Plenary lectures

A. D. Alexandrov and the Birth of the Theory of Tight Surfaces

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Alexander Danilovich Alexandrov was born in 1912, and just over 25 years later in 1938 (the year of my birth), he proved a rigidity theorem for real analytic surfaces of type T with minimal total absolute curvature. He showed in particular that that two real analytic tori of Type T in three-space that are isometric must be congruent. In 1963, twenty-five years after that original paper, Louis Nirenberg wrote the first generalization of that result, for five times differentiable surfaces satisfying some technical hypotheses necessary for applying techniques of ordinary and partial differential equations. Prof. Shiing-Shen Chern, my thesis advisor at the University of California, Berkeley, gave me this paper to present to his graduate seminar and challenged me to find a proof without these technical hypotheses. I read Konvexe Polyeder by A. D. Alexandrov and the works of Pogoreloff on rigidity for convex surfaces using polyhedral methods and I decided to try to find a rigidity theorem for Type T surfaces using similar techniques. I found a condition equivalent to minimal total absolute curvature for smooth surfaces, that also applied to polyhedral surfaces, namely the Two-Piece Property or TPP. An object in three- space has the TPP if every plane separates the object into at most two connected pieces. I conjectured a rigidity result similar to those of Alexandroff and Nirenberg, namely that two polyhedral surfaces in three-space with the TPP that are isometric would have to be congruent. I worked hard on this conjecture for six months and then was disappointed to find a counterexample to my own conjecture. I exhibited two non-congruent polyhedral tori with the same internal metric, made from the same pieces of cardboard!

My first project had failed, and I tried for the next six months to prove a generalization of a theorem of Robert Osserman about minimal surfaces in n-space, which turns out to have been proven by Osserman himself, my second failure in a thesis project.

At this point, Prof. Chern said he wanted to introduce me to a visitor to Berkeley, Nicolaas Kuiper, because You two think alike. It was true. When I showed him my cardboard model, Kuiper told me about his recent theorem on 2-dimensional surfaces with minimal total absolute curvature in n-space, namely that such a surface would have to lie in a 5-dimensional affine subspace. He suggested I should try to prove the analogue of this theorem for polyhedral surfaces. After working for two weeks on this conjecture, while folding laundry in a local Laundromat, I found a counterexample! The next day I brought in a paper model of a polyhedral surface that could be folded into 6-dimensional space, not lying in any 5-dimensional affine subspace but satisfying the Two-Piece Property. Kuiper was astounded. He said, You have a gold mine here, an example that shows that the polyhedral version of this theory is totally different from the smooth theory. Ill give you six months. If you havent figured out your thesis by that time, Ill give the project to one of my students. Its too important not to be done by someone.

It didnt take me six months to prove my main result, a construction of a set of examples of a polyhedral surface in any dimension n not lying in an affine hyperplane and satisfying the TPP. Prof. Chern said that this, together with my earlier counterexample to the polyhedral analogue of the rigidity theorem of Alexandrov, would be my doctoral thesis. He said I should define a descriptive term for such surfaces, and because they satisfied the conditions for Alexandrovs T-surfaces, the term should begin with T. I suggested tight, taut, and turgid and he rejected the third as being too biological and the second because taut was already in use. So my thesis had the title Tightly Embedded Two-Dimensional Polyhedral Surfaces, the first published use of that term. It has become a subcategory in the Mathematical Reviews, with hundreds of papers and articles on tight immersions and embeddings of smooth and polyhedral submanifolds.

I first met A. D. Alexandrov at the ICM in Kyoto in 1990. I saw a group of mathematicians huddled around someone they considered a celebrity, and I was very surprised to learn who it was. I made it a point to go up and introduce myself as a mathematician who had begun his research in response to his 1938 article. I said that I had chosen the term tight embedding to go along with T-surface. To my greater surprise, he indicated that he was not aware of any follow-up to that article. I was fortunate enough to meet him and his wife in Zurich at the next ICM in 1994, and I bought lunch for them as we had a good conversation. I learned about his activity as a champion alpinist in addition to the large number of academic distinctions listed on his professional card. I was very pleased to be able to meet one of my mathematical heroes.

Within in a year or so will be the fiftieth anniversary of my Ph.D. and my seventy-fifth year. I am happy to honor the memory of one of the most distinguished and influential geometers of the twentieth century on the occasion of the hundredth anniversary of his birth.

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On geometry of Carnot–Carathéodory spaces C^1-smooth with vector fields

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Let $\mathbb{M}$ be a smooth N-dimensional manifold. The manifold $\mathbb{M}$ is called the *Carnot-Crathéodory space* if the tangent bundle $T\mathbb{M}$ has a filtration

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \cdots \subsetneq H_i\mathbb{M} \subsetneq \cdots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

by subbundles such that every point $g \in \mathbb{M}$ has a neighborhood $U \subset \mathbb{M}$ equipped with a collection of C^1-smooth vector fields $X_1, \ldots, X_N$, constituting a basis of $T_v\mathbb{M}$ at every point $v \in U$ and meeting the following two properties. For every $v \in U$,

(1) $H_i\mathbb{M}(v) = H_i(v) = \operatorname{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\}$ is a subspace of $T_v\mathbb{M}$ of a constant dimension $\dim H_i$, $i = 1, \ldots, M$;

(2) $[H_i, H_j] \subset H_{i+j}$, $i, j = 1, \ldots, M-1$.

