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Topology of the planar phase of superfluid ^{3}He

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- topological insulators and their classification
- planar phase and an additional symmetry
- topological types for class DIII and the planar phase from homotopy theory

Topological insulators

continuous and band insulators

$H(\mathbf{k})$, where k belongs to R^d or T^d (Brillouin zone)

- gapped
- topologically non-trivial
- edge states
  "index theorem"

Topological classification

Topological classification

Schnyder et al.

classification of robust surface Hamiltonians

Kitaev

K-theory, periodic table

motivation:

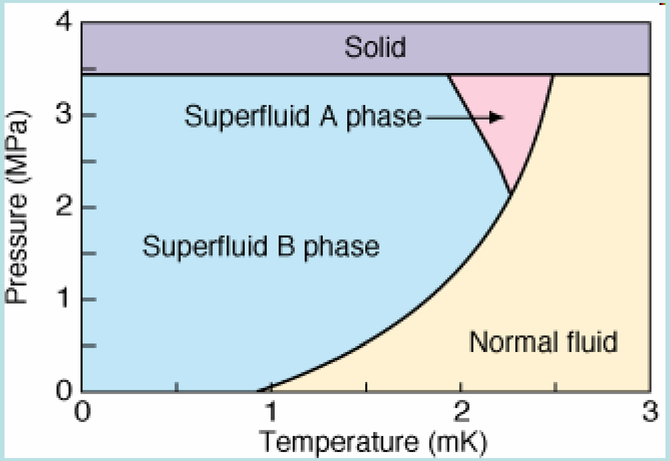
- classification from the bulk properties
- in the basic language of homotopy theory

  cf. Moore, Balents ‘07
- *exhaustive* list of explicit invariants
- planar phase with an additional symmetry
  - non-stable case

Superfluid ^{3}He

$$H = \begin{pmatrix} \varepsilon_k & \hat{\Delta}_k \\ \hat{\Delta}_k^\dagger & -\varepsilon_{-k} \end{pmatrix}$$

$$\hat{\Delta}_k = A_{\alpha i}\hat{\sigma}_\alpha (i\hat{\sigma}_y) k_i$$



LTL, Aalto

B-phase

$$A_{\alpha i} = \Delta R_{\alpha i} e^{i\varphi}$$

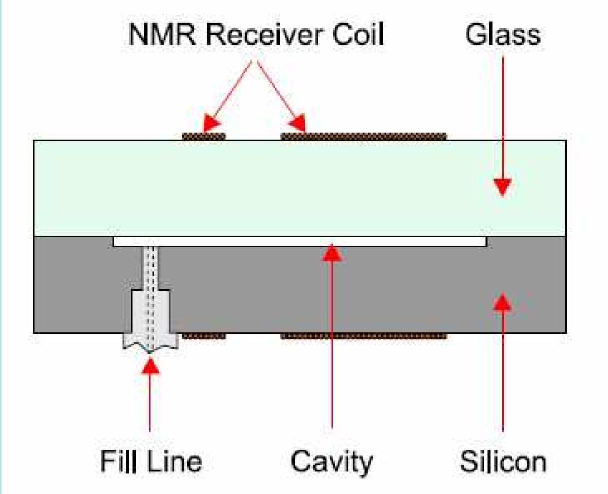
where *R* is orthogonal

$$A_{\alpha i} = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix}$$

symmetries: time reversal and particle-hole, class DIII

^{3}He-B - a topologically non-trivial insulator (N=1 in Z)

Superfluid ^{3}He: planar phase

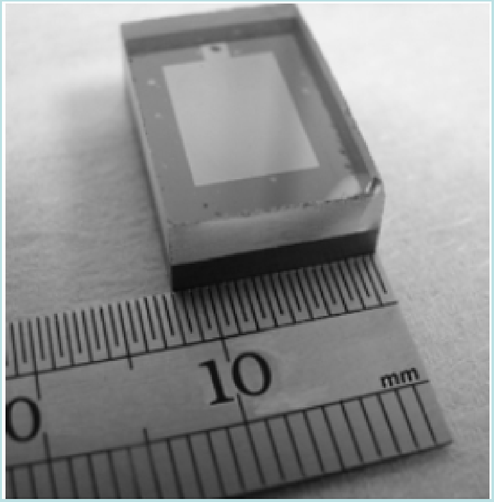


distorted B-phase:

$$A_{\alpha i} = \begin{pmatrix} \Delta_{\parallel} & 0 & 0 \\ 0 & \Delta_{\parallel} & 0 \\ 0 & 0 & \Delta_{\perp} \end{pmatrix}$$

