmion in p-space

3D matter with Weyl points:
Topologically protected flat band in vortex core



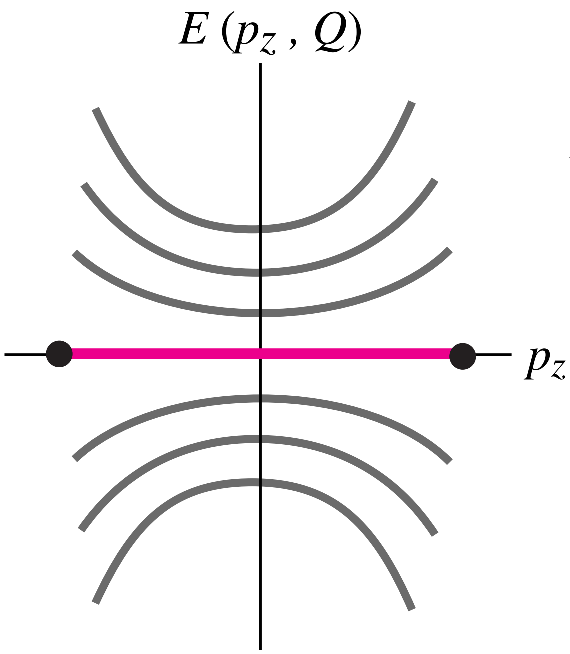
at each p_z between two values:
2D topological Hall insulator:
zero energy states $E\,(p_z)$=0,
in the vortex core
(1D flat band **1011.4665**)



$$\tilde{N}_3(p_z) = \frac{1}{4\pi^2}\,\mathbf{tr}\int dp_x\, dp_y d\omega\, \mathbf{G}\,\partial_\omega\, \mathbf{G}^{-1}\, \mathbf{G}\,\partial_{p_x}\, \mathbf{G}^{-1}\mathbf{G}\,\partial_{p_y}\, \mathbf{G}^{-1}$$

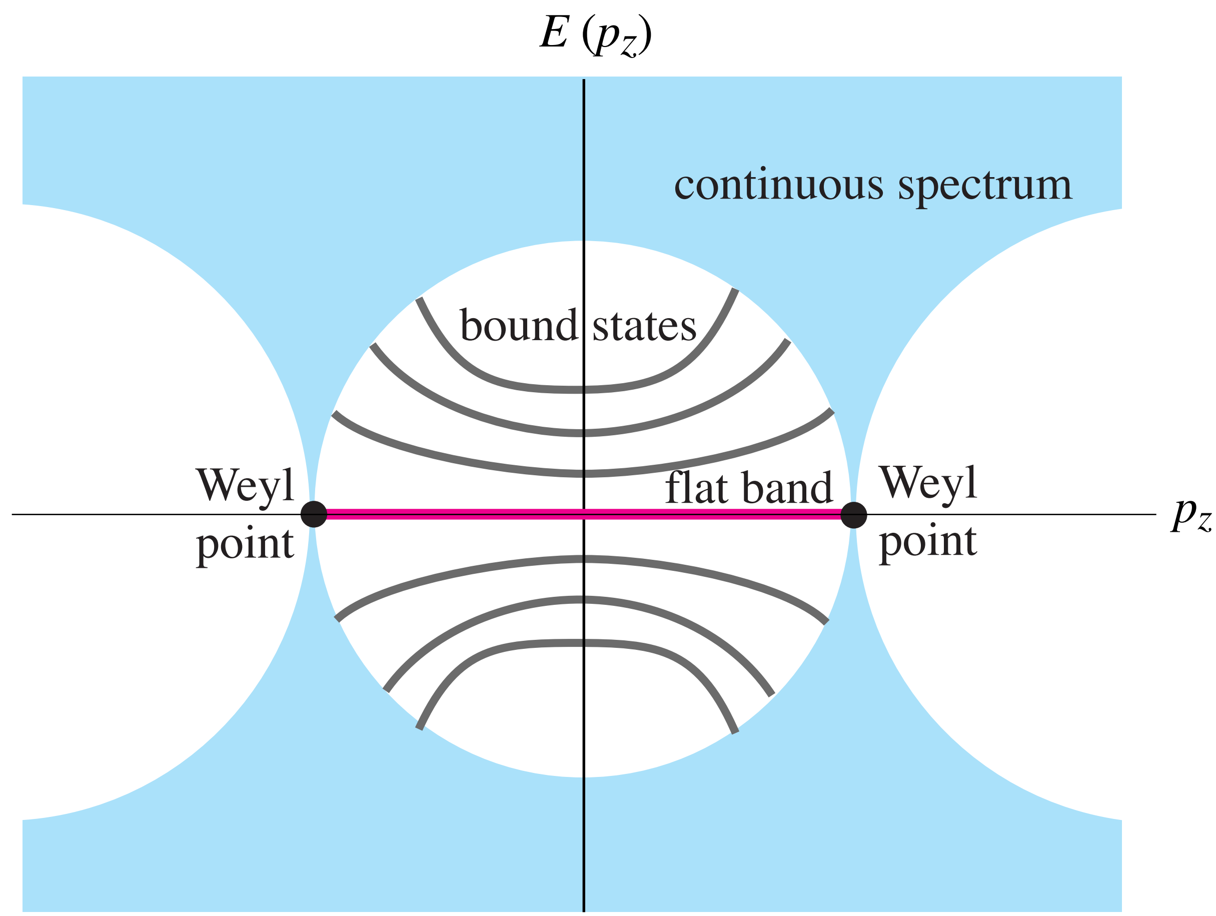
GV & Yakovenko
(1989)

Topologically protected flat band in vortex core of superfluids with Weyl points



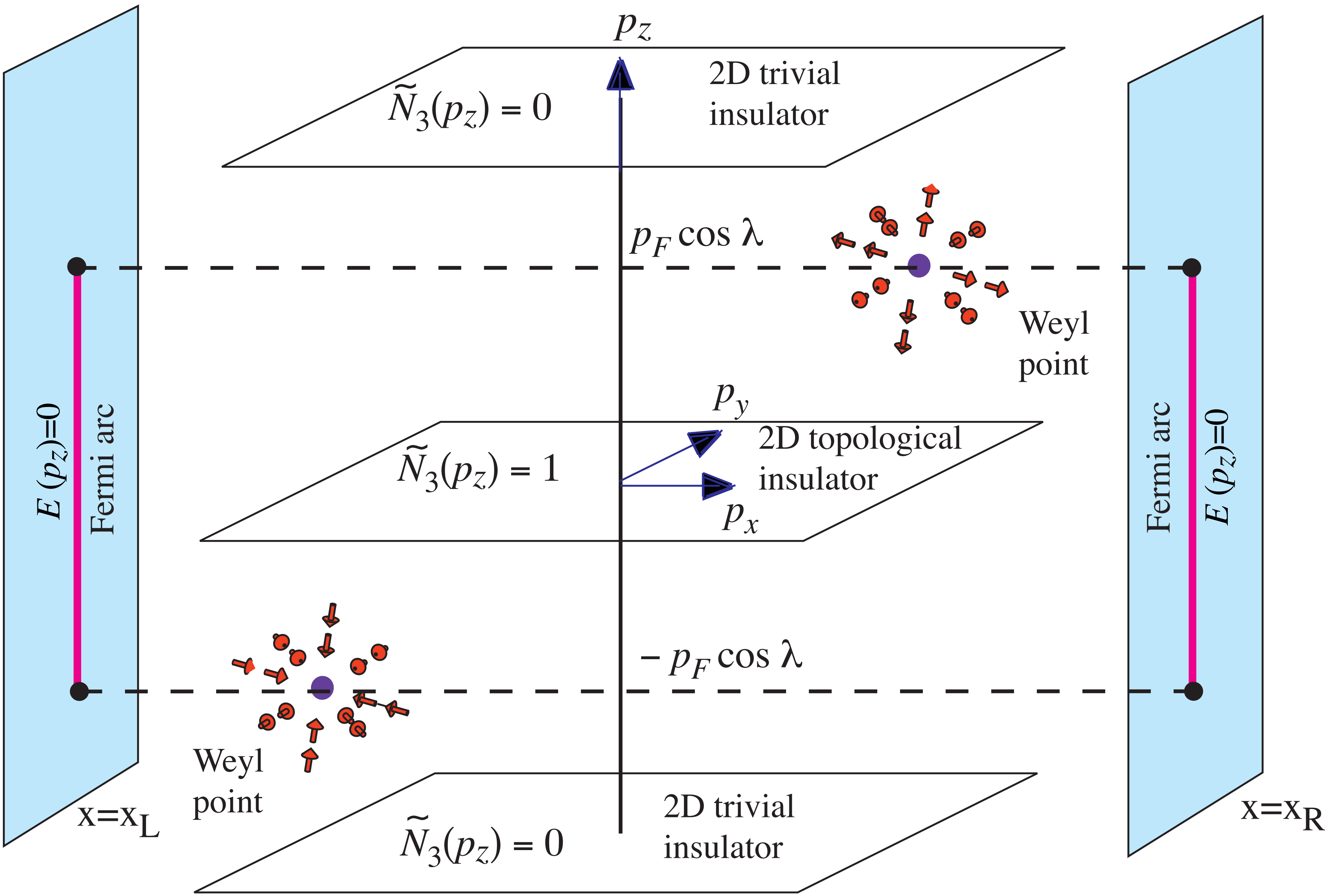
flat band
in spectrum of fermions
bound to core of 3He-A vortex
(Kopnin-Salomaa 1991)

flat band of bound states
terminates on zeroes
of continuous spectrum
(i.e on Weyl points)

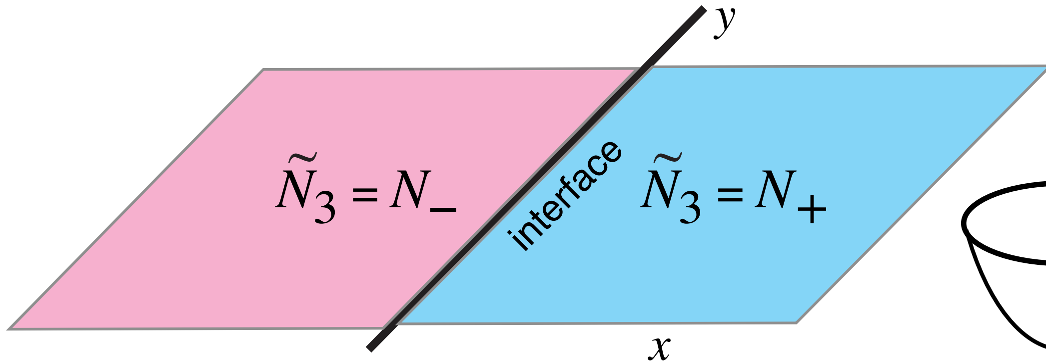


3He-A with Weyl points: Topologically protected Dirac valley (Fermi arc) on surface

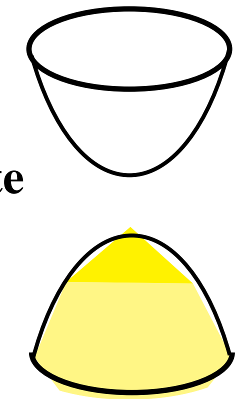
for each $|p_z| < p_F \cos\lambda$
one has 2D topological Hall insulator with
zero energy edge states $E(p_z)=0$
(Dirac valley **PRB 094510** or Fermi arc **PRB 205101**)

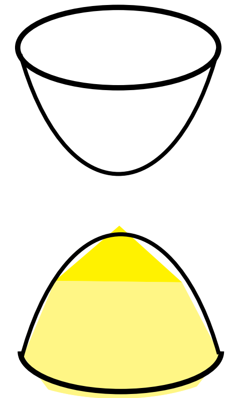


Edge states at interface between effective two 2+1 topological insulators & Fermi arc

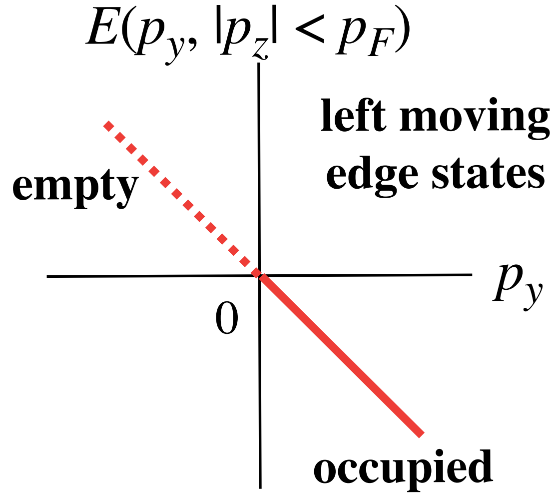


gapped state





gapped state



Index theorem:
number of fermion zero modes
at interface:

$$\nu = N_+ - N_-$$

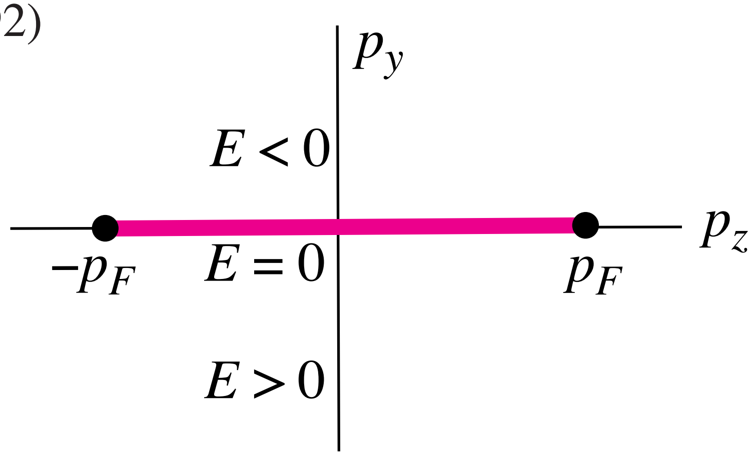
on the edge of insulator with

$$\widetilde{N}_3 = 1$$

one fermion zero mode

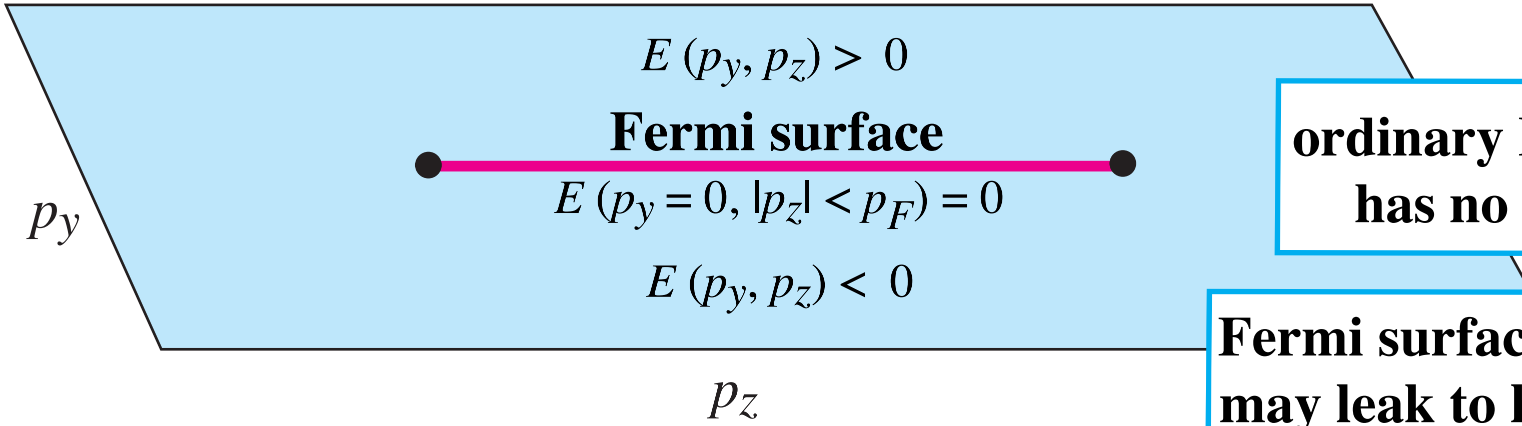
$$\nu = 1$$

GV JETP Lett. **55**, 368 (1992)



Fermi arc in 2D:
Fermi surface which terminates
on two points:
projections of Weyl points

Fermi arc:
Fermi surface separates positive and negative energies, but has boundaries

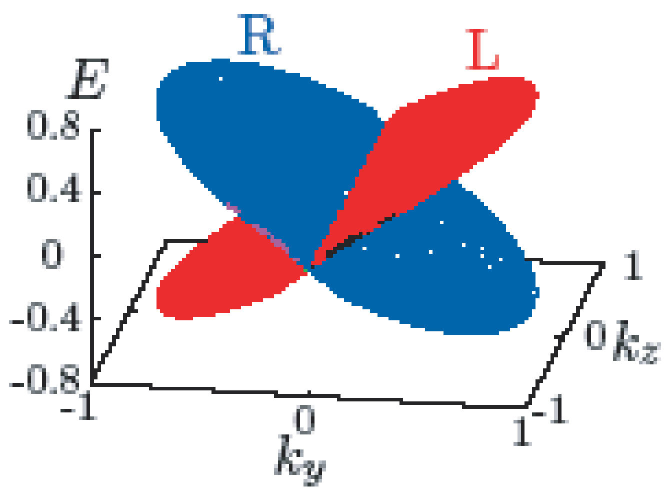


ordinary Fermi surface has no boundaries

Fermi surface of bound states may leak to higher dimension

Fermi surface of localized states is terminated by projections of Weyl points when localized states merge with continuous spectrum