Becides of this, if the next condition holds then the Carnot-Carathéodory space is called the *Carnot manifold*:

(2) $H_{j+1} = \operatorname{span}\{H_j, [H_1, H_j], [H_2, H_{j-1}], \ldots, [H_k, H_{j+1-k}]\}$ where $k = \lfloor \frac{j+1}{2} \rfloor$, $j = 1, \ldots, M-1$.

The subbundle $H\mathbb{M}$ is called *horizontal*. The number M is called the *depth* of the manifold $\mathbb{M}$. The *degree* $\deg X_k$ is defined as $\min\{m \mid X_k \in H_m\}$.

A *sub-Riemannian structure* on $\mathbb{M}$ is a pair $(H\mathbb{M},\ \langle\cdot,\cdot\rangle)$ where $H\mathbb{M} = \{H_g\mathbb{M}\}_{g\in\mathbb{M}}$ and $\langle\cdot,\cdot\rangle = \{\langle\cdot,\cdot\rangle_g\}_{g\in\mathbb{M}}$ is a C^1-smooth in g family of Euclidean inner products $(X, Y) \mapsto \langle X, Y\rangle_g$, $X, Y \in H_g\mathbb{M}$, defined on $H_g\mathbb{M}$.

An absolutely continuous curve $\gamma : [0, T] \to \mathbb{M}$ is said to be *horizontal* if $\dot\gamma(t) \in H_{\gamma(t)}\mathbb{M}$ for almost all $t \in [0, T]$. The length of the horizontal curve equals $\int_0^T |\dot\gamma(t)|\, dt$.

In spite of minimal smoothness of vector fields we are able to prove some new results and derive from them counterparts of the following statements known on Carnot-Carathéodory spaces with smooth enough vector fields:

1) Gromov nilpotentization theorem on convergence of rescaled vector fields;

2) Gromov approximation theorem;

3) Gromov-Mitchell Theorem on the structure of tangent cone;

On Carnot manifold we prove

4) Rashevsky–Chow theorem on existence of horizontal curve with the given endpoints.

5) Ball–Box-theorem; this is a counterpart of the well-known Mitchell-Gershkovich-Nagel-Stein-Wainger theorem.

In addition we show some applications of the above-mentioned results to related domains.

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Hyperpolar actions on noncompact symmetric spaces

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An isometric action of a connected Lie group H on a Riemannian manifold M is called hyperpolar if there exists a connected closed submanifold Σ of M such that Σ meets each orbit of the action and intersects it orthogonally. An elementary example of a hyperpolar action comes from the standard representation of SO_n on $\mathbb{R}^n$. Further examples of hyperpolar actions can be constructed from Riemannian symmetric spaces. Let $M = G/K$ be a Riemannian symmetric space and denote by o a fixed point of the K-action on M. Then the isotropy representation $\pi : K \to O(T_oM)$ of K on the tangent space T_oM of M at o induces a hyperpolar action. Dadok established in 1985 a remarkable, and mysterious, relation between hyperpolar actions on Euclidean spaces and Riemannian symmetric spaces. He proved that for every hyperpolar action on $\mathbb{R}^n$ there exists a Riemannian symmetric space $M = G/K$ with $\dim M = n$ such that the orbits of the action on $\mathbb{R}^n$ and the orbits of the K-action on T_oM are the same via a suitable isomorphism $\mathbb{R}^n \to T_oM$. For symmetric spaces of compact type the hyperpolar actions are reasonably well understood. In the talk I want to focus on symmetric spaces of noncompact type where the situation is much more involved because of the noncompactness of the isometry groups. With my collaborators Díaz-Ramos and Tamaru I developed an approach based on Langlands and Chevalley decompositions of parabolic subalgebras of noncompact semisimple Lie algebras. Geometrically this involves horospherical decompositions of symmetric spaces and boundary components of their maximal Satake compactifications. This approach leads to many new examples, partial classifications, and interesting open problems. I plan to give an overview of these results based on these two papers:

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On Discretization in Riemannian Geometry

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Traditionally, discretization in Riemannian geometry was associated with polyhedral approximations. It seems now clear, due to works of Cheeger, Petrunin, Panov and many others that in dimensions beyond two or maybe three polyhedral structures are too rigid and cannot serve as discrete models of Riemannian spaces. Of course, there are various finite element methods, they do help to solve PDEs but they seem to be just numerical methods not helping us to understand geometry and make models.

In this talk, we will discuss approximating Riemannian manifolds by graphs, of course with additional structures attached to them and with various boundedness conditions. We will discuss both metric and PDE aspects, specifically a comparison of spectral characteristics of the graph and smooth Laplacians. The latter part is a joint work with S. Kurylev.

Reversed Alexandrov-Fenchel inequalities

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We consider the Minkowski addition and the volume of closed convex sets, invariant under the action of a given cocompact lattice of $SO(d, 1)$. Their support functions are defined on compact hyperbolic manifolds rather than on the sphere. In the regular and the polyhedral cases, Alexandrov–Fenchel inequalities are derived. Here the inequalities are reversed and the proofs, although very similar to the original ones by A.D. Alexandrov, are simpler than for the classical case of convex bodies.