planar phase:

$$A_{\alpha i} = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Bennett et al. '10

possible in thin slabs

additional symmetry in planar phase:
spin rotation $R_Z(\pi)$ + U(1) particle-hole sign flip

$$H_k = \sigma_z H_k \sigma_z$$

Class DIII

time-reversal invariance:

$$H(-k) = i\sigma_y H^T(k)(-i\sigma_y)$$

particle-hole symmetry of BdG Hamiltonian:

$$H(-k) = -\tau_x H^T(k)\tau_x$$

hence, a chiral symmetry (their composition):

$$H(k) = -\tau_x\sigma_y H(k)\tau_x\sigma_y$$

The chiral symmetry limits $H(\mathbf{k})$ to a real combination of 8 matrices (instead of initial 16)

$$H = a\sigma_x + b\sigma_z + c\tau_x\sigma_x + d\tau_x\sigma_z + e\tau_y + f\tau_z + g\tau_y\sigma_y + h\tau_z\sigma_y$$

In other words: in the eigenbasis of $\tau_x\sigma_y$ H is off-diagonal

$$H = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}$$

Non-degeneracy of Hamiltonian (gap):

$$\det M \neq 0$$

Particle-hole symmetry:

$$M_{-k} = -M_k^T$$

Can be retracted to unitary matrices: M → U

$M = PU$, where $U \in U(2)$ and *P* is positive hermitian

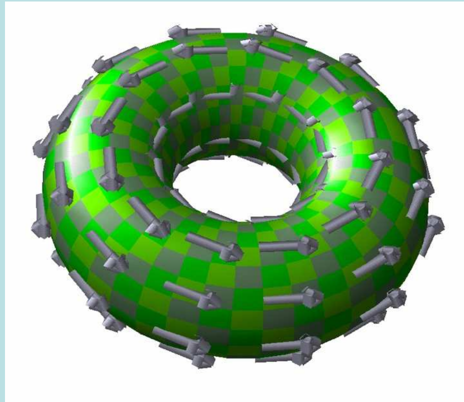
P can be uniformly (linearly) retracted to identity

We have to classify mappings

$$U : BZ \to U(2) \text{ with } U_{-k} = -U_k^T$$

Weak invariants

Continuous systems: <u>**k**-space R^d</u>



Band insulators: BZ is a torus, T^d

Weak invariants: mappings of d faces, $H(\mathbf{k})$

unstable w.r.t. disorder

Example: 2-band Hamiltonians in 3D Brillouin zone

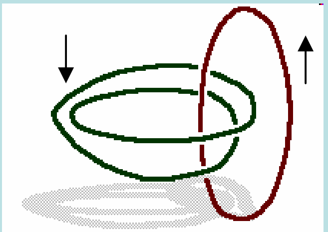
$H(\mathbf{k})$: $T^3 \to S^2$

$H = \mathbf{h}(\mathbf{k})\, \boldsymbol{\sigma}$ $\quad h \neq 0$

Moore, Ran, Wen, '08

Invariants: (a) mappings of three T^2-"faces" to S^2: N_1, N_2, N_3 (weak)

(b) "Hopf" invariant for mapping of the 3D interior $Z_{2*\mathrm{gcd}(N1,N2,N3)}$



$\mathcal{G}_{\omega=0}(\mathbf{k},\mathbf{r})$

Pontrjagin '41

YM, Misirpashaev '95

In the stable case, weak invariants are direct summands (K-theory, Kitaev)

We have to classify mappings

$$U : BZ \to U(2) \text{ with } U_{-k} = -U_k^T$$

"odd parity"

$$U(2) = (S^1 \times S^3)/\mathbb{Z}_2$$

$$U = e^{i\phi} S,\ S \in SU(2)$$

that is det *S* = 1

S belongs to *SU*(2) – 3D sphere S^3

$$m_0 \hat{1} + i(m_x \sigma_x + m_y \sigma_y + m_z \sigma_z)$$

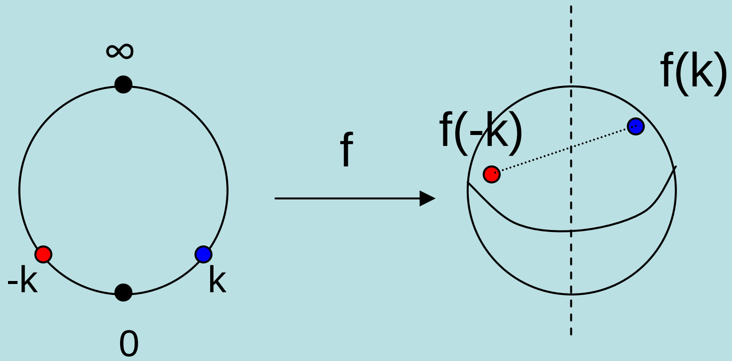
Odd parity implies that either

(a) $\phi_k = \phi_{-k}$ and $m_0, m_x, m_z(-k) = -m_0, -m_x, -m_z(k)$ and $m_y(-k) = m_y(k)$