L spectrum of edge states on left wall



Tsutsumi, et al **PRB 094510**

R spectrum of edge states on right wall

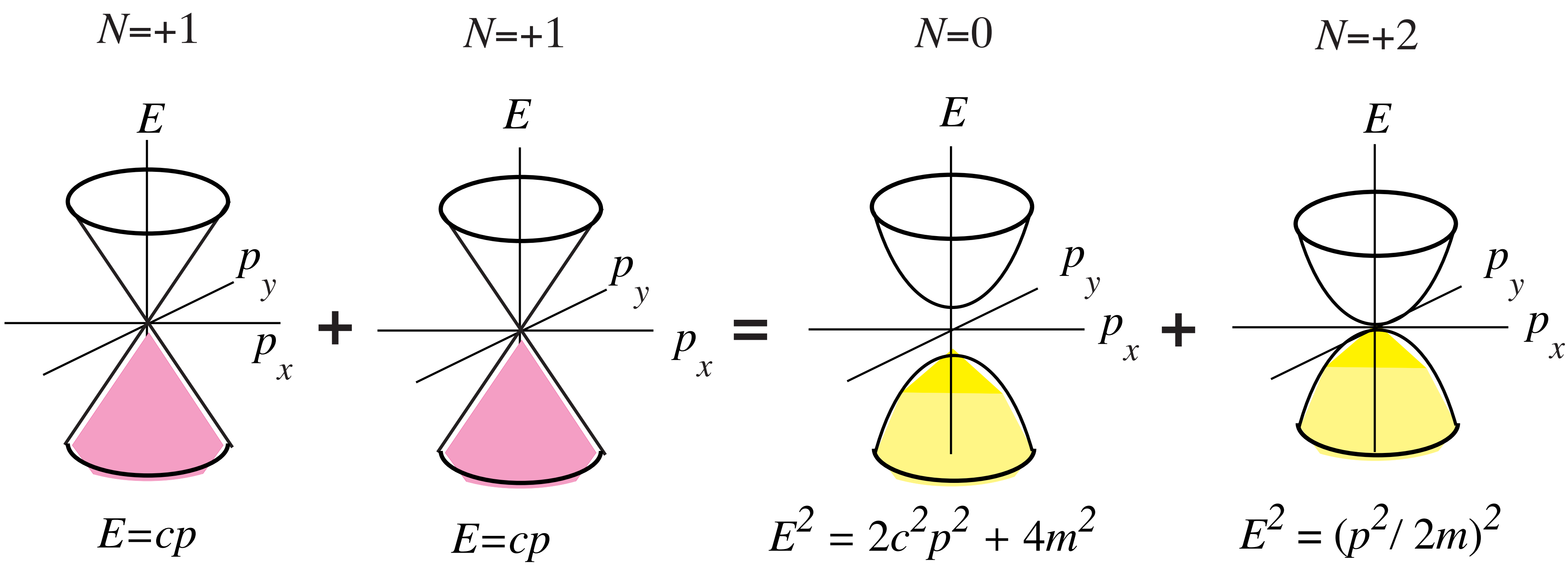
Horava anisotropic scaling gravity

anisotropic z=3 scaling: $x = b\,x'\,,\;\; t = b^{3} t'$

$$S_{\rm grav} = \int \underset{b^3}{d^3x}\; \underset{b^3}{dt}\;\; \underset{b^{-6}}{R^3}$$

Horava anisotropic scaling in bilayered graphene

$$\boxed{N = \frac{1}{4\pi i}\,\mathbf{tr}\left[\,\mathbf{K}\oint dl\,\mathbf{H}^{-1}\,\partial_l\,\mathbf{H}\right]}$$



2+1 massless Dirac fermions

massive fermions

massless Dirac fermions
with quadratic dispersion

relativistic quantum fields and gravity emerging near Weyl point

Atiyah-Bott-Shapiro construction:
linear expansion of Hamiltonian near the nodes in terms of Dirac Γ-matrices

$$\boxed{H = e_i^{\ k}\, \Gamma^i \cdot (p_k - p_k^0)}$$

linear expansion near Weyl point

$e_i^{\ k}$ → effective tetrad: emergent gravity

Γ^i → emergent **Γ**−matrices

$$\boxed{g^{\mu\nu}(p_\mu - eA_\mu - e\tau \cdot \mathbf{W}_\mu)(p_\nu - eA_\nu - e\tau \cdot \mathbf{W}_\nu) = 0}$$

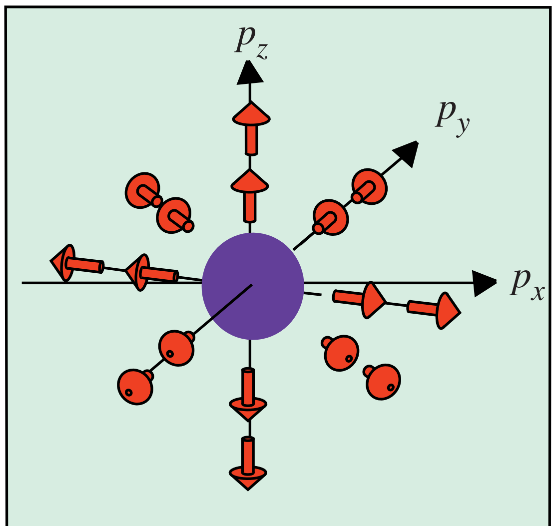
$g^{\mu\nu}$ → effective metric: emergent gravity

A_μ → effective electromagnetic field

$\mathbf{W}_\mu$ → effective $SU(2)$ gauge field

e → effective electric charge $e = +1$ or -1

τ → effective isotopic spin



hedgehog in **p**-space

what gravity & gauge fields emerge in vacua with quadratic Dirac point in bilayer graphene ?

all ingredients of Standard Model :
chiral fermions & gauge fields
emerge in low-energy corner
together with spin, Dirac Γ−matrices, gravity & physical laws:
Lorentz & gauge invariance, equivalence principle, etc

Fermions in 2+1 bylayer graphene

single layer

$$\mathrm{H} = \begin{pmatrix} 0 & p_x + ip_y \\ p_x - ip_y & 0 \end{pmatrix} = \sigma_x p_x + \sigma_y p_y = \begin{pmatrix} 0 & \overbrace{(\mathbf{e}_1(\mathbf{p}) + i\,\mathbf{e}_2)}^{\text{zweibein}}\cdot(\mathbf{p} - \mathbf{A}) \\ (\mathbf{e}_1(\mathbf{p}) - i\,\mathbf{e}_2)\cdot(\mathbf{p} - \mathbf{A}) & 0 \end{pmatrix}$$

double layer

$$\mathrm{H} = \begin{pmatrix} 0 & (p_x + ip_y)^2 \\ (p_x - ip_y)^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & [\overbrace{(\mathbf{e}_1(\mathbf{p}) + i\,\mathbf{e}_2)}^{\text{zweibein}}\cdot(\mathbf{p} - \mathbf{A})]^2 \\ [(\mathbf{e}_1(\mathbf{p}) - i\,\mathbf{e}_2)\cdot(\mathbf{p} - \mathbf{A})]^2 & 0 \end{pmatrix}$$

anisotropic scaling: $\mathrm{x} = \mathrm{b}\ \mathrm{x}'\ ,\ \ \mathrm{t} = \mathrm{b}^2\ \mathrm{t}'$

2+1 anisotropic QED emerging in bylayer graphene

$$H = \begin{pmatrix} 0 & (p_x + ip_y)^2 \\ (p_x - ip_y)^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & [(\mathbf{e}_1(\mathbf{p}) + i\,\mathbf{e}_2)(\mathbf{p} - \mathbf{A})]^2 \\ [(\mathbf{e}_1(\mathbf{p}) - i\,\mathbf{e}_2)(\mathbf{p} - \mathbf{A})]^2 & 0 \end{pmatrix}$$

anisotropic scaling: $x = b\,x'$, $t = b^2\ t'$, $B = b^{-2}\,B'$, $E = b^{-3}\,E'$, $S = S'$

$$S_{\mathrm{QED}} = \int \underset{b^2}{d^2x}\ \underset{b^2}{dt}\ (\ \underset{b^{-4}}{B^2} - \underset{b^{-4}}{E^{4/3}}\ )$$

3+1 isotropic QED emerging in Weyl semimetal

isotropic scaling: $x = b\,x'$, $t = b\,t'$, $B = b^{-2}\,B'$, $E = b^{-2}\,E'$, $S = S'$

$$S_{\mathrm{QED}} = \int \underset{b^3}{d^3x}\ \underset{b}{dt}\ (\ \underset{b^{-4}}{B^2} - \underset{b^{-4}}{E^2}\ )$$

2+1 isotropic QED emerging in single layer graphene

$$S_{\mathrm{QED}} = \int \underset{b^2}{d^2x}\ \underset{b}{dt}\ \underset{b^{-3}}{(\ B^2 - E^2\ )^{3/4}}$$

Conclusion

Momentum-space topology determines:

universality classes of quantum vacua

effective field theories in these quantum vacua

topological quantum phase transitions (Lifshitz, plateau, etc.)

quantization of Hall and spin-Hall conductivity

topological Chern-Simons & Wess-Zumino terms

quantum statistics of topological objects

spectrum of edge states & fermion zero modes on walls & quantum vortices

chiral anomaly & vortex dynamics, etc.

flat band & room-temperature superconductivity

superfuid phases ^{3}He serve as primer for topological matter:

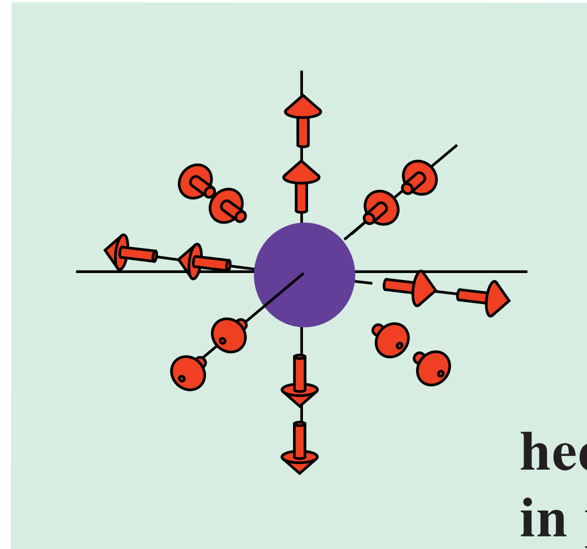
quantum vacuum of Standard Model, topological superconductors & topological insulators, etc.

we need: low T, high H, miniaturization, atomically smooth surface, nano-detectors, ...
and fabrication of samples with room-temperature surface superconductivity

4. From Fermi point to intrinsic QHE & topological insulators

$$N_3 = \frac{1}{8\pi} e_{ijk} \int dS^k \; \hat{\mathbf{g}} \cdot (\partial_{p_i} \hat{\mathbf{g}} \times \partial_{p_j} \hat{\mathbf{g}})$$

over 2D surface S
in 3D momentum space

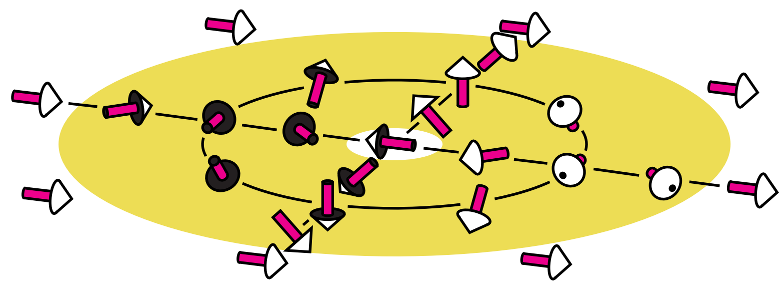


hedgehog
in p-space

3+1vacuum with Fermi point

dimensional reduction

Fully gapped 2+1 system



skyrmion
in p-space

$$\tilde{N}_3 = \frac{1}{4\pi} \int dp_x dp_y \; \hat{\mathbf{g}} \cdot (\partial_{p_x} \hat{\mathbf{g}} \times \partial_{p_y} \hat{\mathbf{g}})$$

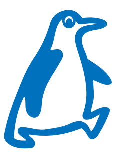
over the whole 2D momentum space
or over 2D Brillouin zone

topological insulators & gapped superconductors in 2+1

topological insulator =
bulk insulator
with topologically protected
gapless states on the boundary

topological gapped superconductor =
superconductor with gap in bulk
but with topologically protected
gapless states on the boundary

p-wave 2D superconductor (Sr_2RuO_4 ?), ^{3}He-A thin film,
CdTe/HgTe/Cd insulator quantum well, planar phase film



who protects gapless states?

generic example:

$$\mathrm{H} = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} \qquad p^2 = p_x^{\,2} + p_y^{\,2}$$

How to extract useful information on energy states from this Hamiltonian
without solving equation

$$\mathrm{H}\psi = E\psi$$

Topological invariant in momentum space

$$\mathrm{H} = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$p^2 = p_x{}^2 + p_y{}^2$$

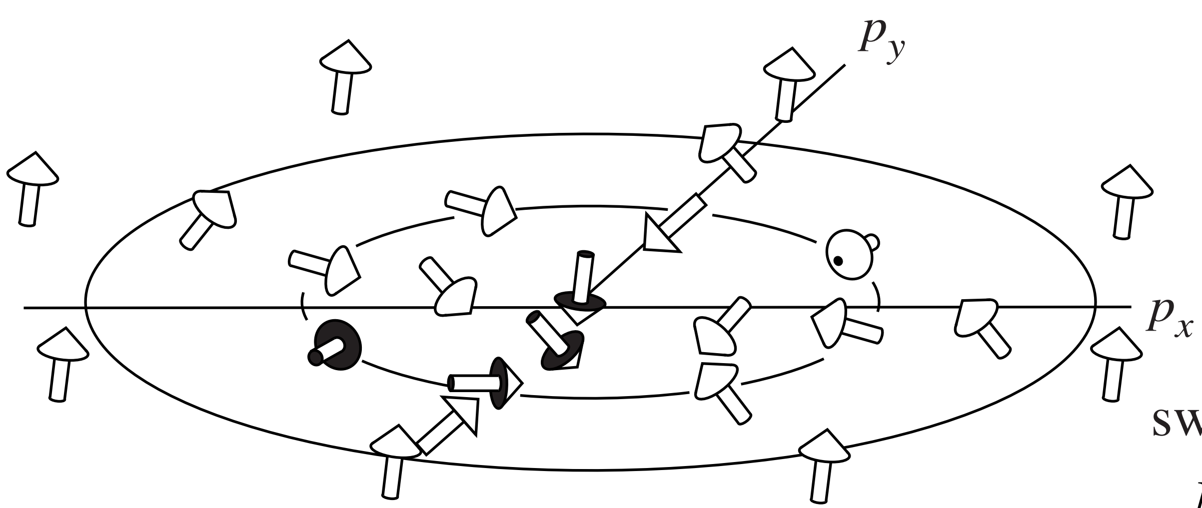
$$\mathrm{H} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \boldsymbol{\tau} \cdot \mathbf{g}(\mathbf{p})$$

fully gapped 2D state at $\mu \neq 0$

$$\boxed{\tilde{N}_3 = \frac{1}{4\pi} \int d^2p \; \hat{\mathbf{g}} \cdot (\partial_{p_x} \hat{\mathbf{g}} \times \partial_{p_y} \hat{\mathbf{g}})}$$

GV, JETP **67**, 1804 (1988)

Skyrmion (coreless vortex) in momentum space at $\mu > 0$

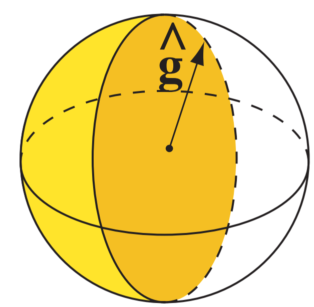


unit vector

$\hat{\mathbf{g}}(p_x, p_y)$

sweeps unit sphere

$\tilde{N}_3\ (\mu > 0) = 1$

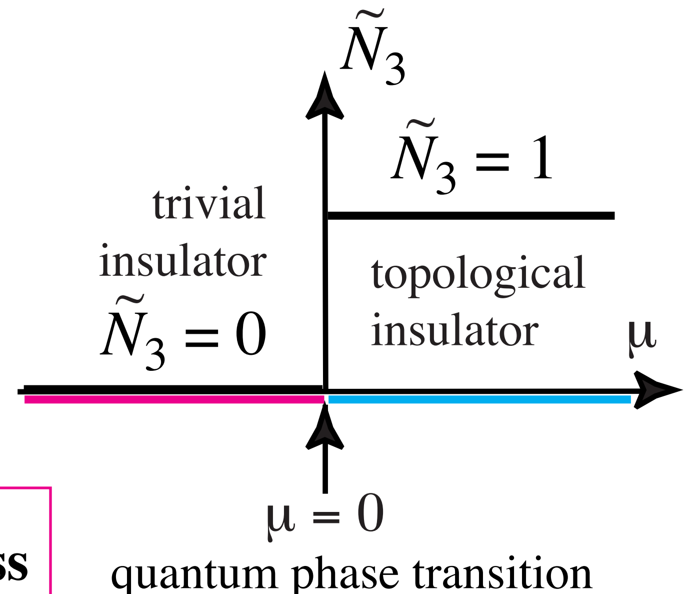


quantum phase transition:
from topological to non-topologicval insulator/superconductor

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \boldsymbol{\tau} \cdot \mathbf{g}(\mathbf{p})$$

Topological invariant in momentum space

$$\widetilde{N}_3 = \frac{1}{4\pi} \int d^2p \;\hat{\mathbf{g}} \cdot (\boldsymbol{\partial}_{p_x} \hat{\mathbf{g}} \times \boldsymbol{\partial}_{p_y} \hat{\mathbf{g}})$$