Sabitov Polynomials for Volumes of Four-dimensional Polyhedra

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In the end of the 19th century R. Bricard constructed his famous flexible octahedra. These are (non-embedded) octahedra in $\mathbb{R}^3$ that can be deformed continuously so that the edge lengths are constant, but the polyhedron does not remain congruent to itself in the process of deformation. An example of an embedded flexible polyhedron in $\mathbb{R}^3$ was constructed by R. Connelly only in 1977. Surprisingly, it appeared that, for all known examples, the volume of the flexible polyhedron remains constant. This was posed as a conjecture, called *Bellows Conjecture*, that this phenomenon actually holds for every flexible polyhedron in $\mathbb{R}^3$. Bellows Conjecture was proved in 1996 by I. Kh. Sabitov [3], [4], see also [1]. His approach was to show that the volume of the polyhedron of given combinatorial type with given edge lengths can take only finitely many values. More precisely, his result was as follows.

Sabitov Theorem. *The volume of an arbitrary (not necessarily convex) simplicial polyhedron in a 3-dimensional Euclidean space is a root of a monic polynomial whose coefficients depends on thecombinatorial type and the edge lengths of the polyhedron only:*

$$V^N + a_1(\ell)V^{N-1} + \ldots + a_N(\ell) = 0,$$

where by ℓ we denote the set of lengths of edges of the polyhedron. Besides, the coefficients $a_j(\ell)$ are polynomials in the squares of lengths of edges of the polyhedron.

Since then it has been unknown whether the same holds in arbitrary dimension $n \geq 3$. We prove the direct analog of Sabitov Theorem for polyhedra in the 4-dimensional Euclidean space. The main corollary is that the volume of any flexible polyhedron in the 4-dimensional space is constant.

Our proof contains two principally new features with respect to the 3-dimensional case. The first one is the extension of the notion of a polyhedron based on the concept of a simplicial cycle. The second one has algebraic geometrical nature. We use certain new lemma concerning the properties of the variety consisting of possible sets of edge lengths for the polyhedra of the given combinatorial type.

These results are contained in [2].

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Ricci curvature and angles

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We give the definition of angles on a Gromov-Hausdorff limit space of a sequence of complete n-dimensional Riemannian manifolds with a lower Ricci curvature bound. We apply this to prove there is a weakly second differentiable structure on these spaces and prove there is a unique Levi-Civita connection allowing us to define the Hessian of a second differentiable function.

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Infinitesimal rigidity of convex surfaces: variational methods and duality

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Both in the discrete and in the smooth context, we investigate two kinds of rigidity for surfaces in $\mathbb{R}^3$: the one with respect to the induced metric and the one with respect to the Gauss curvature parametrized by the Gauss map.

We discuss two different duality relations between the both and connect variations of the volume to variations of the Hilbert-Einstein functional. This allows us to interpret Blaschke's proof of the infinitesimal rigidity of smooth convex surfaces in the spirit of Minkowski's proof of the infinitesimal rigidity in his theorem.

The talk is based on the preprints *arXiv:1105.5066* and *arXiv:1105.5067*.

Soap Bubbles and Polynomials

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Soap bubbles and fluid droplets are modeled by complete embedded constant mean curvatures (CMC) surfaces in $\mathbb{R}^3$. Besides the round sphere and cylinder, these CMC surfaces are highly transcendental objects, whose moduli spaces are generally understood in only a few special cases, thanks to work beginning with A. D. Alexandrov over half a century ago. In this talk, we will reveal a surprising connection with $\mathbb{C}P^1$-structures and holomorphic solutions of Hill's equation $U_{zz} + q(z)U = 0$, where $q(z)$ is the Schwarzian of the developing map for the $\mathbb{C}P^1$-structure. In the special case where $q(z)$ is a (normalized, monic) polynomial of degree $k-2$, the corresponding CMC surface must have genus 0 and k ends, as well as a plane of reflection symmetry which cuts the surface into a pair of graphical pieces. This correspondence allows us to explicitly work out the moduli space of all these coplanar CMC surfaces. The special cases of the sphere and the cylinder (indeed, all Delaunay unduloids) correspond to $q(z) = 0$ and 1, respectively; all triunduloids ($k = 3$) correspond to $q(z) = z$. If time and taste permit, we'll also discuss some related potential applications, including an explicit description of minimal surfaces in S^3.

The infimum of the volumes of convex polyhedra of any given facet areas is 0

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In [2] the following theorem was proved.

Theorem A ([2]). *Let $m \geq 3$, and $s_m \geq s_{m-1} \geq \cdots \geq s_1 > 0$, and $s_m < s_{m-1} + \ldots + s_1$. Then the infimum of the areas of convex m-gons in $\mathbb{R}^2$, having side lengths these s_i, is the minimal area of those triangles, whose sides have lengths $\sum_{i \in I_1} s_i$, $\sum_{i \in I_2} s_i$, $\sum_{i \in I_3} s_i$ (supposing these lengths satisfy the non-strict triangle inequality), where $\{I_1, I_2, I_3\}$ is an arbitrary partition of $\{1, \ldots, m\}$ into non-empty parts. If the cyclic order of the sides is fixed, then an analogous statement holds, where the sides with indices in I_j (for each $j \in \{1, 2, 3\}$) form an arc of the polygonal curve.*

An analogue of Theorem A was given in [5] and [2].