or (b) $\phi_k = \phi_{-k} + \pi$ and $m_0, m_x, m_z(-k) = m_0, m_x, m_z(k)$ and $m_y(-k) = -m_y(k)$

Hence, mapping to $e^{i\varphi}$ is topologically trivial

Mapping to *SU*(2) is “odd”:



$= SU(2) = S^3$

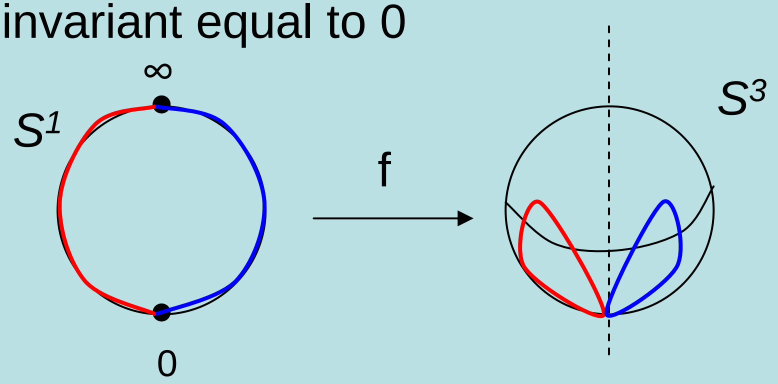
Poles (k=0 and ∞) are mapped to poles !

Z_2 invariant: to the same poles or not

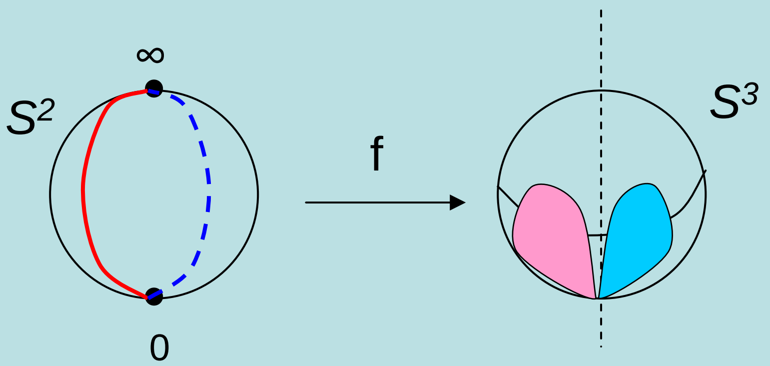
1D and 2D: no further invariants!

3D: *Z*-degree of the mapping (its parity is that Z_2 invariant !)