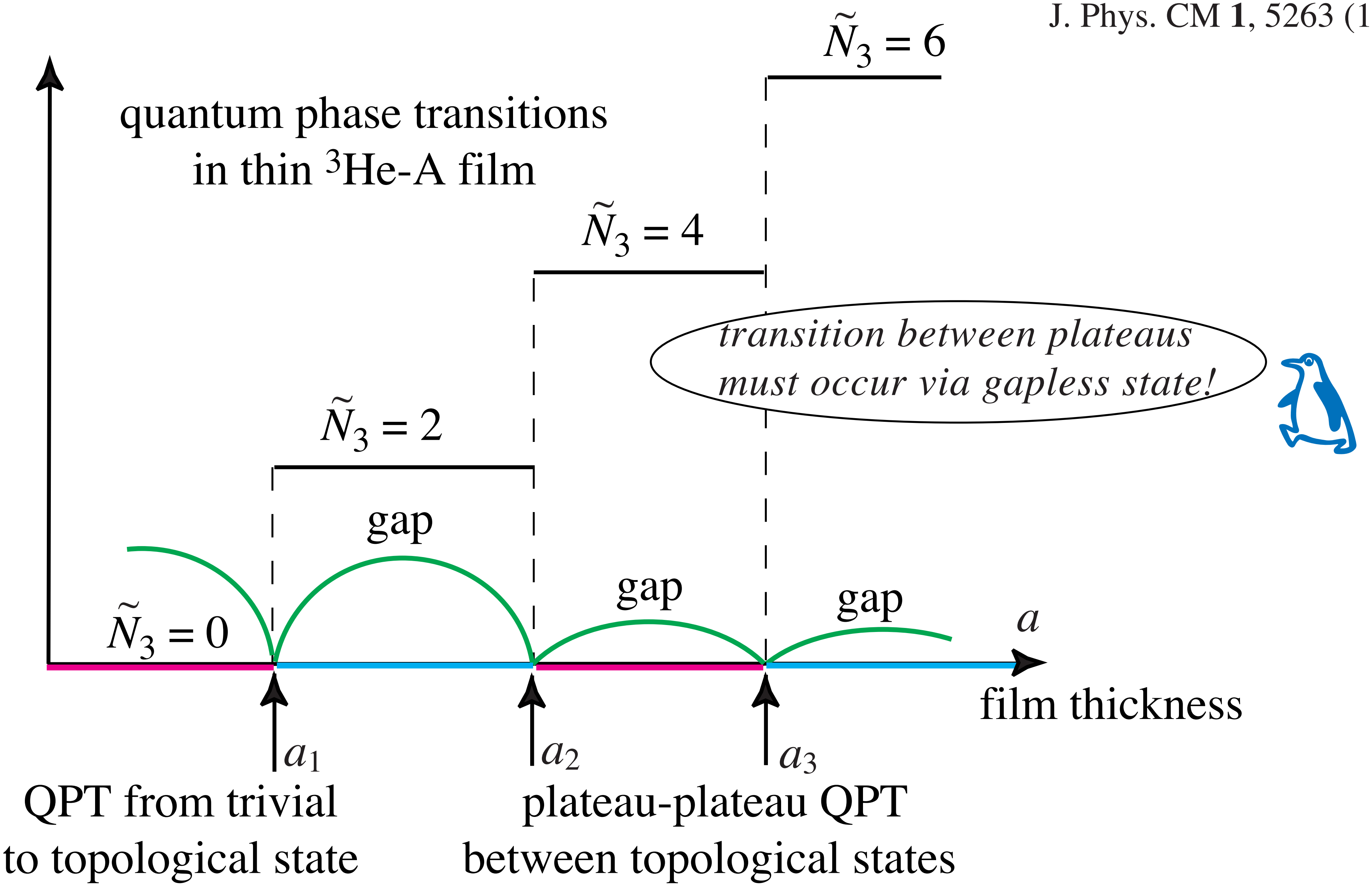
intermediate state at $\mu = 0$ must be gapless

$\Delta\widetilde{N}_3 \neq 0$ is origin of fermion zero modes
at the interface between states with different $\widetilde{N}_3$

p-space invariant in terms of Green's function & topological QPT

$$\tilde{N}_3 = \frac{1}{24\pi^2} e_{\mu\nu\lambda}\, \mathbf{tr} \int d^2p\, d\omega\, \mathbf{G}\, \partial^\mu \mathbf{G}^{-1}\, \mathbf{G}\, \partial^\nu \mathbf{G}^{-1} \mathbf{G}\, \partial^\lambda \mathbf{G}^{-1}$$

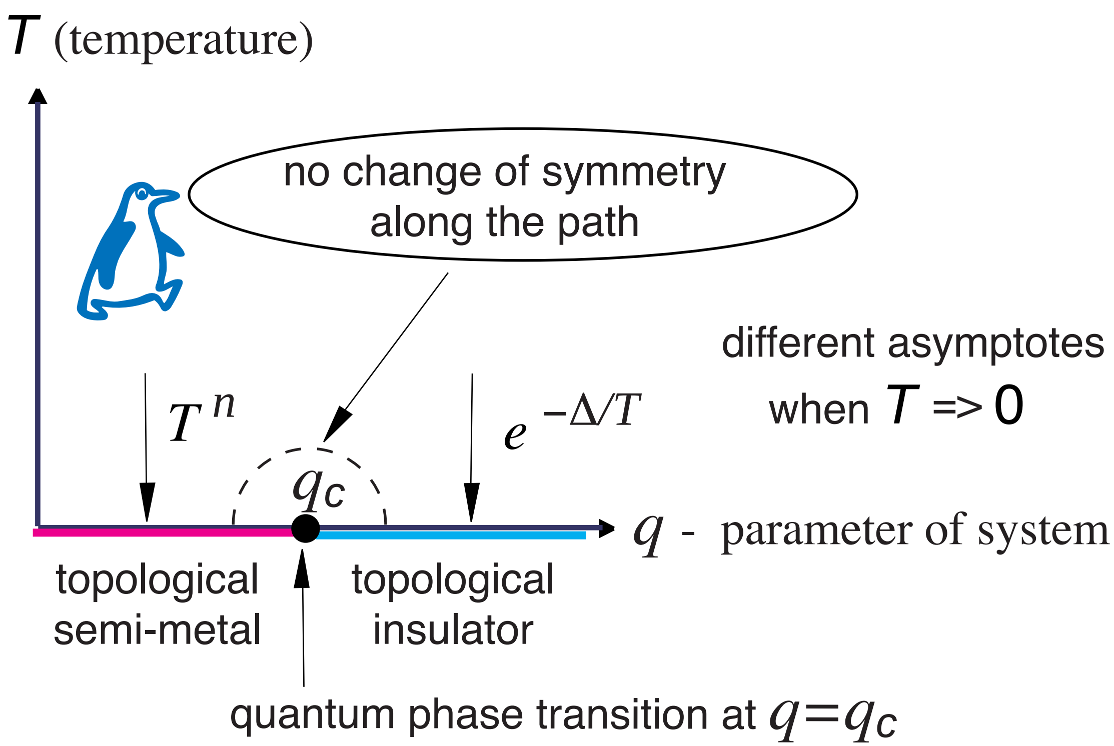
GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)



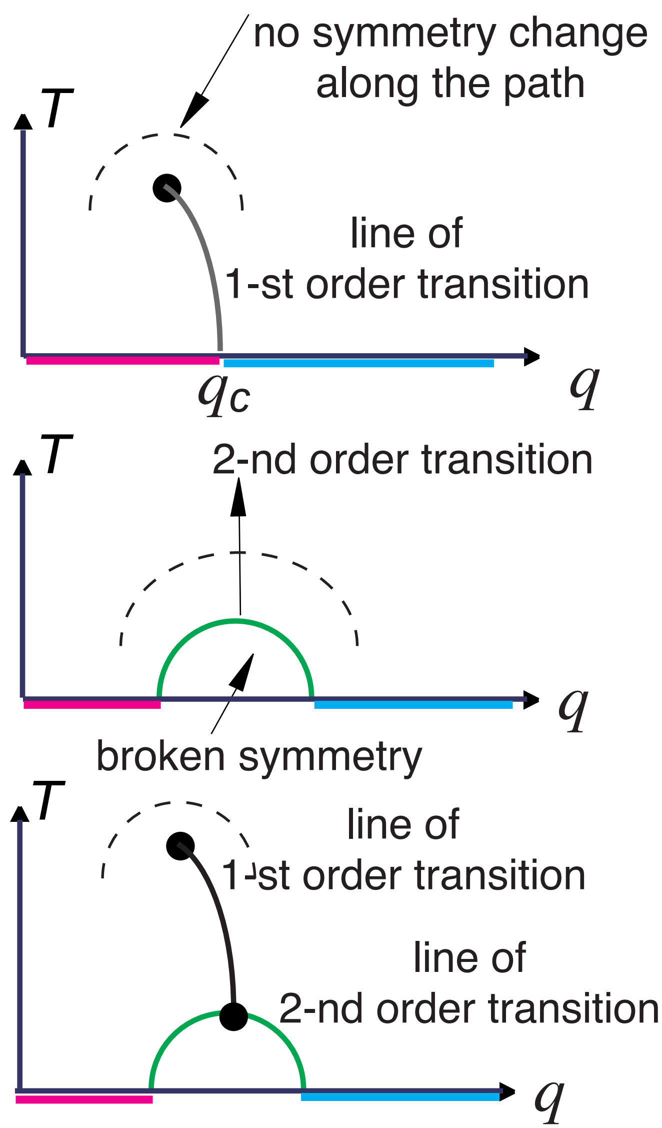
topological quantum phase transitions

transitions between **ground states (vacua)** of the **same symmetry**,
but **different topology** in **momentum space**

example: QPT between gapless & gapped matter



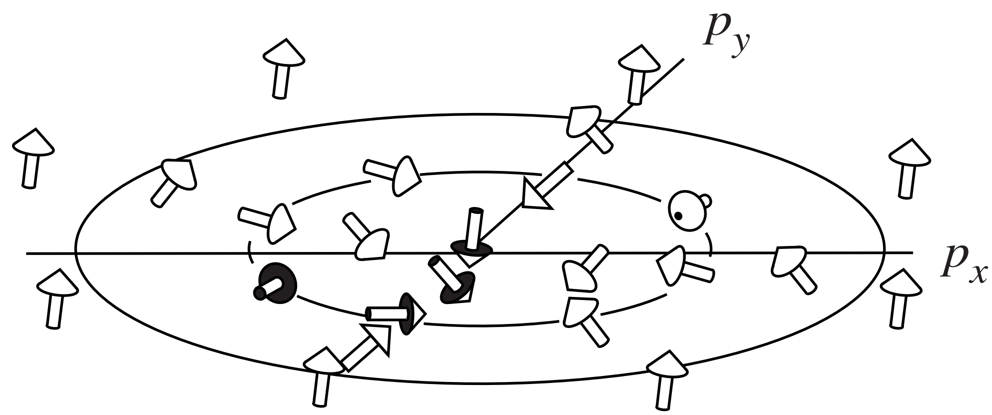
QPT interrupted by thermodynamic transitions



other topological QPT:
Lifshitz transition,
transtion between topological and nontopological superfluids,
plateau transitions,
confinement-deconfinement transition, ...

Zero energy states on surface of topological insulators & superfluids

Fully gapped 2+1 system

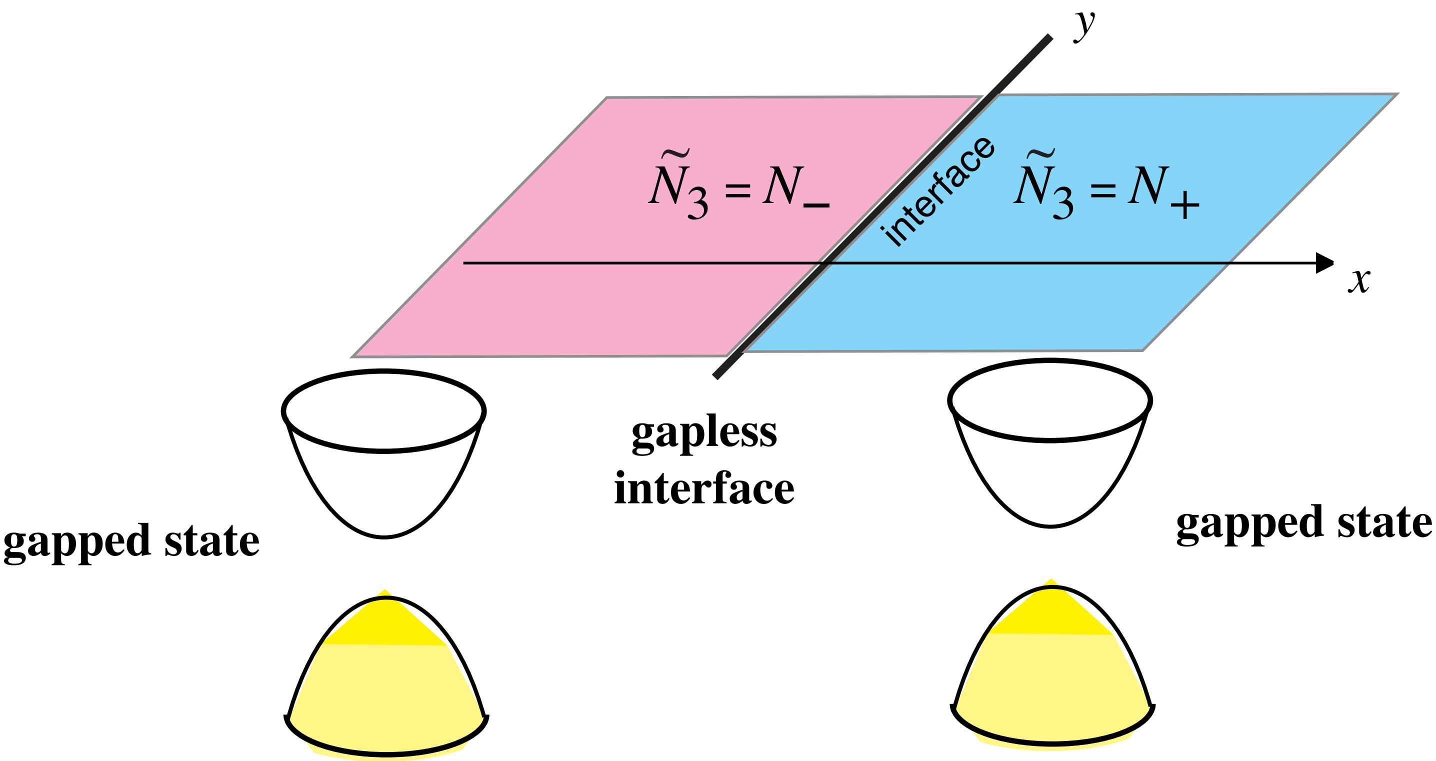


$$\tilde{N}_3 = \frac{1}{4\pi} \int dp_x dp_y \; \hat{\mathbf{g}} \cdot (\partial_{p_x} \hat{\mathbf{g}} \times \partial_{p_y} \hat{\mathbf{g}})$$

Fully gapped 3+1 system

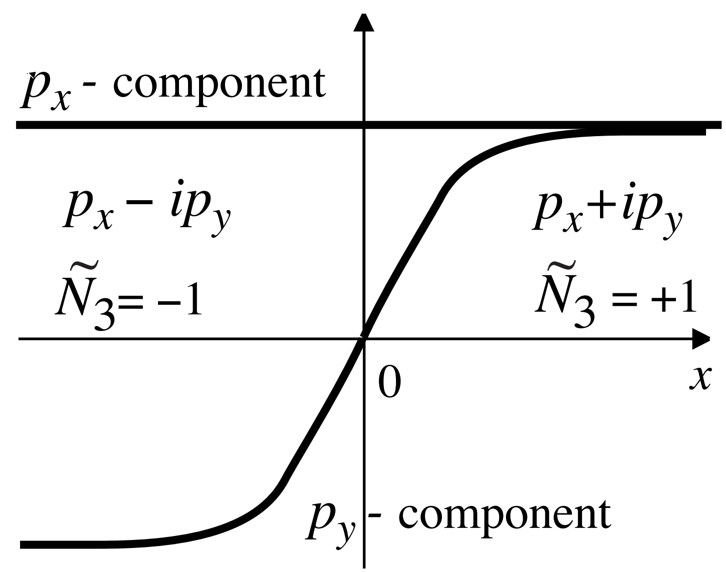
Majorana fermions on the surface and in the vortex cores

interface between two 2+1 topological insulators or gapped superfluids

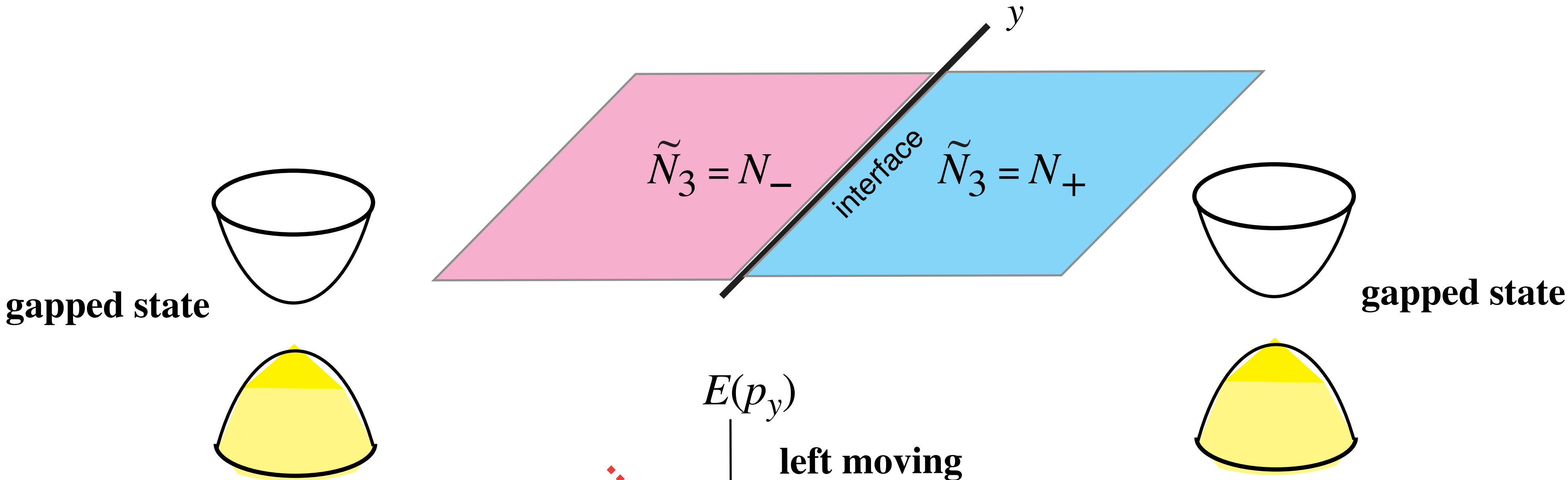


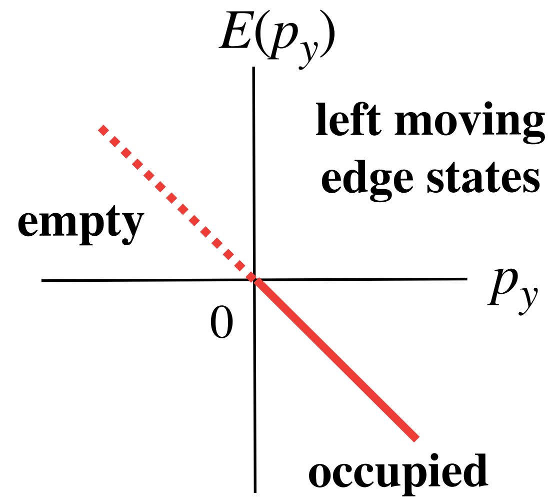
* domain wall in 2D chiral superconductors:

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + i\, p_y \tanh x) \\ c(p_x - i\, p_y \tanh x) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$



Edge states at interface between two 2+1 topological insulators or gapped superfluids



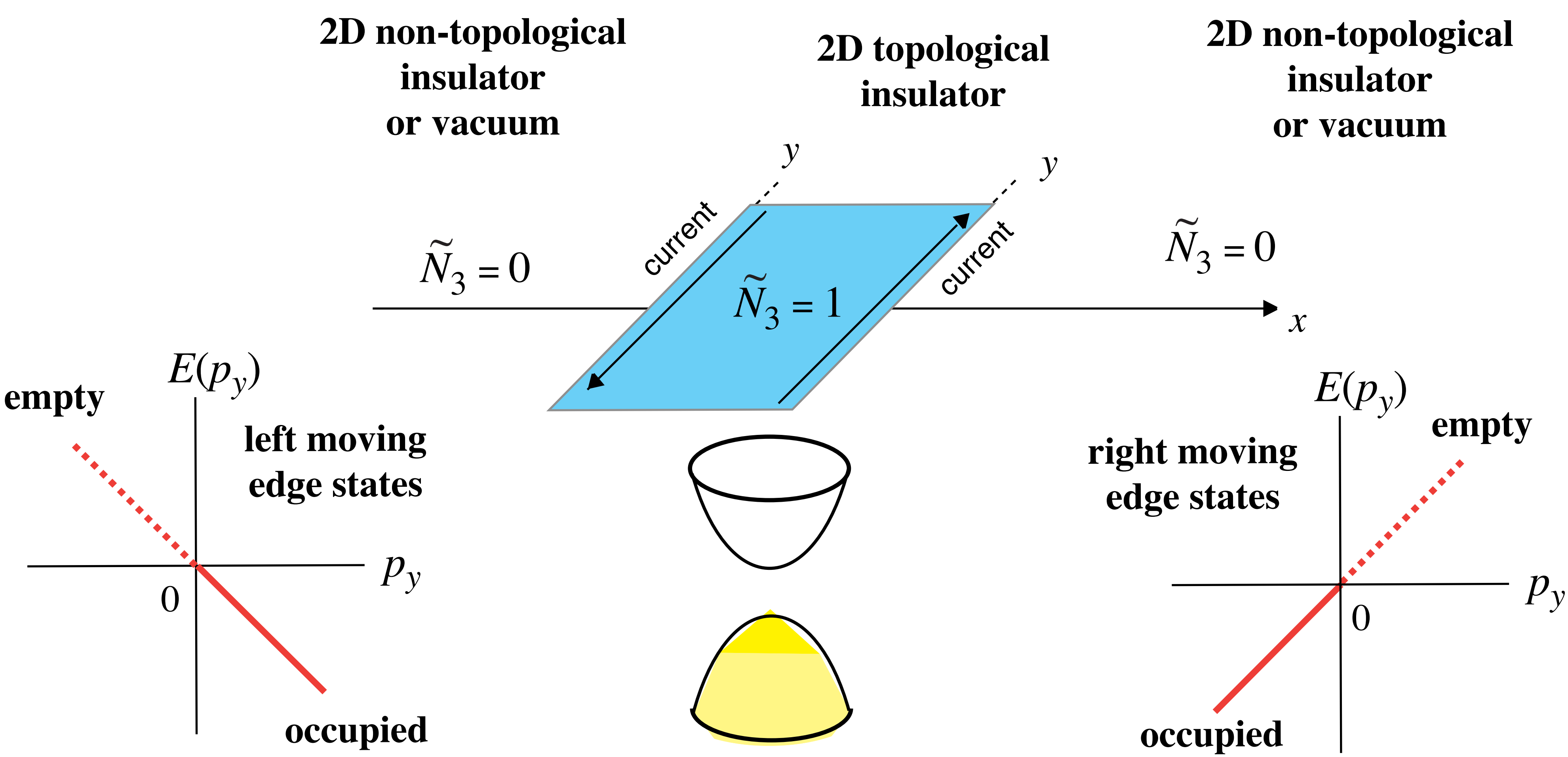


GV JETP Lett. **55**, 368 (1992)

Index theorem:
number of fermion zero modes
at interface:

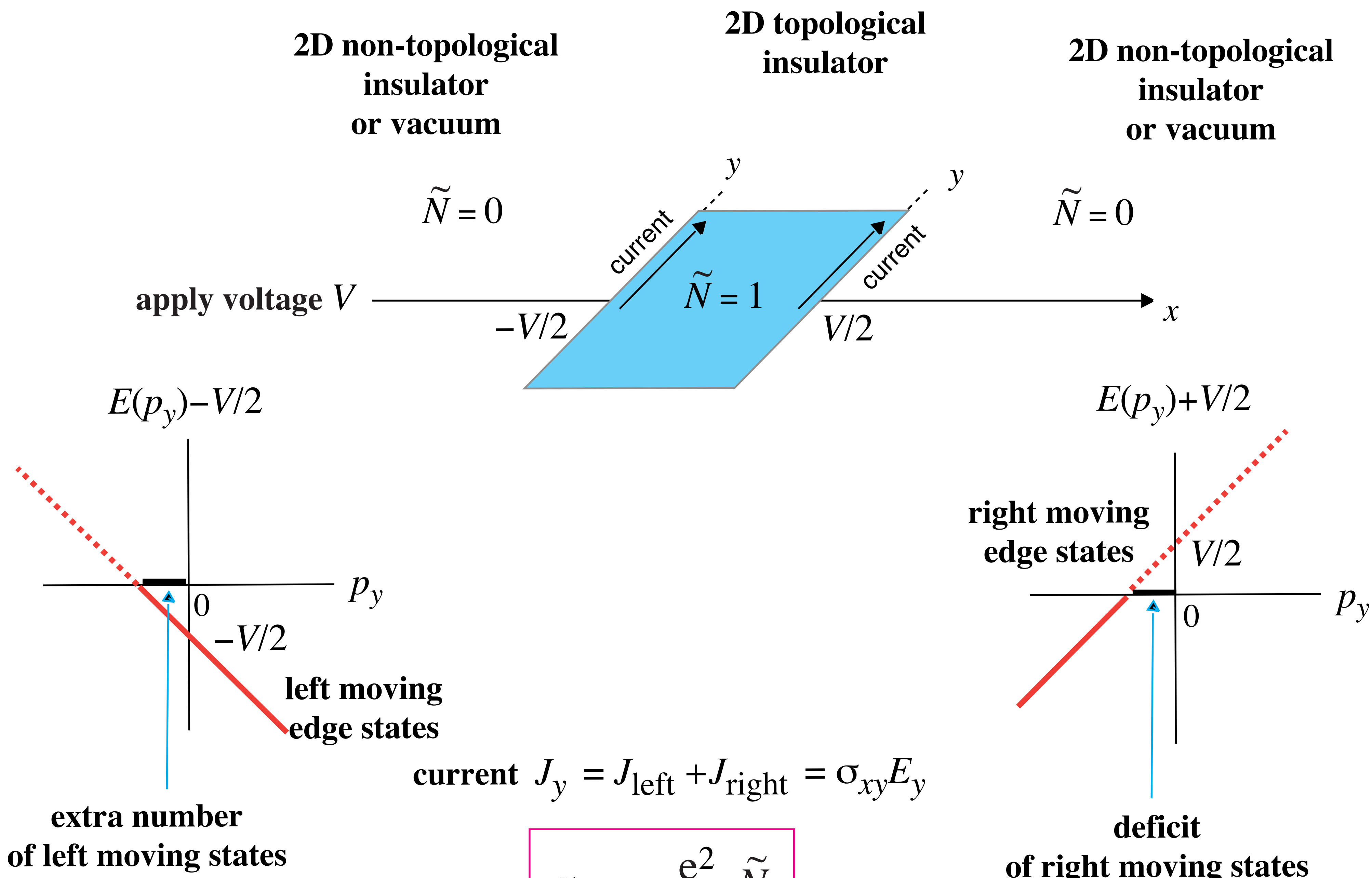
$$\nu = N_+ - N_-$$

Edge states and currents



current $J_y = J_{\text{left}} + J_{\text{right}} = 0$

Edge states & intrinsic QHE: topological invariant determines Hall quantization



$$\boxed{\sigma_{xy} = \frac{e^2}{4\pi} \tilde{N}}$$

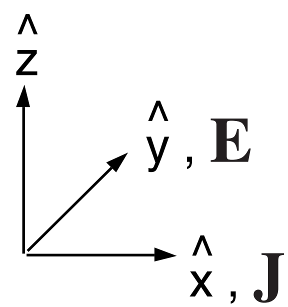
Intrinsic quantum Hall effect & momentum-space invariant

$$\boxed{S_{CS} = \frac{e^2}{16\pi} \tilde{N}_3 e^{\mu\nu\lambda} \int d^2x\, dt\, A_\mu F_{\nu\lambda}}$$

p-space invariant — **r**-space invariant

A_μ - electromagnetic field

electric current $J_x = \delta S_{CS} / \delta A_x = \dfrac{e^2}{4\pi} \tilde{N}_3 E_y$

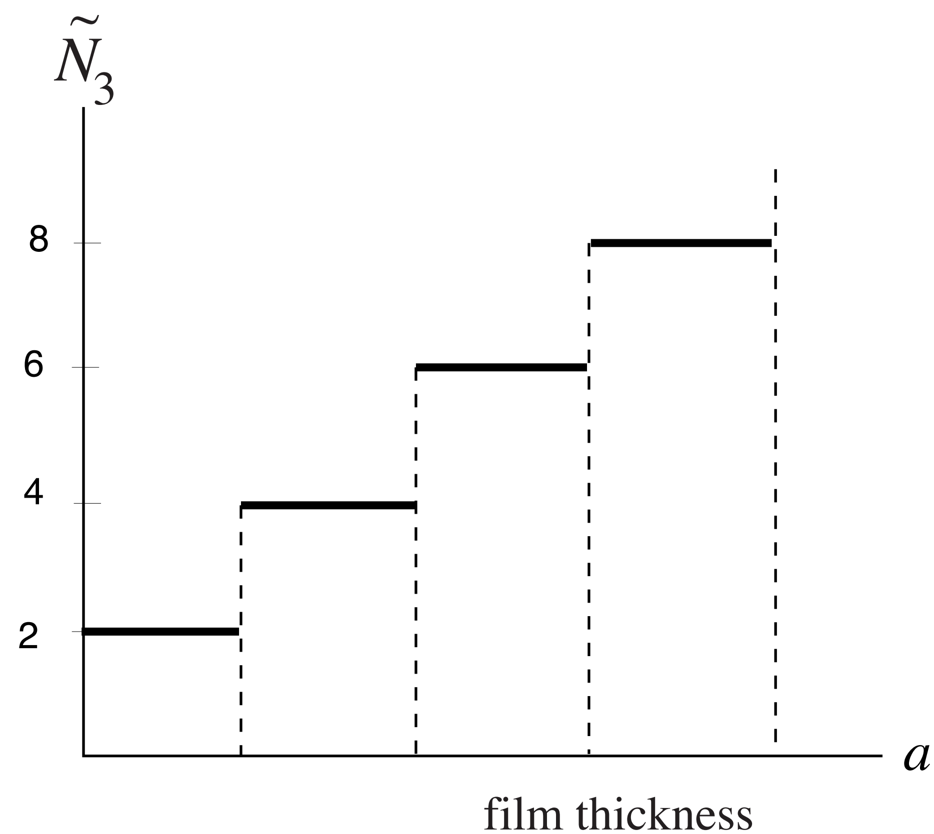


quantized intrinsic Hall conductivity
(without external magnetic field)

$$\boxed{\sigma_{xy} = \frac{e^2}{4\pi} \tilde{N}_3}$$

GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

film of topological quantum liquid



general Chern-Simons terms & momentum-space invariant

(interplay of ***r***-space and ***p***-space topologies)

$$S_{CS} = \frac{1}{16\pi}\ \tilde{N}_{3IJ}\, e^{\mu\nu\lambda} \int d^2x\, dt\, A_\mu^{\ I}\, F_{\nu\lambda}^{\ \ J}$$

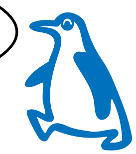
***r*-space invariant**

***p*-space invariant protected by symmetry**

$$\tilde{N}_{3IJ} = \frac{1}{24\pi^2} e_{\mu\nu\lambda}\, \mathbf{tr}\left[\int d^2p\, d\omega\, K_I K_J\, \mathbf{G}\, \boldsymbol{\partial}^\mu\, \mathbf{G}^{-1}\, \mathbf{G}\, \boldsymbol{\partial}^\nu\, \mathbf{G}^{-1} \mathbf{G}\, \boldsymbol{\partial}^\lambda\, \mathbf{G}^{-1}\right]$$

K_I - charge interacting with gauge field $A_\mu^{\ I}$

gauge fields can be real, artificial or auxiliary



$K = e$ for electromagnetic field A_μ

$K = \hat{\sigma}_z$ for effective spin-rotation field $A_\mu^{\ z}$ $\quad (A_0^{\ z} = \gamma H^{\ z})$

$$id/dt - \gamma\hat{\sigma}\cdot\mathbf{H} = id/dt - \hat{\sigma}\cdot\mathbf{A}_0$$

applied Pauli magnetic field plays the role of components of effective SU(2) gauge field $A_\mu^{\ i}$

Intrinsic spin-current quantum Hall effect & momentum-space invariant

spin current $J_x^z = \dfrac{1}{4\pi}\,(\gamma N_{ss}\, dH^z/dy + N_{se}\, E_y)$

spin-spin QHE

spin-charge QHE

2D singlet superconductor:

$$\boxed{\sigma_{xy}^{\mathrm{spin/spin}} = \frac{N_{ss}}{4\pi}}$$

s-wave: $N_{ss} = 0$

$p_x + ip_y$: $N_{ss} = 2$

$d_{xx-yy} + id_{xy}$: $N_{ss} = 4$

film of planar phase of superfluid ^{3}He

$$\boxed{\sigma_{xy}^{\mathrm{spin/charge}} = \frac{N_{se}}{4\pi}}$$

GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

planar phase film of 3He & 2D topological insulator

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + i\, p_y \sigma_z) \\ c(p_x - i\, p_y \sigma_z) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$\boxed{\tilde{N}_3 = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \mathbf{tr}\left[\int d^2p\, d\omega\, \mathbf{G}\, \boldsymbol{\partial}^\mu\, \mathbf{G}^{-1}\, \mathbf{G}\, \boldsymbol{\partial}^\nu\, \mathbf{G}^{-1} \mathbf{G}\, \boldsymbol{\partial}^\lambda\, \mathbf{G}^{-1}\right] = 0}$$

$$\boxed{\tilde{N}_{se} = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \mathbf{tr}\left[\int d^2p\, d\omega\, \sigma_z\, \mathbf{G}\, \boldsymbol{\partial}^\mu\, \mathbf{G}^{-1}\, \mathbf{G}\, \boldsymbol{\partial}^\nu\, \mathbf{G}^{-1} \mathbf{G}\, \boldsymbol{\partial}^\lambda\, \mathbf{G}^{-1}\right]}$$

$$\tilde{N}_3^+ = +1 \qquad \tilde{N}_3^- = -1$$

$$\tilde{N}_3 = \tilde{N}_3^+ + \tilde{N}_3^- = 0 \qquad \tilde{N}_{se} = \tilde{N}_3^+ - \tilde{N}_3^- = 2$$

spin quantum Hall effect

spin current $J_x^z = \frac{1}{4\pi} N_{se} E_y$ $\boxed{\text{spin-charge QHE}}$

$$\boxed{\sigma_{xy}^{\text{spin/charge}} = \frac{N_{se}}{4\pi}}$$

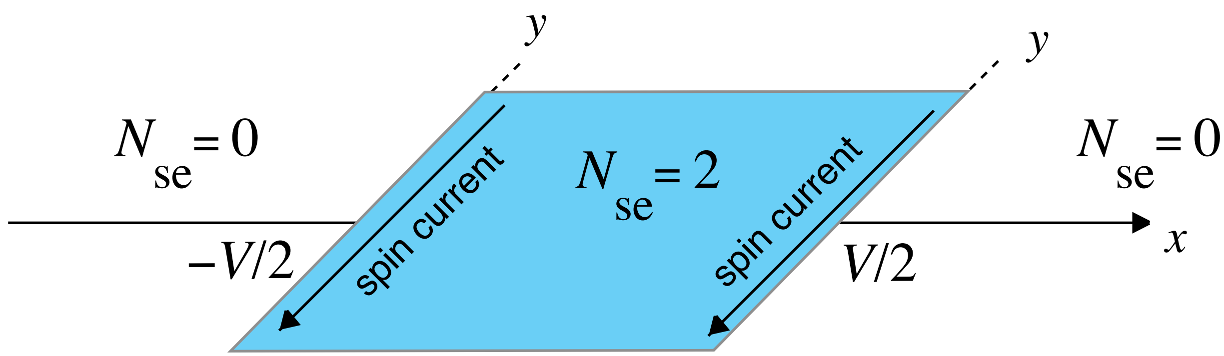
$N_{se} = 2$

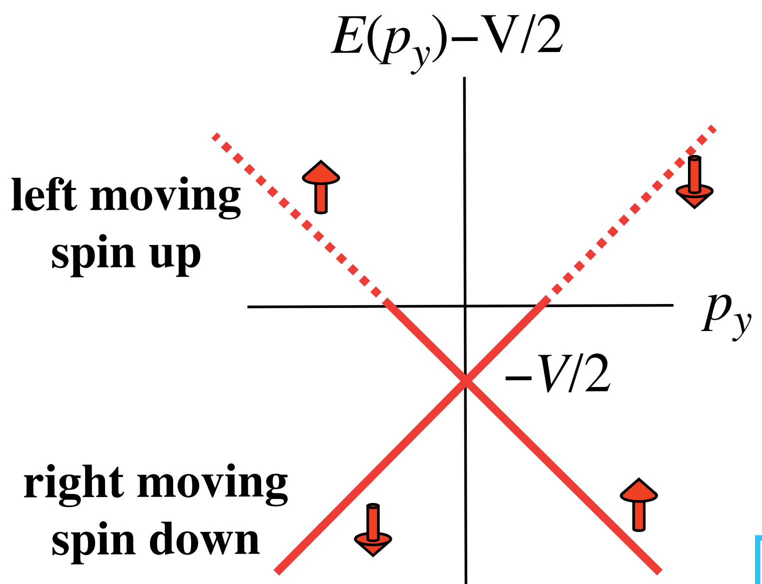
GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