Theorem B ([5, 2]). *Let $m \geq 3$, and $s_m \geq s_{m-1} \geq \cdots \geq s_1 > 0$, and $s_m < s_{m-1} + \ldots + s_1$. Then the infimum of the areas of simple m-gons in $\mathbb{R}^2$, having side lengths these s_i, is the minimal area of those triangles, whose sides have lengths $\sum_{i \in I_1} \varepsilon_i s_i$, $\sum_{i \in I_2} \varepsilon_i s_i$, $\sum_{i \in I_3} \varepsilon_i s_i$ (supposing these lengths are non-negative, and satisfy the non-strict triangle inequality), where the ε_i's are arbitrary signs, and $\{I_1, I_2, I_3\}$ is an arbitrary partition of $\{1, \ldots, m\}$ into non-empty parts. (Moreover, if this minimum is not 0, we may additionally suppose that, for each $j \in \{1, 2, 3\}$, the sum $\sum_{i \in I_j} \varepsilon_i s_i$ cannot be written as $\sum_{i \in I_j'} \varepsilon_i s_i + \sum_{i \in I_j''} \varepsilon_i s_i$, where $\{I_j', I_j''\}$ is a partition of I_j, and where both these partial summands are positive.)*

We remark that the proofs in the two papers were different. Also, in [5] the result is formulated in a special case only, but all the ingredients of the proof of the general case are present in this paper as well.

We write $\mathbb{S}^2$, or $\mathbb{H}^2$, for the unit sphere of $\mathbb{R}^3$, or the hyperbolic plane, respectively. [2] extended Theorems A and B to $\mathbb{S}^2$ and $\mathbb{H}^2$ as follows:

Theorem C ([2]). *Let $m \geq 3$, and $s_m \geq s_{m-1} \geq \cdots \geq s_1 > 0$, and $s_m < s_{m-1} + \ldots + s_1$. We consider $\mathbb{S}^2$ and $\mathbb{H}^2$, but, in case of $\mathbb{S}^2$, we additionally suppose $\sum_{i=1}^{m} s_i \leq \pi$. Then, for both cases, the word-for-word analogues of Theorems 1 and 2 hold in $\mathbb{S}^2$ and $\mathbb{H}^2$.*

Thus, in each of these three theorems, the question of finding the infimum is reduced to finding the minimum of a set of non-negative numbers, the cardinality of this set being bounded by a function of m. (For Theorem A, or B, this bound is at most 3^m, or 6^m, respectively. For Theorem A, for given cyclic order of the sides, this bound is $\binom{m}{3}$.)

[2] posed the question, if it is possible to extend these theorems to $\mathbb{R}^n$, for $n \geq 3$ [2]. Their conjecture was that, analogously to the two-dimensional case, the solutions would be given as the volumes of some simplices. Unfortunately, they were unaware of the fact, that the case of simplices already had long ago been solved.

In fact, A. Narasinga Rao posed the following problem ([6]):

"The areas of the four facets of a tetrahedron are $\alpha, \beta, \gamma, \delta$. Is the volume determinate? If not, between what limits does it lie?"

This problem was solved independently in the papers [7, 3, 1, 4]. In fact, under the above condition, the volume is not determinate (if such tetrahedra exist, which we suppose). Moreover, there is, up to congruence, exactly one tetrahedron of maximal volume with the given facet areas, and there is a tetrahedron with volume as small as we want. A generalization of these results for multi-dimensional Euclidean spaces was obtained in the papers [7, 3, 4].

Theorem D ([7, 3, 1, 4]). *Consider real numbers $S_{n+1} \geq S_n \geq \cdots \geq S_1 > 0$, where $n \geq 3$. Then there is a simplex Σ in $\mathbb{R}^n$, the n-dimensional Euclidean space, with these areas of the facets, if and only if*

$$S_{n+1} < S_1 + S_2 + \ldots + S_n.$$

Supposing that this inequality holds, let $\Sigma(S_1, S_2, \ldots, S_{n+1})$ be the set of all such simplices in $\mathbb{R}^n$. Then, up to congruence, there is exactly one simplex $\Sigma \in \Sigma(S_1, S_2, \ldots, S_{n+1})$ of maximal volume, and for any $\varepsilon > 0$, there is a simplex $\Sigma \in \Sigma(S_1, S_2, \ldots, S_{n+1})$, with volume $\mathrm{Vol}\,(\Sigma) < \varepsilon$.

1 Main result

Theorem 1. *For any $\varepsilon > 0$, and for every numbers $S_m \geq S_{m-1} \geq \cdots \geq S_1 > 0$, such that $S_m < S_1 + S_2 + \cdots + S_{m-1}$ (where $m > n \geq 3$), there is a polytope $P \subset \mathbb{R}^n$ with m facets, with facet areas $S_1, S_2, \ldots, S_m$, and with volume $\mathrm{Vol}\,(P) < \varepsilon$. Even, there is a convex polytope with this property.*

For this theorem, we have three different proofs. (1): an existence proof, by obtaining a contradiction; (2): by reduction to the case of the simplices; (3): a geometric proof, showing that our examples with small volumes are like "needles".

Thus, there is a very interesting dichotomy. In Theorems A and B, for $\mathbb{R}^2$, we have some definite functions of the side lengths, as infima. In Theorem 1, for $\mathbb{R}^n$, with $n \geq 3$, the infimum does not depend at all on the facet areas.