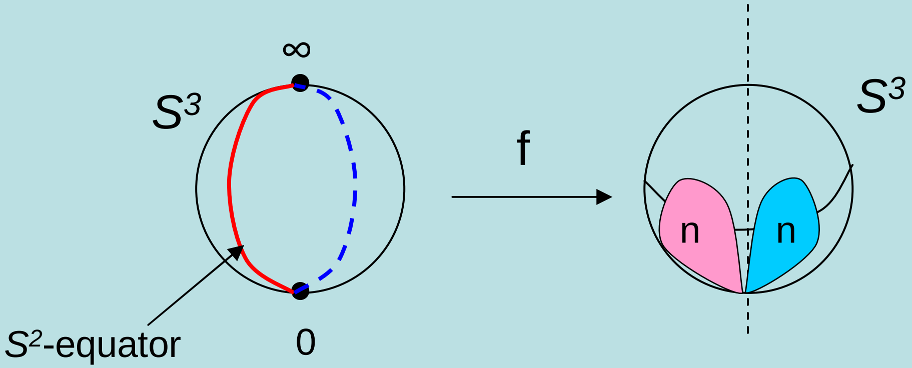
1D: for Z_2-invariant equal to 0



2D: for Z_2-invariant equal to 0



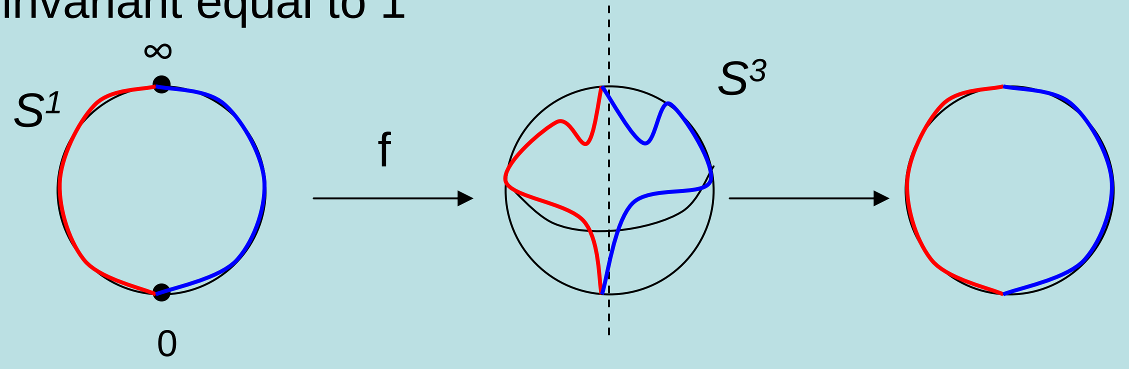
3D: for Z_2-invariant equal to 0



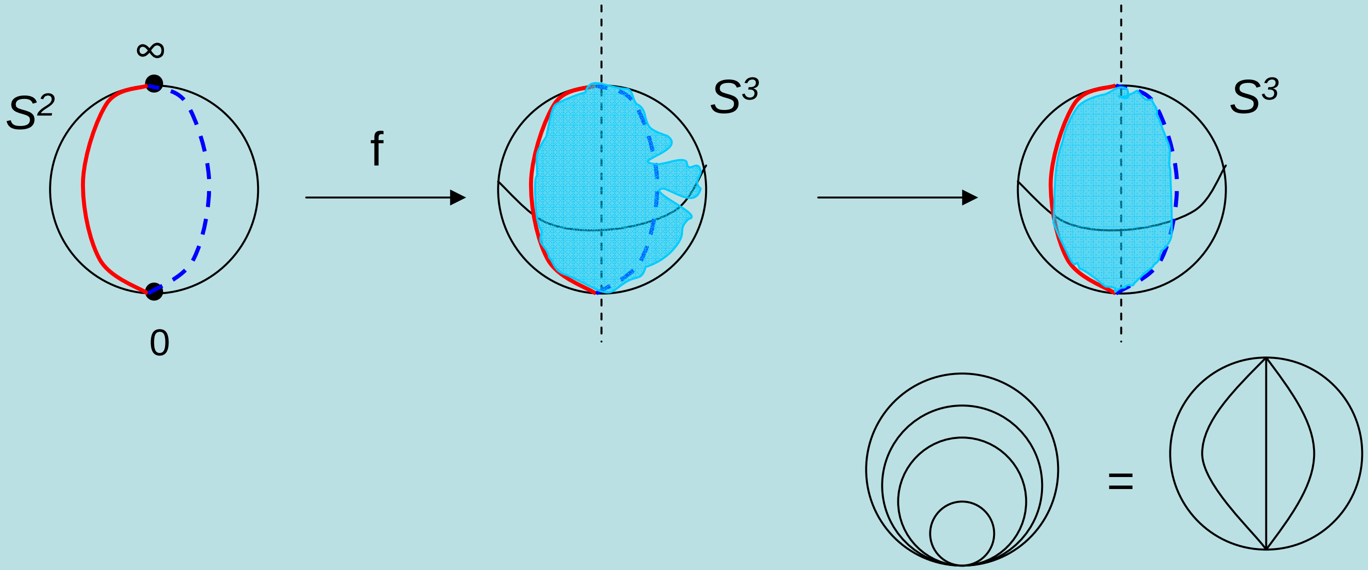
total degree = n + n = 2n

Similarly, for Z_2-invariant equal to 1, the total degree is 2n+1

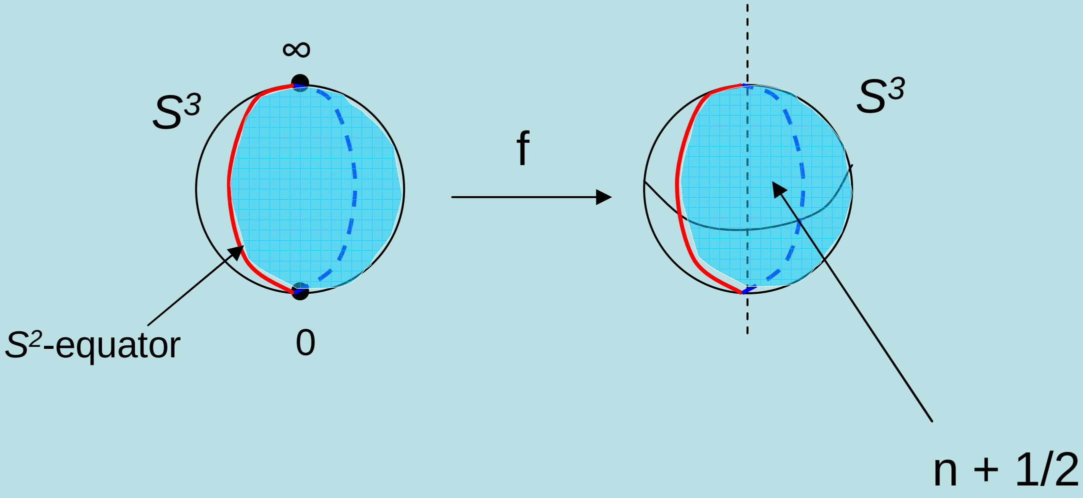
1D: for Z_2-invariant equal to 1



2D: for Z_2-invariant equal to 1



3D: for Z_2-invariant equal to 1



total degree = 2*(n + ½) = 2n + 1

In ^{3}He-B this map is $\mathbf{k} \to (\Delta\hat{k}_x, -\Delta\hat{k}_y, \Delta\hat{k}_z, \xi = \frac{k^2}{2m} - \varepsilon_F)$ & degree=1

	d=1	d=2	d=3
DIII	Z_2	Z_2	Z

$$N = \# \oint Tr\left[\Sigma\ H^{-1}dH \wedge H^{-1}dH \wedge H^{-1}dH\right]$$

Fix asymtotics away from Fermi surface: another Z-invariant

Planar phase

$$H_k = \sigma_z H_k \sigma_z$$

Thus, $U^\dagger = -U$