Intrinsic spin-current quantum Hall effect & edge state

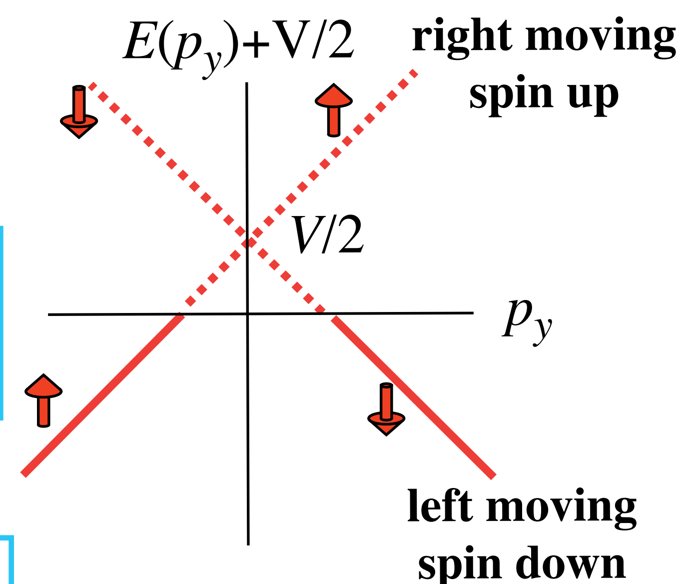
spin current $J_x^{\,z} = \dfrac{1}{4\pi}\left(\gamma N_{ss}\, dH^z/dy + N_{se}\, E_y\right)$

spin-charge QHE





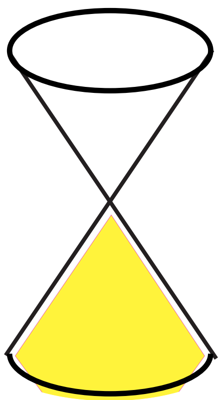
$$\sigma_{xy}^{\text{spin/charge}} = \frac{N_{se}}{4\pi}$$



electric current is zero
spin current is nonzero

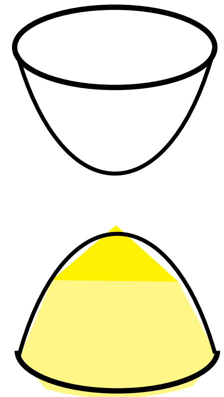
3D topological superfluids / insulators / semiconductors / vacua

gapless topologically
nontrivial vacua

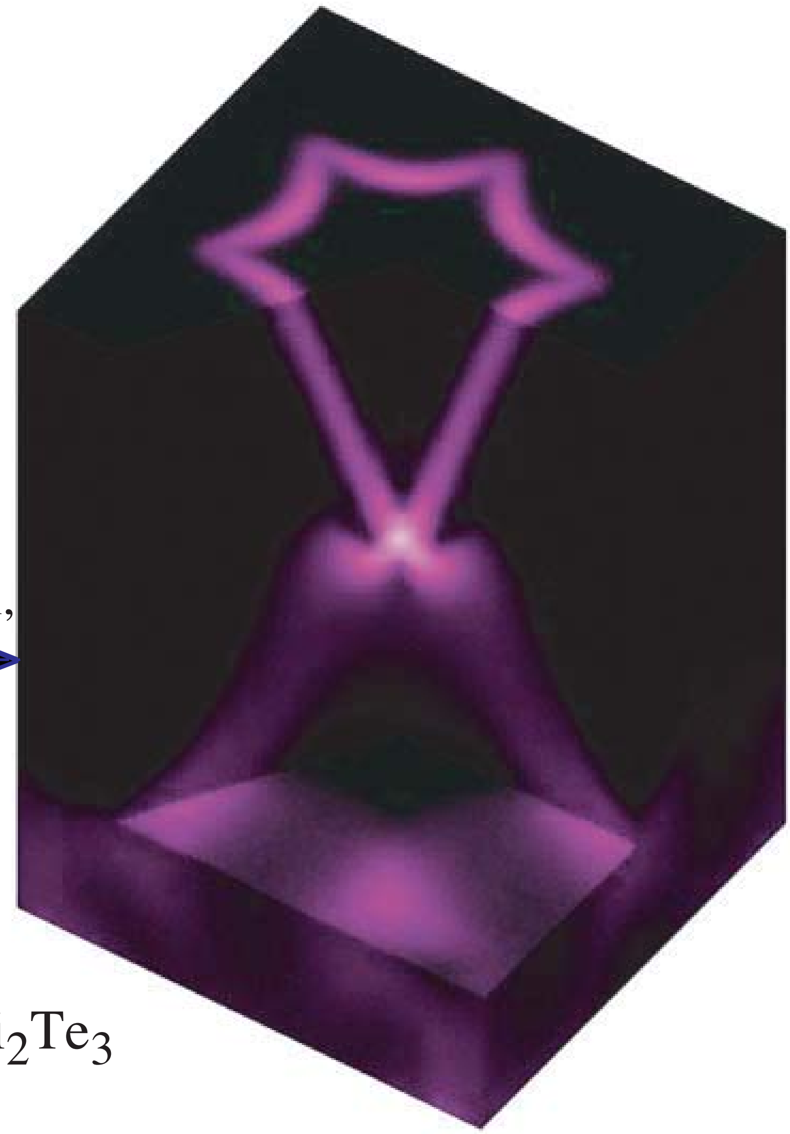


3He-A,
Standard Model
above electroweak transition,
semimetals,
4D graphene
(cryocrystalline vacuum)

fully gapped topologically
nontrivial vacua

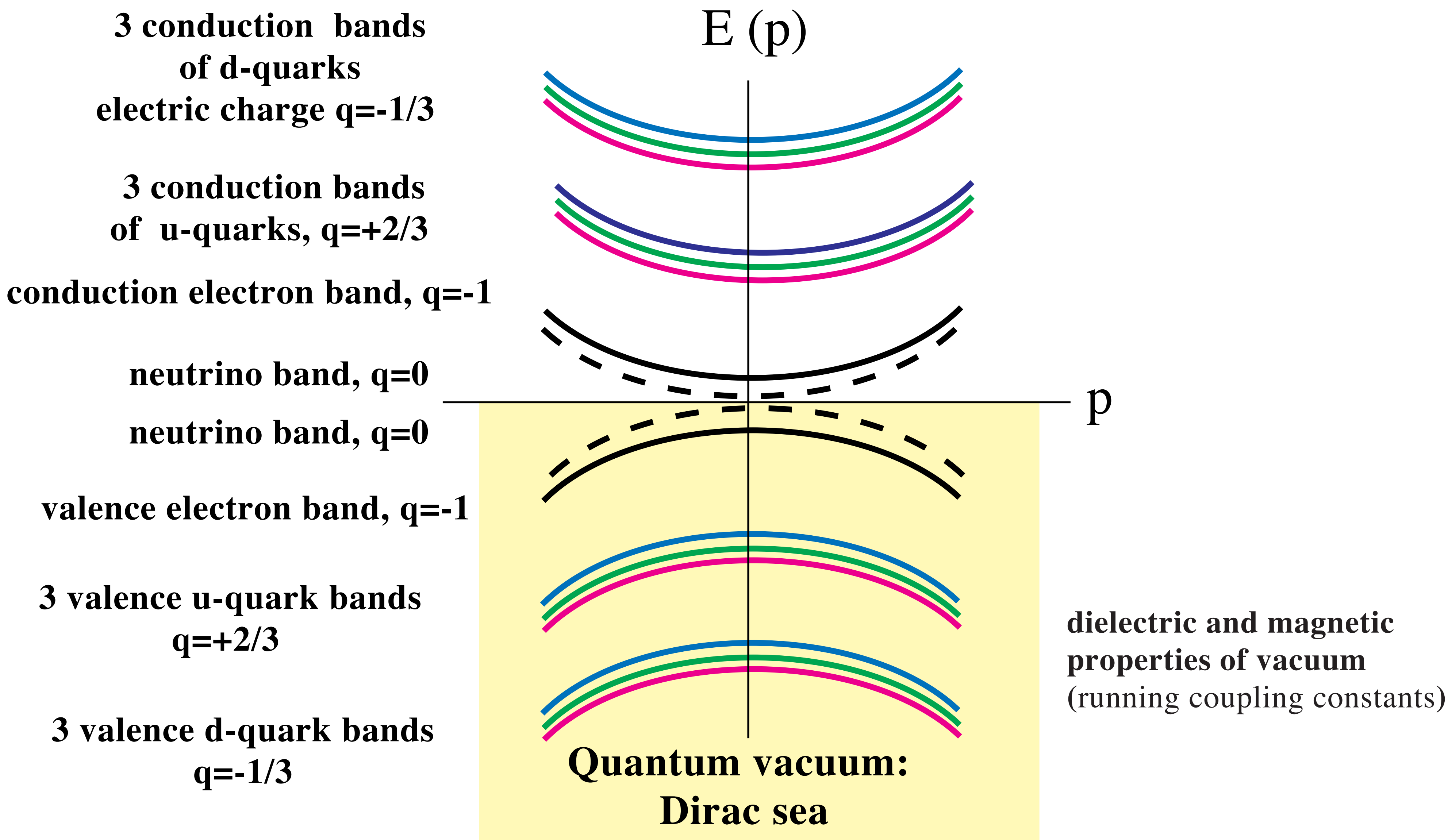


3He-B,
Standard Model
below electroweak transition,
topological insulators,
triplet & singlet
chiral superconductor, ...



Bi_2Te_3

Present vacuum as semiconductor or insulator



electric charge of quantum vacuum

$$Q = \sum_a q_a = N\,[-1 + 3\times(-1/3) + 3\times(+2/3)\,] = 0$$

fully gapped 3+1 topological matter

superfluid ^{3}He-B, topological insulator Bi_2Te_3 , present vacuum of Standard Model

*** Standard Model vacuum as topological insulator**

Topological invariant protected by symmetry

$$N_K = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \, \mathbf{tr} \int dV \, K \, \mathbf{G} \, \boldsymbol{\partial}^{\mu} \, \mathbf{G}^{-1} \, \mathbf{G} \, \boldsymbol{\partial}^{\nu} \, \mathbf{G}^{-1} \mathbf{G} \, \boldsymbol{\partial}^{\lambda} \, \mathbf{G}^{-1}$$

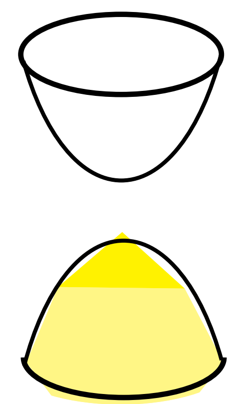
over 3D momentum space

$\mathbf{G}$ is Green's function at $\omega=0$, K is symmetry operator $\mathbf{GK} = +/- \, \mathbf{KG}$

Standard Model vacuum: $K=\gamma_5$ $\mathbf{G}\gamma_5 = -\gamma_5\mathbf{G}$

$$N_K = 8n_g$$

8 massive Dirac particles in one generation

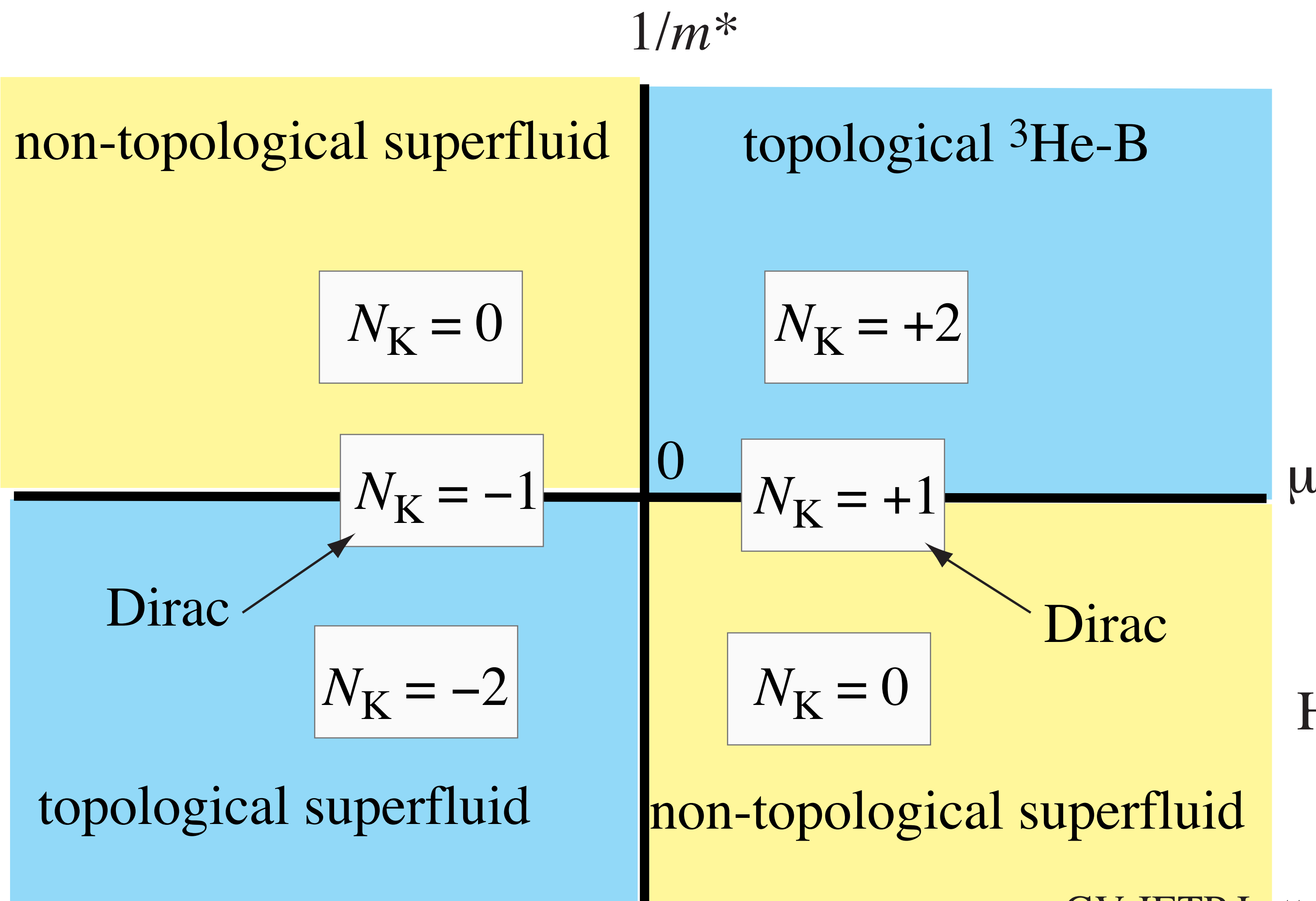


topological superfluid ^{3}He-B

$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu \end{pmatrix} = \left(\frac{p^2}{2m^*} - \mu\right)\tau_3 + c_B \sigma\cdot\mathbf{p}\,\tau_1$$

$$\mathbf{H}\tau_2 = -\tau_2\mathbf{H}$$

$$K = \tau_2$$



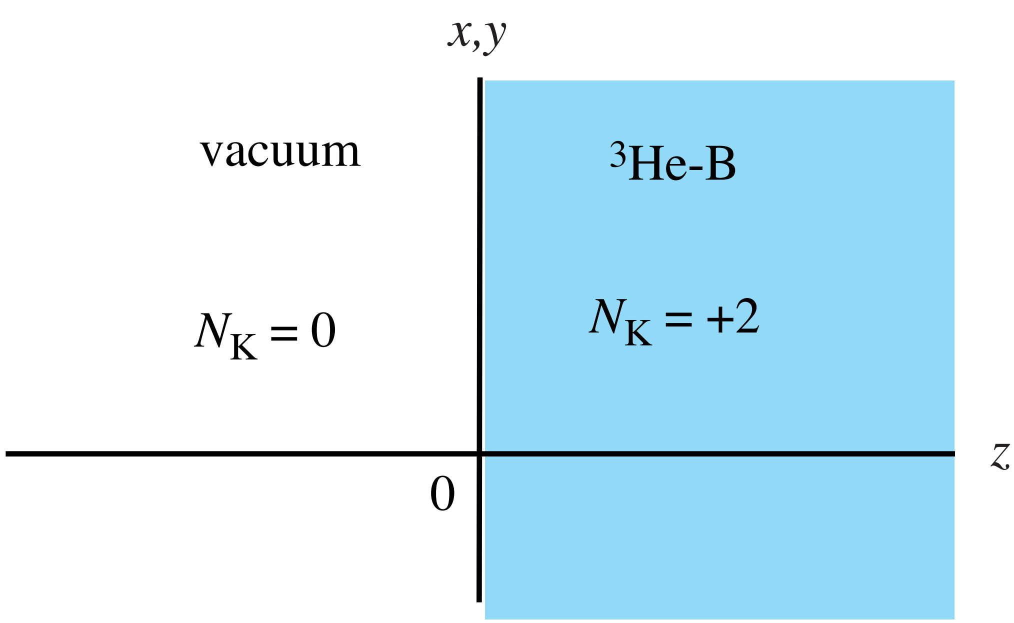
Dirac vacuum

$$1/m^* = 0$$

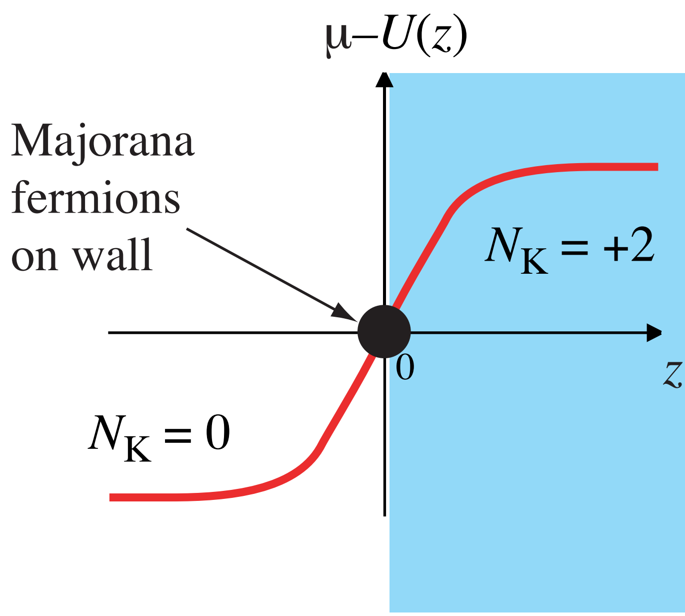
$$H = \begin{pmatrix} -M & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & +M \end{pmatrix}$$