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Section reports

On SO(3,3) fractional linear action on SO(3)

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We introduce the fractional linear action of SO(3,3) on SO(3) and we prove that this action is correctly (and globally) defined on SO(3). Same holds for the SO(n,n) fractional linear action on SO(n). For n=3, we introduce a commutative diagram which intertwines the SL(4) standard linear action in R^4 with the above mentioned SO(3,3)-action on SO(3): here we have in mind SU(2) identification with the space of rays emanating from the origin in R^4, as well as the 2-cover of SO(3,3) by SL(4) and the 2-cover of SO(3) by SU(2). We then prove that fractional linear action of SO(3,3) on SO(3) is projective. The equivalence of the above two actions is demonstrated explicitly.

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Continuous deformations of polyhedra that do not alter the dihedral angles

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We study polyhedra (more precisely, boundary-free polyhedral surfaces) the spatial shape of which can be changed continuously in such a way that all dihedral angles remain constant.

These polyhedra may be considered as a natural "dual object" for the flexible polyhedra. The latter are defined as polyhedra whose spatial shape can be changed continuously due to changes of their dihedral angles only, i. e., in such a way that every face remains congruent to itself during the flex. Since 1897, it was shown that flexible polyhedra do exist and have numerous nontrivial properties. Among the authors contributed to the theory of flexible polyhedra, we should mention R. Bricard, R. Connelly, I.Kh. Sabitov, and A.A. Gaifullin. For more details, the reader is referred to the survey article [3].

In 1986, M.Eh. Kapovich brought our attention to the fact that polyhedra, admitting nontrivial deformations that keep all dihedral angles fixed, may be of some interest for the theory of hyperbolic manifolds, where Andreev's theorem [1] plays an important role. The latter reads that a compact convex hyperbolic polyhedron with nonobtuse dihedral angles is uniquely determined by its dihedral angles.

The case of the Euclidean 3-space is somewhat special [2] and we restrict our study by the Lobachevskij and spherical 3-spaces only.

Of course, in the both spaces we can immediately propose the following example. Consider the boundary P of the union of a convex polytope Q and a small tetrahedron T (the both are treated as solid bodies for a moment) located such that (i) a face τ of T lies inside a face of Q and (ii) T and Q lie on the different sides of the plane containing τ. Obviously, the nonconvex compact polyhedron P has no self-intersections and admits nontrivial (i.e., not generated by a rigid motion of the whole space) continuous deformations preserving all dihedral angles. In order to construct such a deformation we can keep Q fixed and continuously move (e.g., rotate) T in such a way that the condition (i) is satisfied. In this example, many quantities associated with P remain constant. To name a few, we can mention (1) the volume; (2) the surface area; (3) the Gauss curvature of every vertex (i.e., the difference between 2π and the sum of all plane angles of P incident to this vertex); (4) the total mean curvature of P (i.e., the sum $\frac{1}{2}\sum_{\ell}(\pi-\alpha(\ell))|\ell|$ calculated over all edges ℓ of P, where $\alpha(\ell)$ is the dihedral angle of P at ℓ and $|\ell|$ is the length of ℓ); (5) every separate summand $(\pi-\alpha(\ell))|\ell|$ in the formula for the total mean curvature.

We study whether the above example provides us with the only possibility to construct nonconvex compact polyhedra that admit nontrivial continuous deformations preserving all dihedral angles? We prove that the answer is negative.

We study also what quantities associated with nonconvex compact polyhedra necessarily remain constant during continuous deformations leaving the dihedral angles fixed. For exmaple, we prove that the volume is necessarily constant while the Gauss curvature of a vertex and the expression $(\pi-\alpha(\ell))|\ell|$ may be nonconstant.

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Complex hyperbolic equidistant loci

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The talk is related to a project of constructing many compact $\mathbb{C}$-surfaces uniformizable by a holomorphic 2-ball. Similarly to the Poincaré disc, the holomorphic ball B is equipped with a natural metric such that the group $\mathrm{PU}(2,1)$ of holomorphic automorphisms of B is an index 2 subgroup in the group of isometries, and the corresponding geometry is called *complex hyperbolic.* The compact complex hyperbolic surfaces are known to be rigid and difficult to construct. They can be characterized by the relation $c_1^2 = 3c_2$ for their Chern numbers. The most famous examples are the (nonarithmetic) Mostow examples and the (arithmetic) fake projective planes.

The idea of the mentioned project is to construct polyhedra that will *a posteriori* become Dirichlet polyhedra. It turns out that the combinatorics of such polyhedra determines them almost uniquely and, moreover, if there exists a polyhedron with a given combinatorics, then it automatically satisfies the conditions of the 4-dimensional Poincaré's polyhedron theorem discussed in the talk of Carlos H. Grossi. Consequently, we need to study the loci equidistant from finitely many points because the faces of the polyhedra in question lie on such loci.

G. Giraud's theorem (1916+ε) claims that there are at most 3 bisectors (a bisector is the locus equidistant from 2 points) that contain a nonempty locus equidistant from generic 3 points. This fact imposes a strong restriction on the combinatorics of a Dirichlet polyhedron. In order to collect all the information concerning the combinatorics of Dirichlet polyhedra, we studied in detail equidistant loci from finitely many points. For example, it was established the following analog of Giraud's rigidity for 4 points:

Theorem. *The nonempty locus equidistant from* 4 *generic points is a real rational curve* C*. The bisectors containing an infinite subset of* C *form a real elliptic curve.*

Andreev theorem, combinatorically symmetric polyhedra and varieties of representations