$U = i\,\boldsymbol{n}\,\boldsymbol{\sigma}$ – spin rotation by π around any axis **n**

or $U = \pm i$

- Classify <u>odd</u> mappings from **k**-space to S^2

Similarly to DIII, we have:

Z_2 in 1D

Z in 2D – for the planar phase

unlike Z_2 in Yip '10

In the planar phase $\mathrm{k} \to (\Delta \hat{k}_x, -\Delta \hat{k}_y,\ \xi = \frac{k^2}{2m} - \varepsilon_F)$ & degree=1

Planar phase

$$N = \frac{1}{8\pi} \oint e_{\alpha\beta\gamma} n_\alpha dn_\beta dn_\gamma$$

cf. Volovik & Yakovenko ’89:

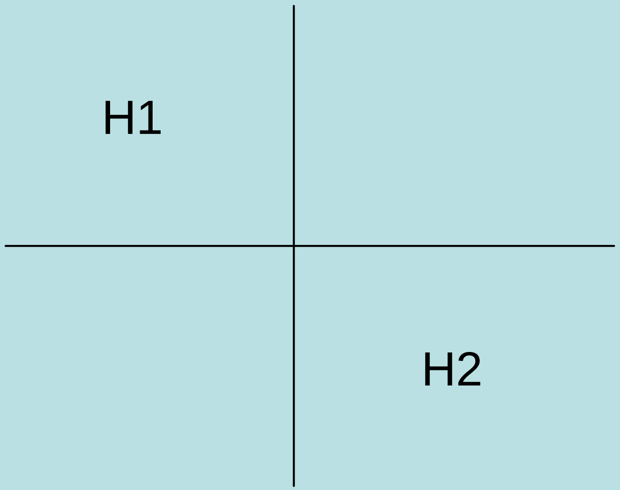
$$N = \frac{1}{24\pi^2} \int Tr \left[P \ \mathcal{G} d\mathcal{G}^{-1} \wedge \mathcal{G} d\mathcal{G}^{-1} \wedge \mathcal{G} d\mathcal{G}^{-1} \right]$$

parity(N) = Z_2-invariant

Higher-dimensional Hamiltonians

Extra symmetry $P^2=1$

Heinzner, Huckleberry, Zirnbauer `05



Conclusion

topological classes
for DIII and
the planar phase
from basic homotopy theory