GV JETP Lett. **90**, 587 (2009)

Boundary of 3D gapped topological superfluid



$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu + U(z) & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu - U(z) \end{pmatrix}$$



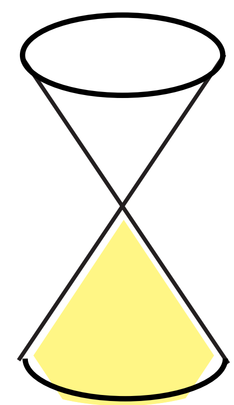
Majorana particle = Majorana anti-particle

1/2 of fermion: $b = b^+$

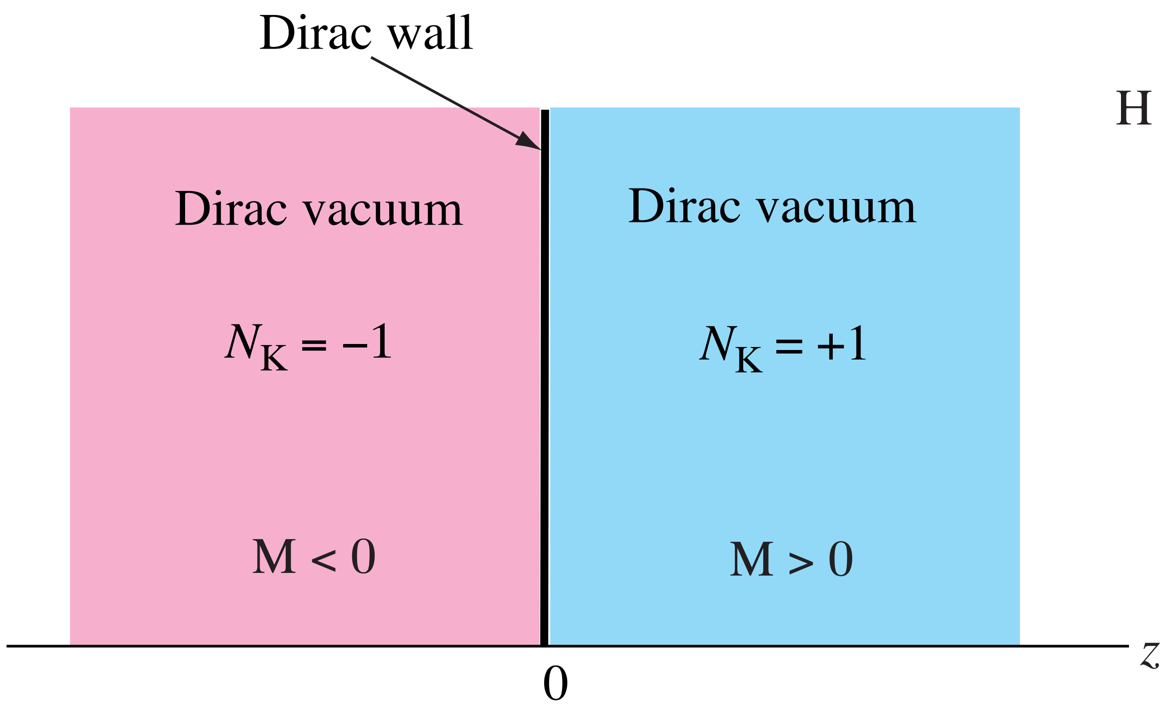
spectrum of Majorana zero modes

$$H_{zm} = c_B\, \hat{\mathbf{z}} \cdot \sigma \times \mathbf{p} = c_B\,(\sigma_x p_y - \sigma_y p_x)$$

helical fermions

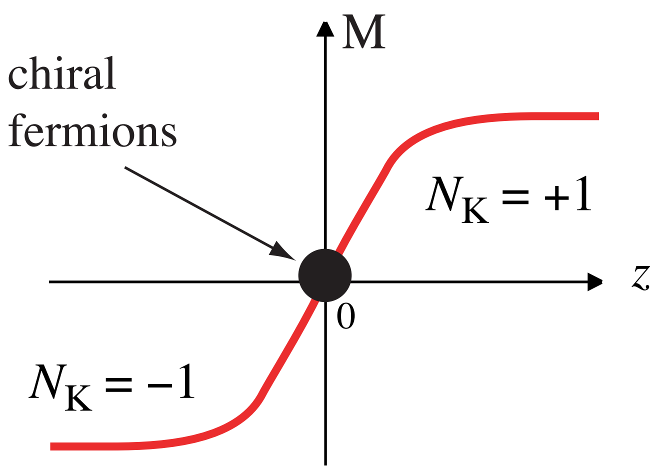


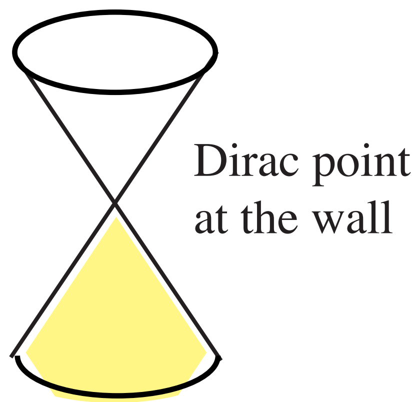
fermion zero modes on Dirac wall

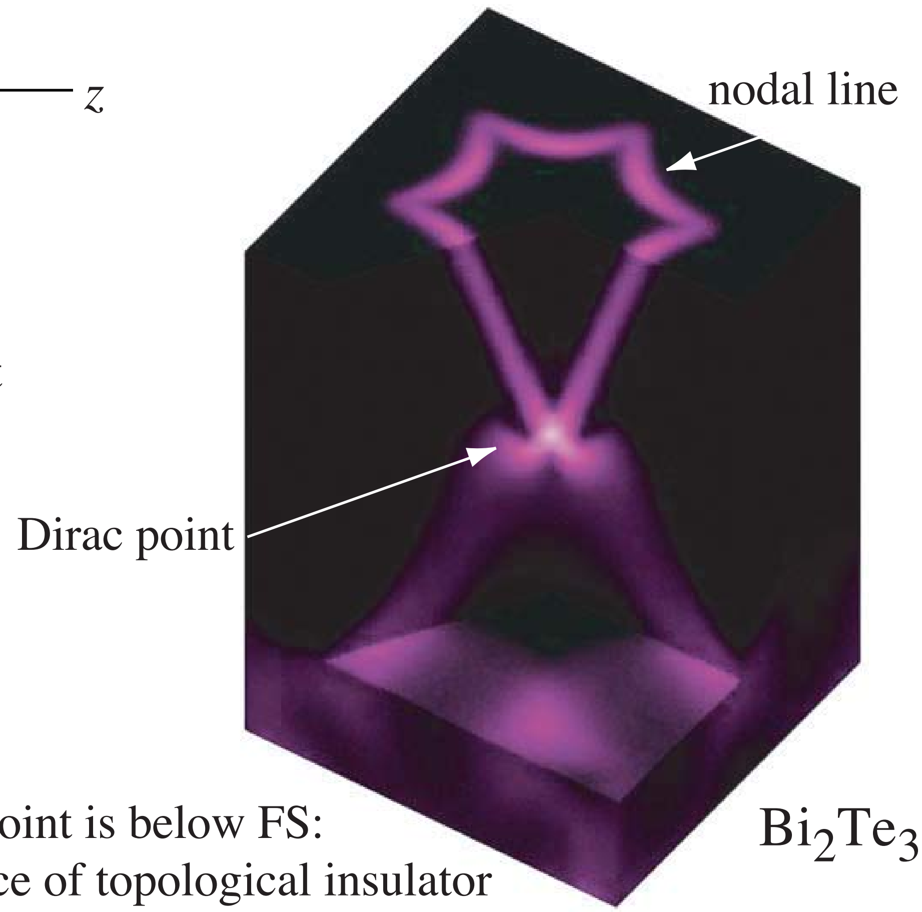


$$H = \begin{pmatrix} -M(z) & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & +M(z) \end{pmatrix}$$

Volkov-Pankratov,
2D massless fermions
in inverted contacts
JETP Lett. **42**, 178 (1985)





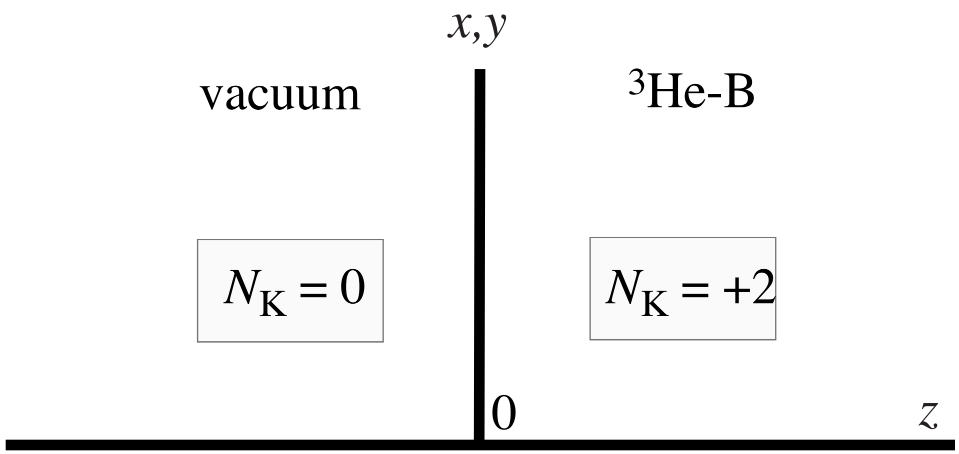


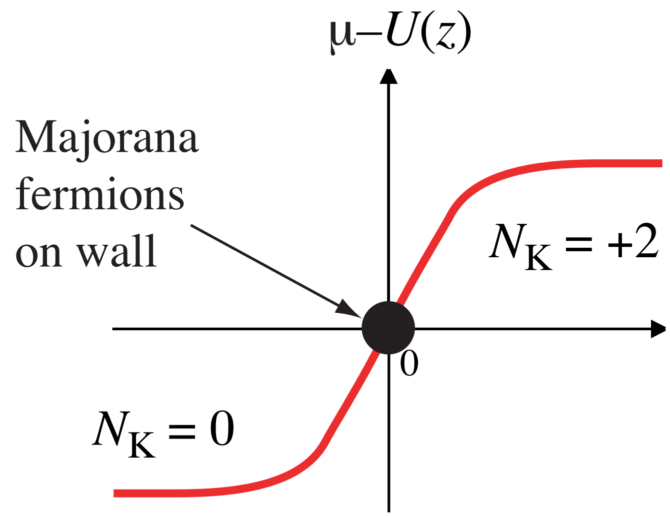
in Bi_2Te_3 Dirac point is below FS:
nodal line on surface of topological insulator

Majorana fermions: edge states
on the boundary of 3D gapped topological matter

$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu + U(z) & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu - U(z) \end{pmatrix}$$

* boundary of topological superfluid ^{3}He-B



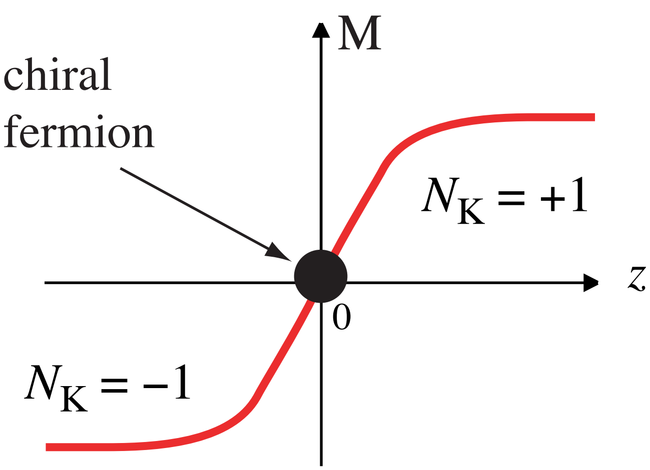


spectrum of fermion zero modes

$$H_{zm} = c\,(\sigma_x p_y - \sigma_y p_x)$$

helical fermions

* Dirac domain wall

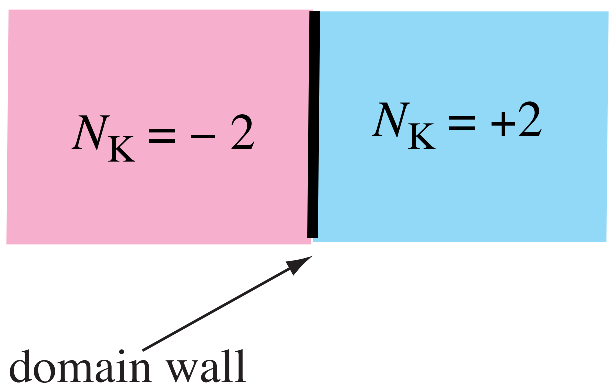


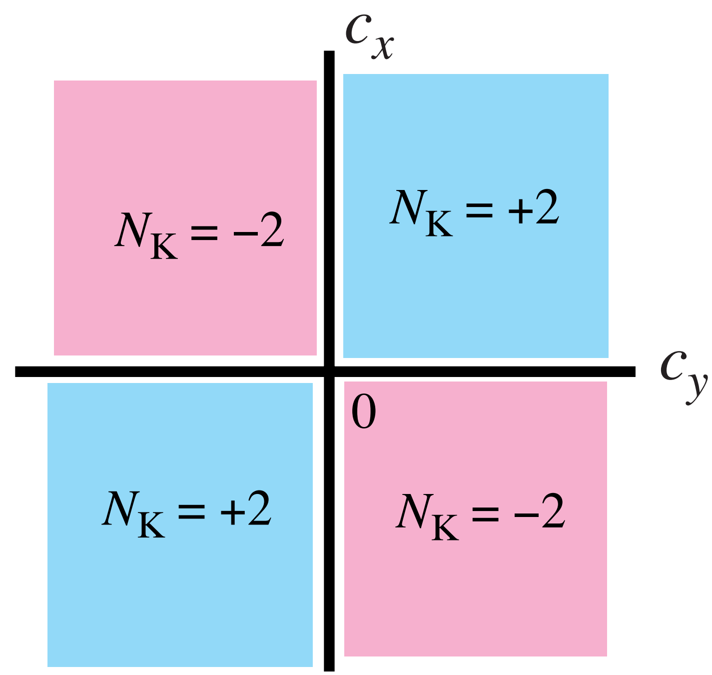
$$H = \begin{pmatrix} -M(z) & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & +M(z) \end{pmatrix}$$

Volkov-Pankratov,
2D massless fermions
in inverted contacts
JETP Lett. **42**, 178 (1985)

Majorana fermions on interface in topological superfluid ^{3}He-B

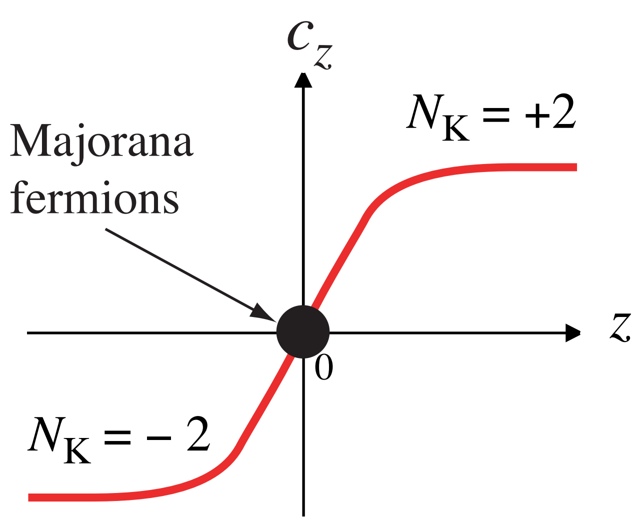
$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & \sigma_x c_x p_x + \sigma_y c_y p_y + \sigma_z c_z p_z \\ \sigma_x c_x p_x + \sigma_y c_y p_y + \sigma_z c_z p_z & -\frac{p^2}{2m^*} + \mu \end{pmatrix}$$