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We will discuss how the Andreevs hyperbolic polyhedron theorem is related to global geometry and topology of a non-trivial compact 4-dimensional cobordism M whose interior has a complete hyperbolic structure, as well as to properties of the variety of discrete representations of the fundamental group of its 3-dimensional boundary ∂M. We construct such hyperbolic 4-cobordisms M whose boundary components are covered by the discontinuity set $\Omega(G) \subset S^3$ with two connected components Ω_1 and Ω_2, where the action Γ of the fundamental group $\pi_1(\partial M)$ is symmetric and has contractible fundamental polyhedra of the same combinatorial (3-hyperbolic) type. Nevertheless we show that such great geometric symmetry of boundary components of our hyperbolic 4-cobordism M(G) is not enough to ensure that the group $G = \pi_1(M)$ is quasi-Fuchsian, and in fact our 4-cobordism M is non-trivial. This is related to non-connectedness of the variety of discrete representation of the uniform hyperbolic lattice Γ and the property of kernels of the constructed homomorphisms $\Gamma \to G$ to be k-generated free groups F_k.

A method for the construction and solution of the Einstein equation on generalized flag manifolds

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A Riemannian manifold (M, g) is called Einstein if the Ricci tensor satisfies $\mathrm{Ric}(g) = \lambda g$ for some $\lambda \in \mathbb{R}$. A generalized flag manifold is a compact homogeneous space $M = G/K = G/C(S)$, where G is a compact semisimple Lie group and $C(S)$ is the centralizer of a torus in G. Equivalently, it is the orbit of the adjoint representation of G. At first, I will review certain aspects about G-invariant Einstein metrics on generalized flag manifolds, stressing the fact that when the number of the irreducible summands of the isotropy representation of $M = G/K$ increases, then finding explicitely the Ricci tensor (and moreover solving the Einstein equation) becomes a difficult task. Based on collaboration with Ioannis Chrysikos and Yusuke Sakane I will discuss the aspect of the classification of flag manifolds by using the painted Dynking diagrams, and I will present a new method for the construction of the Einstein equation, by using Riemannian submersions. In this way it is possible to obtain invariant Einstein metrics on flag manifolds with five or six isotropy summands.

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Min-Density Covering of Areas by Disks and Applications in Sensor Networks

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Complete Riemannian metrics with holonomy group G_2 on deformations of cones over[1] $S^3 \times S^3$

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One of the interesting problems of differential geometry is to construct complete Riemannian metrics with holonomy group G_2. The main idea is to consider standard conical metric on a Riemannian manifold with a special geometry [1, 2, 3]. Then the deformation of this metric depends on a number of functional parameters, which allow to define the G_2 structure explicitly. Let us consider $M = S^3 \times S^3$ as a base of the cone [4], then the cone metric can be written as

$$d\bar{s}^2 = dt^2 + \sum_{i=1}^{3} A_i(t)^2 \left(\eta_i + \tilde{\eta}_i\right)^2 + \sum_{i=1}^{3} B_i(t)^2 \left(\eta_i - \tilde{\eta}_i\right)^2,$$

where $\eta_i, \tilde{\eta}_i$ are 1-forms and the functions $A_i(t), B_i(t)$ define the deformation of the conical singularity. In [4] was obtained the system of differential equations, which guarantees that the metric $d\bar{s}^2$ has the holonomy group contained in G_2. In [4] was also found a particular solution of this system, corresponding to the metric with holonomy G_2 on $S^3 \times \mathbb{R}^4$.

We continue to study this class of metrics, putting $A_2 = A_3$, $B_2 = B_3$. Note that in this case the system can be written as:

$$\begin{aligned}
\frac{dA_1}{dt} &= \frac{1}{2}\left(\frac{A_1^2}{A_2^2} - \frac{A_1^2}{B_2^2}\right)\\
\frac{dA_2}{dt} &= \frac{1}{2}\left(\frac{B_2^2 - A_2^2 + B_1^2}{B_1 B_2} - \frac{A_1}{A_2}\right)\\
\frac{dB_1}{dt} &= \frac{A_2^2 + B_2^2 - B_1^2}{A_2 B_2}\\
\frac{dB_2}{dt} &= \frac{1}{2}\left(\frac{A_2^2 - B_2^2 + B_1^2}{A_2 B_1} + \frac{A_1}{B_2}\right)
\end{aligned}$$

We consider different from [4] boundary condition, which leads to metrics with other topological structure. Namely, we require that only one function B_1 vanishes at the vertex of the cone. This leads to the fact that the Riemannian metric $d\bar{s}^2$ is defined on $H^4 \times S^3$, where H is the space of the canonical complex line bundle over S^2, and H^4 - it‘s fourth tensor power. The main result can be formulated as

Theorem. *There is a one-parameter family of pairwise non-homothetic complete Riemannian metrics of the form $d\bar{s}^2$ with holonomy G_2 on $H^4 \times S^3$, and the metric can be parametrized by a set of initial data $(A_1(0), A_2(0), B_1(0), B_2(0)) = (\mu, \lambda, 0, \lambda)$, where $\lambda, \mu > 0$ and $\mu^2 + \lambda^2 = 1$.*

For $t \to \infty$ metrics of this family are arbitrarily close approximated by the direct product $S^1 \times C(S^2 \times S^3)$, where $C(S^2 \times S^3)$ is a cone over a product of spheres. Here, sphere S^2 arises as a factorization of the diagonal embedded in $S^3 \times S^3$ three-dimensional sphere by the action of the circle corresponding to the vector field $\xi^1 + \tilde{\xi}_1$

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Radial graphs over spherical domains with prescribed mean curvature

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We are interested in hypersurfaces in $\mathbb{R}^{n+1}$, characterized as radial graphs over a given domain Ω of $\mathbb{S}^n$, having boundary contained in $\mathbb{S}^n$ and whose mean curvature is a prescribed function $H\colon \mathbb{R}^{n+1} \to \mathbb{R}$. We assume that there exist $r_1, r_2 > 0$ with $r_1 \le 1 \le r_2$, such that H is of class C^1 in the region $A = \{\rho q \mid \rho \in [r_1, r_2],\ q \in \overline{\Omega}\}$ and satisfies:

- a barrier condition: $H(r_1 q) \ge r_1^{-1}$ and $H(r_2 q) \le r_2^{-1}$ for every $q \in \overline{\Omega}$,
- a monotonicity condition in the radial direction: $\frac{\partial}{\partial \rho} \rho H(\rho q) \le 0$ for every $q \in \Omega$ and $\rho \in (r_1, r_2)$.

Under this condition we prove existence and uniqueness in A of a radial graph Σ over Ω such that $\partial\Sigma = \partial\Omega$ and whose mean curvature is H.

Our result is related to some papers by I. Bakelman and B. Kantor [1] and by A. Treibergs and W. Wei [4], concerning the problem of hyperspheres with prescribed mean curvature. Moreover our result is in some sense complementary to a seminal result by J. Serrin [3] who first studied the problem of radial graphs over a given domain in $\mathbb{S}^n$.

This work is presented in a joint article with Giovanni Gullino [2].

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Smooth Cosmic Censorship and Determining Causality From Linking

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The talk is based on joint work with Stefan Nemirovski Globally hyperbolic spacetimes form probably the most important class of spacetime. It is observed that on many 4-manifolds there is a unique smooth structure underlying a globally hyperbolic Lorentz metric. For instance, every contractible smooth 4-manifold admitting a globally hyperbolic Lorentz metric is diffeomorphic to the standard $\mathbb{R}^4$. Similarly, a smooth 4-manifold homeomorphic to the product of a closed oriented 3-manifold N and $\mathbb{R}$ and admitting a globally hyperbolic Lorentz metric is in fact diffeomorphic to $N \times \mathbb{R}$. Thus one may speak of a censorship imposed by the global hyperbolicty assumption on the possible smooth structures on $(3+1)$-dimensional spacetimes.

We also show that for a large class of globally hyperbolic spacetimes causal relation between two events can be interpreted as the statement that the spheres of light rays through the two events are linked in the manifold of all light rays. In particular this solves the Low conjecture and the Legendrian Low conjecture formulated by Natario and Tod.

The Prym differentials on variable compact Riemann surfaces

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The space of leftinvariant orthogonal almost complex structures on 6-dimensional Lie groups

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Let G is 6-dimensional Lie group. Any leftinvariant almost complex structure on Lie group is identified with Lie algebra $\mathfrak{g}$ endomorphism I, such that $I^2 = -1$. Fix leftinvariant metric on the Lie group, then one can define set of all leftinvariant almost complex structures, which keep the metric. We will call such almost complex structures as orthogonal. The space of all orthogonal almost complex structures, with additional property to keep orientation is homogeneous space $\mathcal{Z} = SO(6)/U(3)$ and it is isomorphic to $\mathbb{C}P^3$.

In different problems with Hermitian structures on manifolds, and particulary on Lie groups, sometimes we need in explicit formulas for almost complex structures, instead implicit condition $I^2 = -1$. There exists only one orthogonal leftinvariant almost complex structure on 2-dimensional Lie group. In dimension 4 these structures form 2-parametric family equal to S^2. In 6-dimensional case, we have no explicit formulas of this sort of structures.

The space $\mathcal{Z}$ is researched. Using the visualization of this space as 6-dimensional tetrahedron $\mathbb{C}P^3$, with "edges" $\mathbb{C}P^1$ and "faces" $\mathbb{C}P^2$ [1] we can get explicit description for arbitrary point in $\mathbb{C}P^3$. As result any almost complex structure from $\mathcal{Z}$ is represented explicitly as composition of rotations.

For further details see [2].

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Manifolds of Morphisms between Foliations with Transverse Linear Connection

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Manifolds of Morphisms in Foliations with Transverse Linear Connection

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We investigate foliations with transverse linear connection. Works of Molino, Kamber and Tondeur, Bel'ko and others are devoted to different aspects of this class of foliations.

Let $\mathcal{F}ol$ be the category whose objects are foliations and morphisms are smooth maps between foliated manifolds transforming leaves to leaves. The class of smoothness is C^{∞}.

The goal of our work is an introduction of the structure of a smooth infinite-dimensional manifold on the full automorphism group $\mathcal{D}(M, F)$ of a foliation with transverse linear connection in the category of foliations $\mathcal{F}ol$.

In order to introduce the structure of a smooth infinite-dimensional manifolds modeled on LF-spaces (i.e. inductive limites of Frechet spaces) in the group of diffeomorphisms of a smooth manifold Michor used the construction of a local addition [1]. Macias-Virgos and Sanmartin applied this method to foliations by introducing the concept of an adapted local addition for a foliation [2].

Applying the results of Willmore and Walker we have proved the existence of special connection for foliated manifold M such that the foliation (M, F) with transverse linear connection becomes totally geodesic.