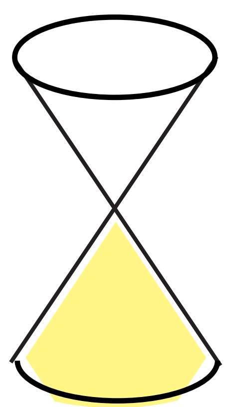




phase diagram

one of 3 "speeds of light" changes sign across wall





spectrum of fermion zero modes

$$H_{zm} = c\,(\sigma_x p_y - \sigma_y p_x)$$

Zero energy states in the core of vortices in topological superfluids

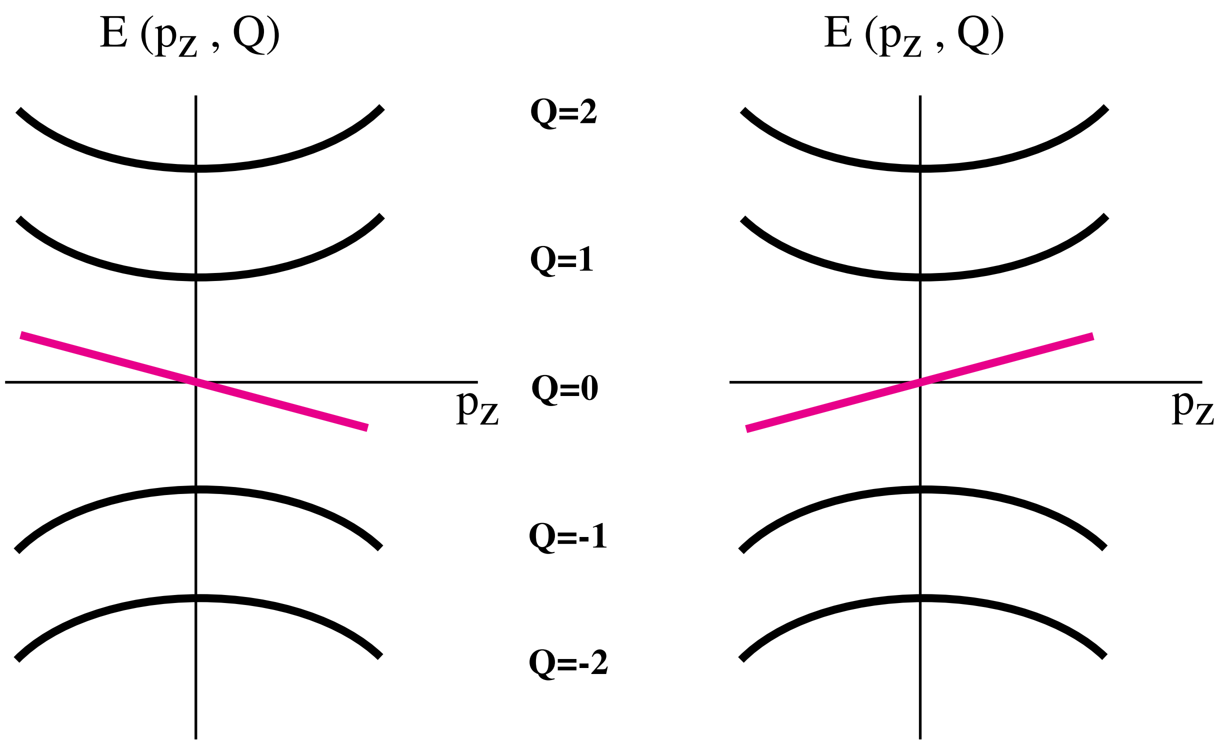
vortices in fully gapped 3+1 system

fermion zero modes in vortex core

Bound states of fermions on cosmic strings and vortices

Spectrum of quarks in core of electroweak cosmic string

quantum numbers: Q - angular momentum & p_z - linear momentum



$E(p_z) = - cp_z$ for d quarks $E(p_z) = cp_z$ for u quark

asymmetric branches cross zero energy

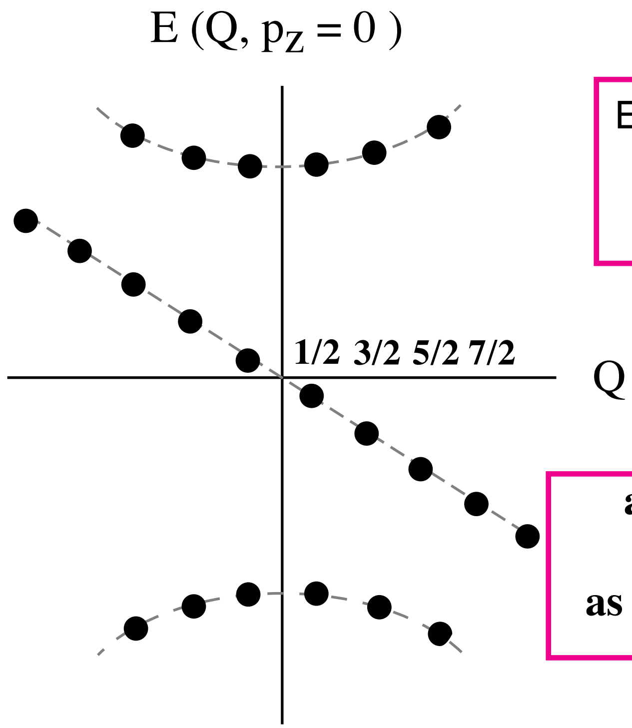
Index theorem: $\boxed{\text{Number of asymmetric branches = N; N is vortex winding number}}$

Jackiw & Rossi
Nucl. Phys. B**190**, 681 (1981)

Bound states of fermions on vortex in s-wave superconductor

Caroli, de Gennes & J. Matricon, Phys. Lett. **9** (1964) 307

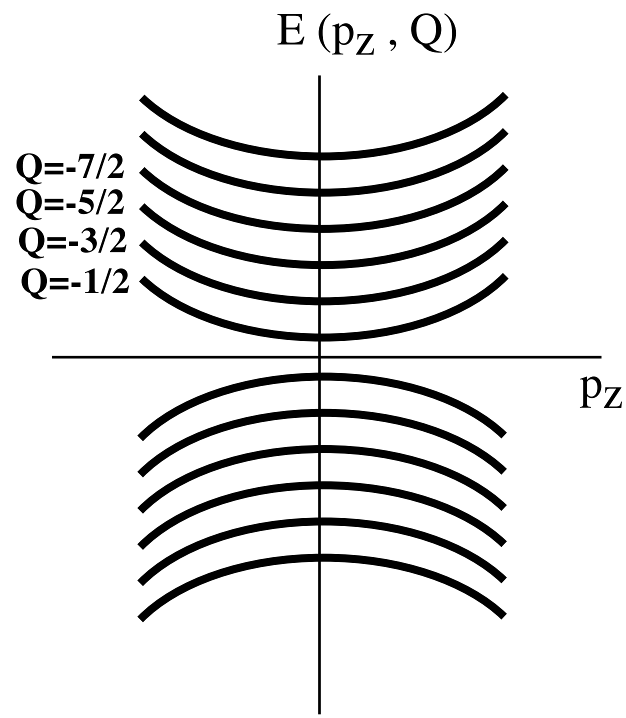
$N_K = 0$



$$E(Q,p_z) = -Q\,\omega_0\,(p_z)$$

$$\omega_0 = \Delta^2/E_F \ll \Delta$$

asymmetric branch as function of Q



Angular momentum Q is half-odd integer
in s-wave superconductor

no true fermion zero modes:
no asymmetric branch as function of p_Z

Index theorem for approximate fermion zero modes:

Number of asymmetric Q-branches = 2N
N is vortex winding number

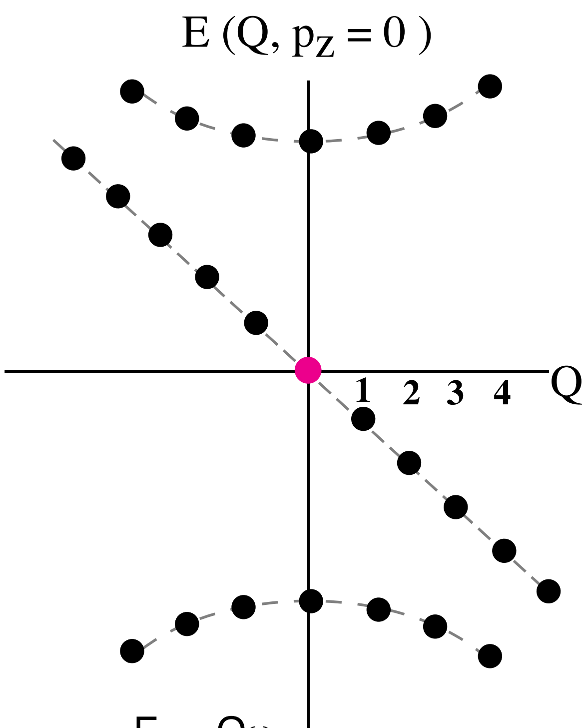
GV JETP Lett. **57**, 244 (1993)

Index theorem for true fermion zero modes?

is the existence of fermion zero modes
related to topology in bulk?

fermions zero modes on symmetric vortex in 3He-B

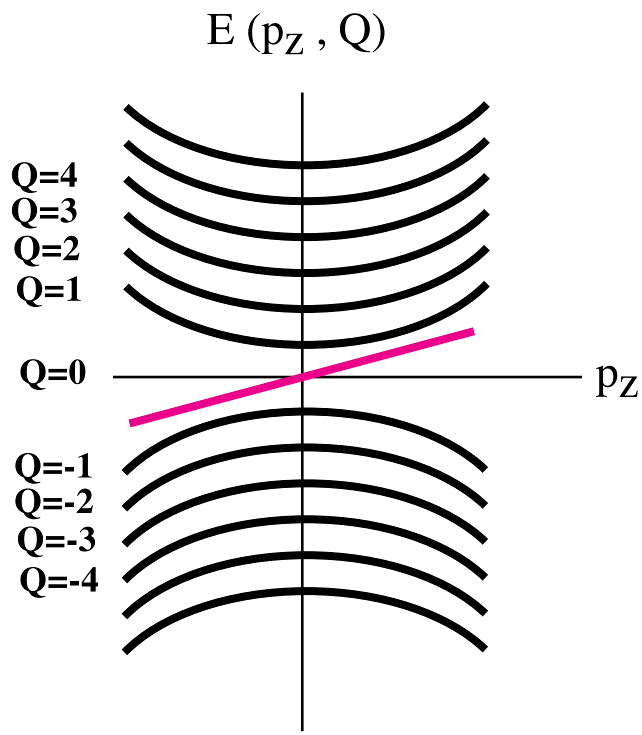
topological ^{3}He-B at $\mu > 0$: $N_K = 2$



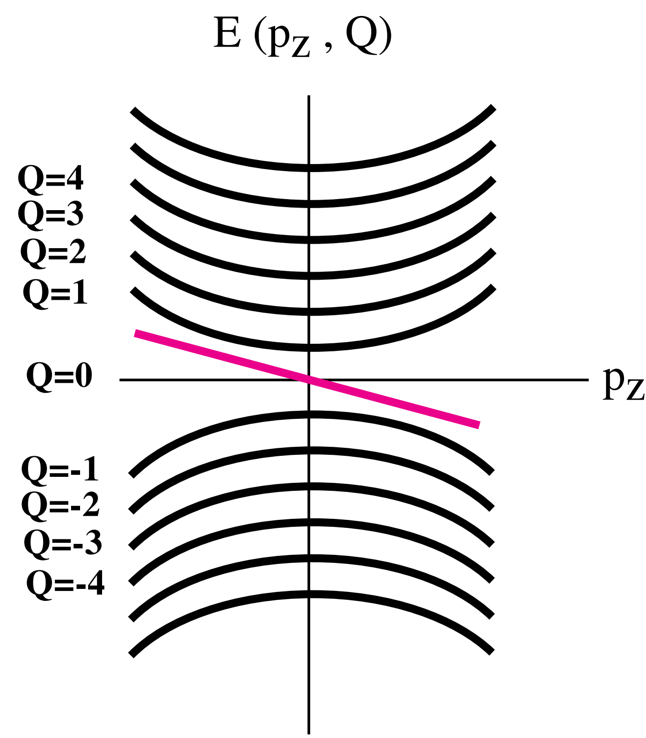
$E_Q = -Q\omega_0$

$\omega_0 = \Delta^2/E_F << \Delta$

Q is integer
for p-wave superfluid ^{3}He-B



helicity +



helicity –

gapless fermions on Q=0 branch form

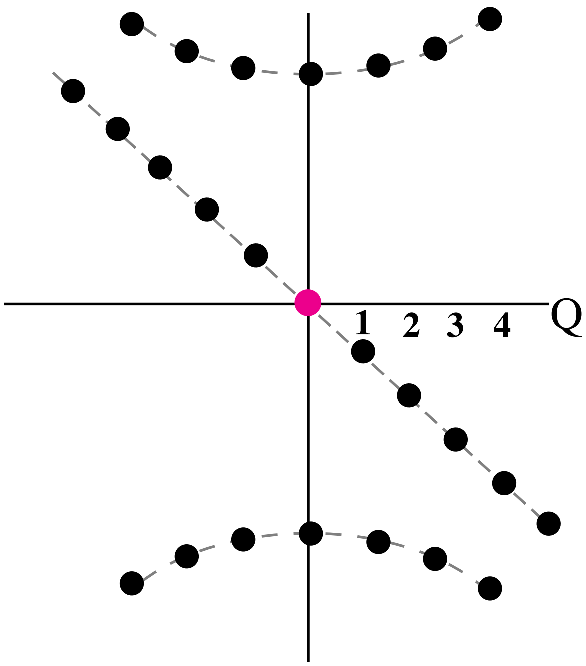
$\boxed{\text{1D Fermi-liquid}}$

Misirpashaev & GV
Fermion zero modes in symmetric vortices in superfluid 3He,
Physica B **210**, 338 (1995)

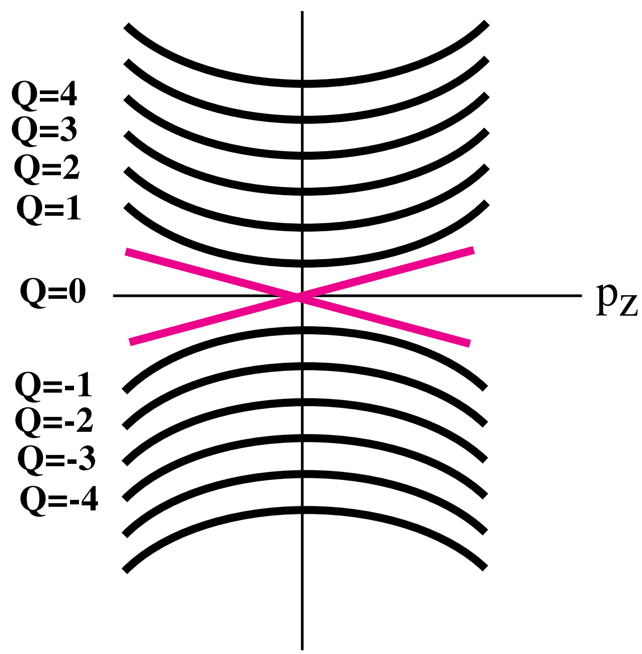
fermions zero modes on symmetric vortex in 3He-B

topological ^{3}He-B at $\mu > 0$: $\boxed{N_K = 2}$

$E\,(Q, p_z = 0\,)$



$E\,(p_z\,,\,Q)$



$E_Q = -Q\omega_0$

$\omega_0 = \Delta^2/E_F \ll \Delta$

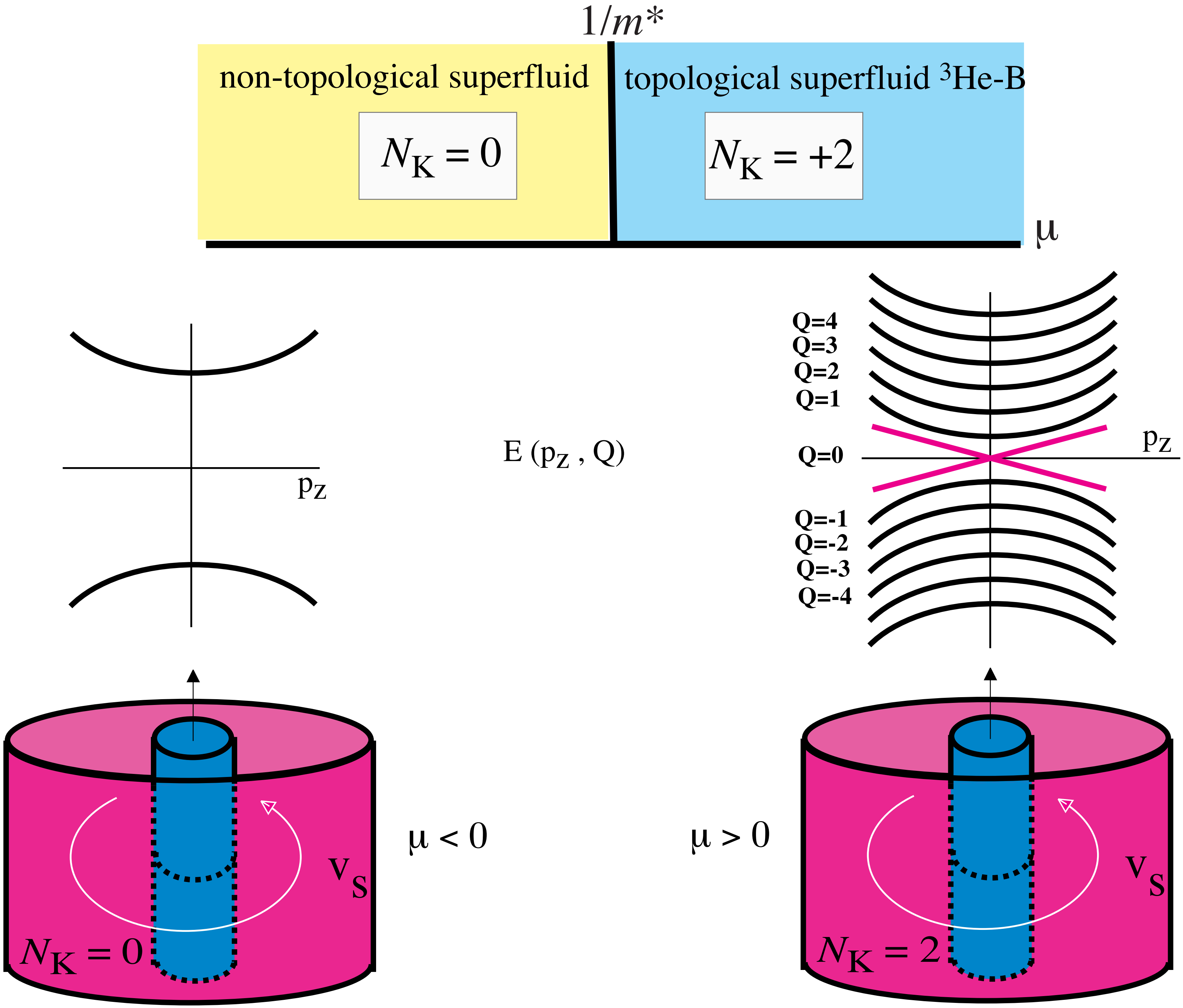
Q is integer
for p-wave superfluid ^{3}He-B

gapless fermions on Q=0 branch form

$\boxed{\text{1D Fermi-liquid}}$

Misirpashaev & GV
Fermion zero modes in symmetric vortices in superfluid 3He,
Physica B **210**, 338 (1995)

topological quantum phase transition in bulk & in vortex core



superfluid ^{3}He-B as non-relativistic limit of relativistic triplet superconductor

$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5 \Delta \\ \gamma_5 \Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$

relativistic triplet superconductor

$$\Big\downarrow \quad \begin{matrix} cp << M \\ \mu << M \end{matrix}$$

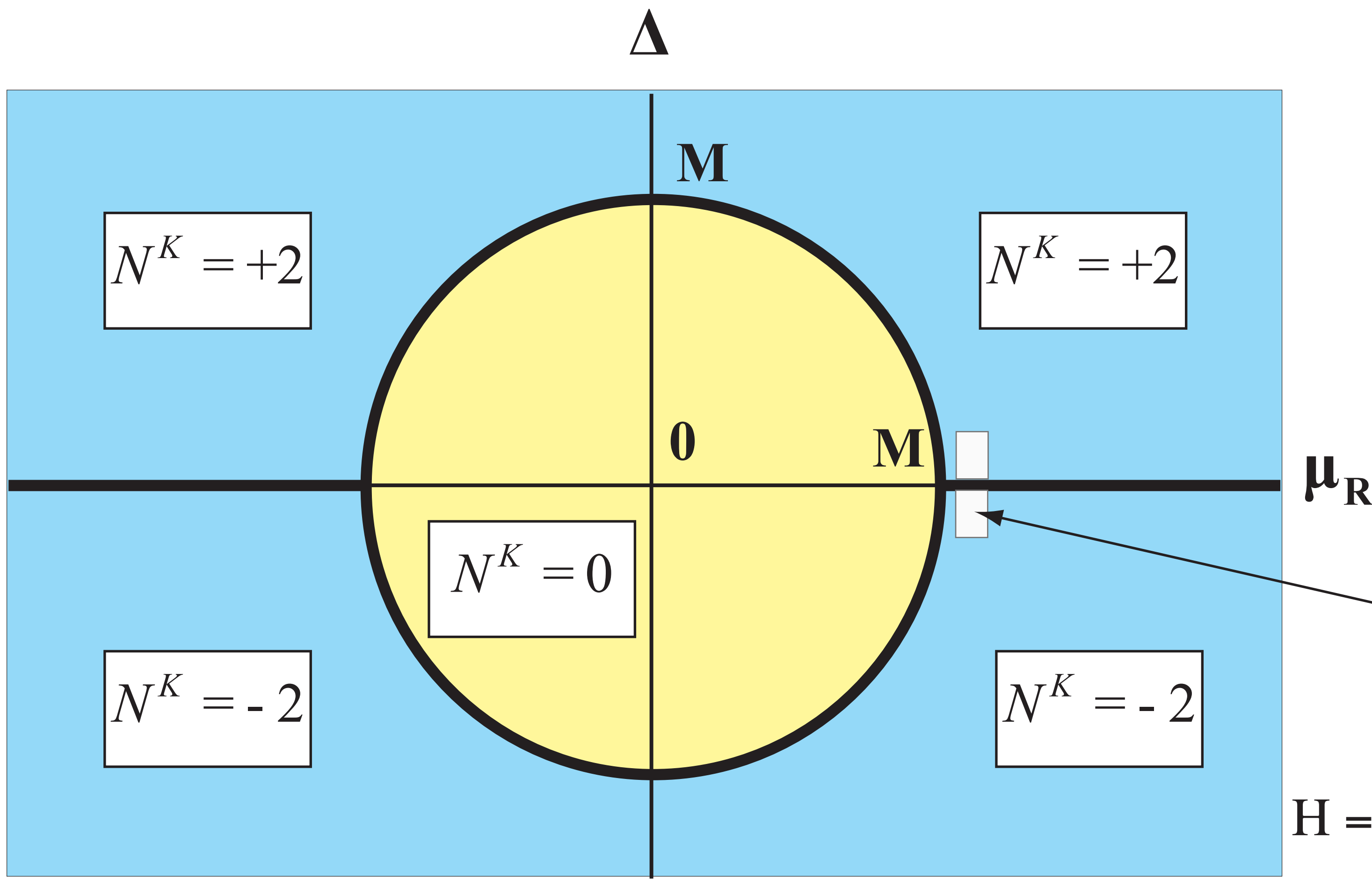
$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

superfluid ^{3}He-B

$$c_B = c\,\Delta / M \qquad m = M / c^2$$

$$(\mu + M)^2 = \mu_R^2 + \Delta^2$$

phase diagram of topological states of relativistic triplet superconductor

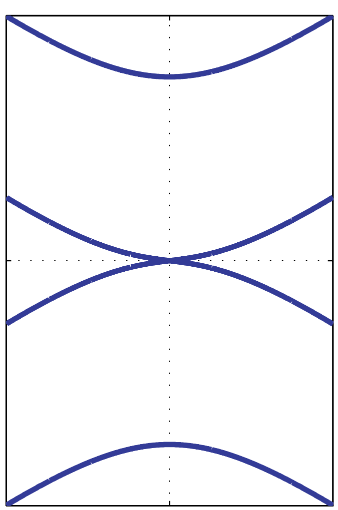


^{3}He-B

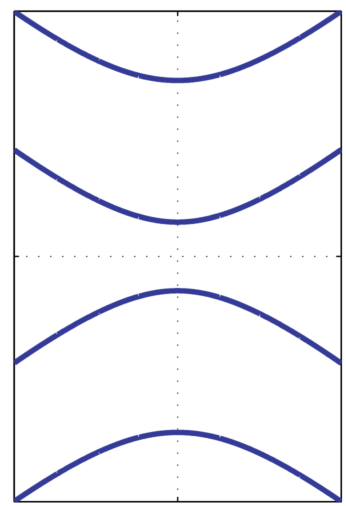
$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5 \Delta \\ \gamma_5 \Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$

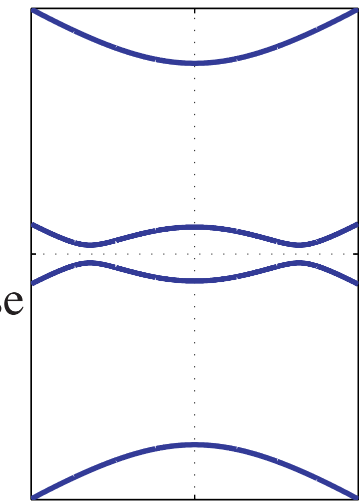
energy spectrum in relativistic triplet superconductor

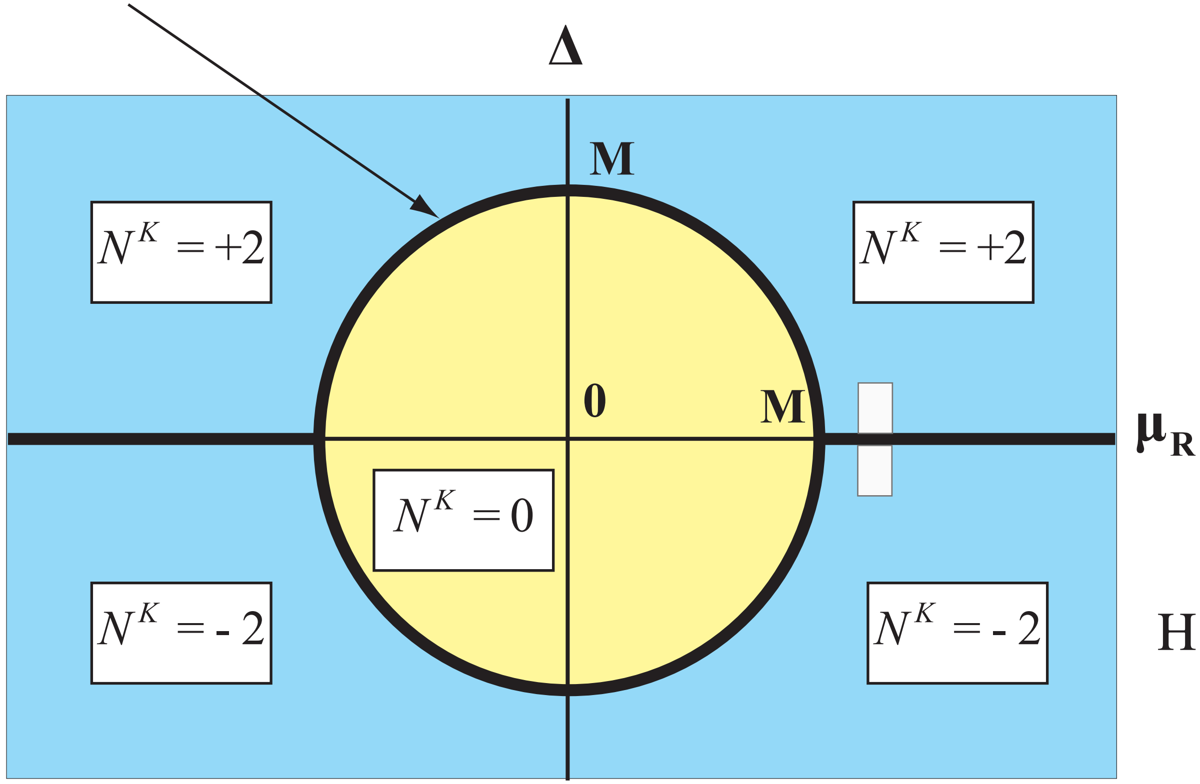


gapless spectrum
at topological
quantum phase
transition



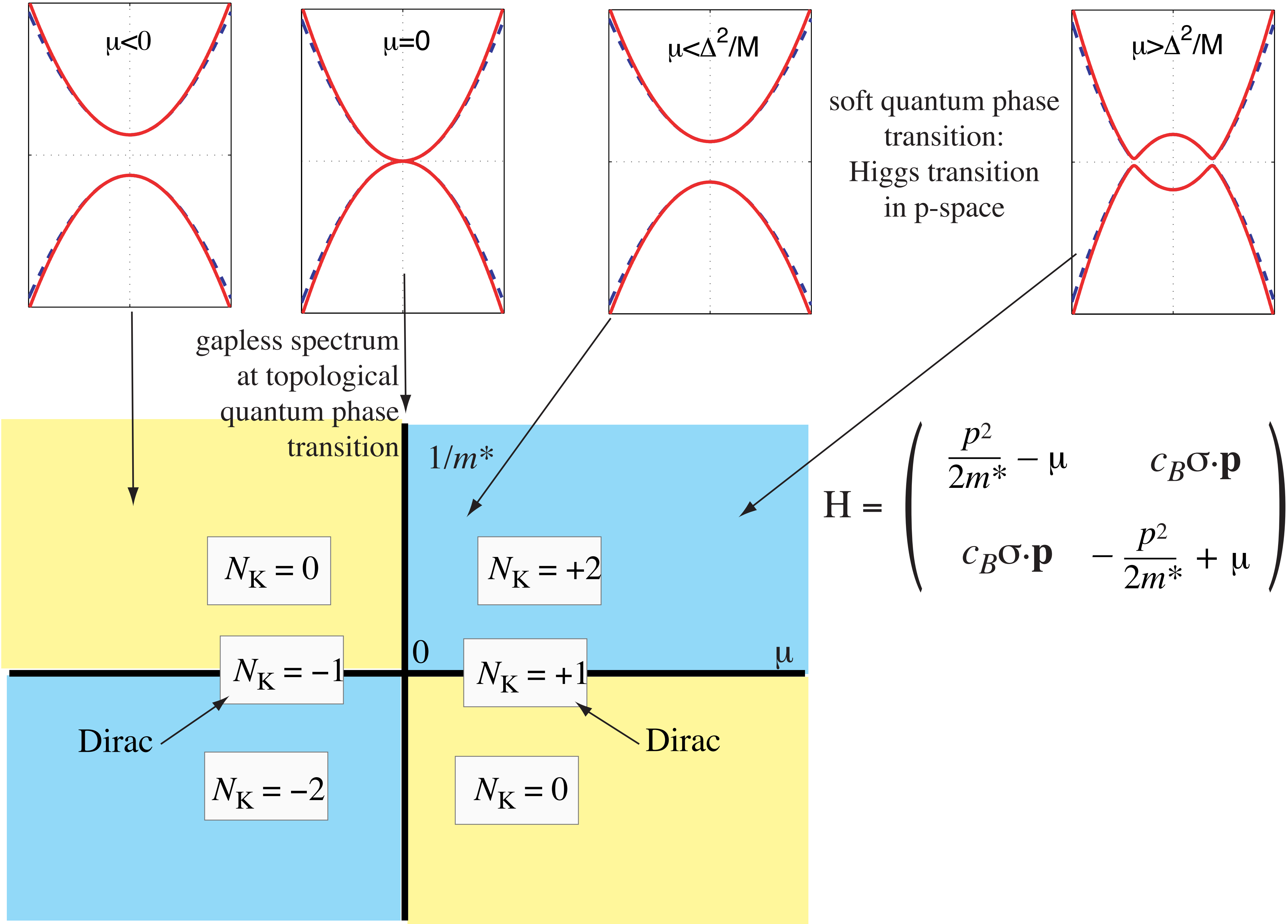
soft quantum phase
transition:
Higgs transition
in p-space





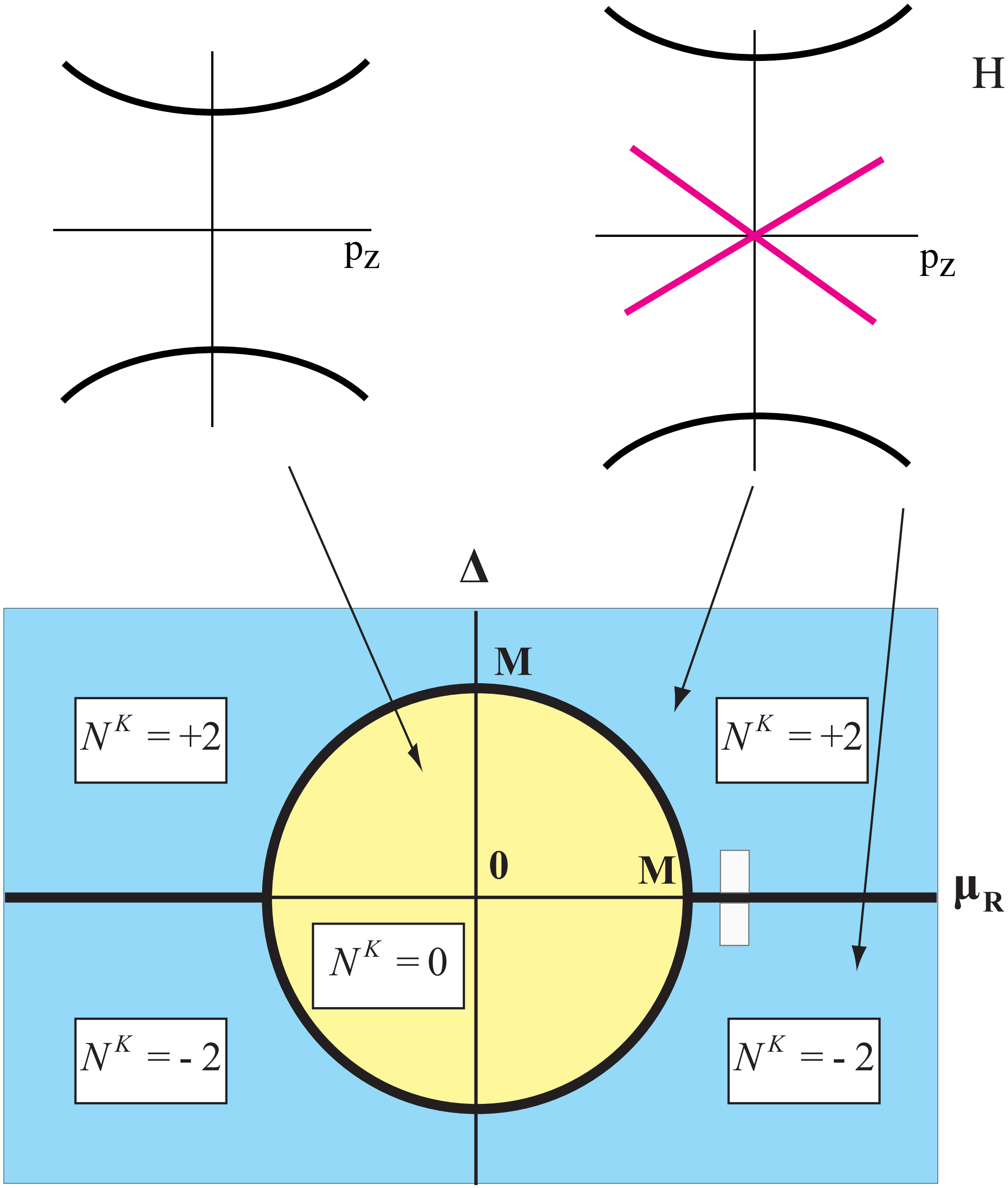
$$H=\begin{pmatrix} c\alpha\cdot\mathbf{p}+\beta M-\mu_R & \gamma_5\Delta \\ \gamma_5\Delta & -c\alpha\cdot\mathbf{p}-\beta M+\mu_R \end{pmatrix}$$

spectrum of non-relativistic ^{3}He-B



$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu \end{pmatrix}$$

fermion zero modes in relativistic triplet superconductor



$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5\Delta \\ \gamma_5\Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$

vortices in topological superconductors have fermion zero modes

generalized index theorem ?

possible index theorem for fermion zero modes on vortices

(interplay of ***r***-space and ***p***-space topologies)

$$\boxed{N_5 = \frac{1}{4\pi^3 i}\mathbf{tr}\left[\int d^3p\, d\omega\, d\phi\ \mathbf{G}\,\boldsymbol{\partial}_\omega \mathbf{G}^{-1}\,\mathbf{G}\,\boldsymbol{\partial}_\phi \mathbf{G}^{-1}\mathbf{G}\,\boldsymbol{\partial}_{p_x} \mathbf{G}^{-1}\mathbf{G}\,\boldsymbol{\partial}_{p_y} \mathbf{G}^{-1}\mathbf{G}\,\boldsymbol{\partial}_{p_z} \mathbf{G}^{-1}\right]}$$

for vortices in Dirac vacuum

$N_5 = N$ **winding number**