Thank to this our proof of the existence of an adapted local addition for (M, F) is simpler than the proof of the analogous statement of Virgos and Sanmartin for Riemannian foliations.

The following Theorem is the main result of this work. Emphasize that compactness of foliated manifolds are not assumed.

Main Theorem. *Let (M, F) and (M, F') be foliations with transverse linear connection. Then the set of morphisms $Mor(F, F')$ between these foliations in the category $\mathcal{F}ol$ admits the structure of an infinite-dimensional manifild modeled on LF-spaces.*

Corrollary. *Let (M, F) be a foliation with transverse linear connection of an arbitrary codimension on an n-dimensional manifold M. Then the automorphism group $\mathcal{D}(M, F)$ of this foliation in the category $\mathcal{F}ol$ admits the structure of an infinite-dimensional Lie group modeled on LF-spaces.*

The main Theorem extends the corresponding result of Macias-Virgos and Sanmartin-Carbon for Riemannian foliations [2]. In particular, our result is valid for Lorentzian and pseudo-Riemannian foliations.

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Minimal surfaces and hyperkaehler geometry

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The space of quasifuchsian groups can be interpreted as is the space of hyperbolic 3-manifolds homeomorphic to a product of a closed surface with a real line. Provide every such manifold with a minimal section. We show that the resulting space is isomorphic to the space of solution of sine-Gordon equation on Riemann surface and show some properties of this space, in particular the existence of a canonical hyperkaehler structure on it.

On the complexity of 3-manifolds

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Dynamics of Convex-Cocompact Subgroups of Mapping Class Groupsk

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On Half Conformally Flat Lie Groups With Left-Invariant Riemannian Metrics

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Let M be an oriented 4-dimensional Riemannian manifold. The Weyl tensor W is decomposed under the Hodge star operator $*$ as $W = W^+ \oplus W^-$. Riemannian manifold M is called half conformally flat if either $W^+ = 0$ or $W^- = 0$. A classification of conformally flat manifolds ($W = 0$) is given in [1]. The 4-dimensional half conformally flat Lie groups with left-invariant Riemannian metrics are investigated in this paper.

Theorem. *Let G be a real 4-dimensional Lie group with left-invariant Riemannian metric. Then*
1) $W^+ = 0$ if and only if $W = 0$.
2) $W^- = 0$ if and only if either $W = 0$ or Lie algebra of Lie group G contains in the following list:

$\mathbb{A}^{\beta}_{4,9}$ $(-1 < \beta \leq 1)$ *with structure constants* $c^1_{1,4} = 2A$, $c^1_{2,3} = c^2_{2,4} = c^3_{3,4} = A > 0$, $\beta = 1$;

$\mathbb{A}^{\beta}_{4,9}$ $(-1 < \beta \leq 1)$ *with structure constants* $c^1_{1,4} = c^1_{2,3} = 2A$, $c^2_{2,4} = c^3_{3,4} = A > 0$, $\beta = 1$;

$\mathbb{A}^{\alpha}_{4,11}$ $(\alpha > 0)$ *with structure constants* $c^1_{1,4} = 2A\alpha$, $c^1_{2,3} = c^2_{2,4} = c^3_{3,4} = A\alpha$, $c^3_{2,4} = -c^2_{3,4} = -A$, $A > 0$;

$\mathbb{A}^{\alpha}_{4,11}$ $(\alpha > 0)$ *with structure constants* $c^1_{1,4} = c^1_{2,3} = 2A\alpha$, $c^2_{2,4} = c^3_{3,4} = A\alpha$, $c^3_{2,4} = -c^2_{3,4} = -A$, $A > 0$.

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About $L_{q,p}$-cohomology of noncompact manifolds

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We study Sobolev type classes of differential forms $\Omega^k_{q,p}(M)$ on an n-dimensional Riemannian manifold M, g . Any q-integrable k-forms ω belongs to the class $\Omega^k_{q,p}(M)$ if its weak exterior differential $d\omega$ is p-integrable. The $L_{q,p}$-cohomology ($H^k_{q,p}(M)$ and its reduced version $\overline{H}^k_{q,p}(M)$) are cohomology and reduced cohomology of the Banach complex $\left\{\Omega^k_{q,p}(M), d\right\}$. The Banach complex $\Omega^k_{q,p}(M)$ is an invariant of Lipschitz structure. Under some reasonable resrtrictions the $L_{q,p}$-cohomology are quasi-isometric invariants. For $q = \frac{n}{k}, p = \frac{n}{k+1}$ the cohomology are quasiconformal invariant.

In the paper [1] we introduced a version of Sobolev-Poincaré inequality for differential forms. Vanishing of $H^k_{q,p}(M)$ is equivalent to existence of a corresponding Sobolev-Poincaré inequality for classes $\Omega^k_{q,p}(M)$. The classical Sobolev-Poincaré inequality corresponds to $k = 1$. We show some applications to quasilinear equations.

For $p > 1, q > 1$ the $L_{q,p}$-cohomology of compact manifolds coincide with the de Rham cohomology if and only if $\frac{1}{p} - frac1q \leq 1/n$. Under the same restrictions on p, q any $L_{q,p}$-cohomology class has a smooth representative for any (noncompact) manifold.

The $L_{q,p}$-cohomology were applied to Lipschitz classification of simply connected compact manifolds with nonnegative pinched curvature. A typical result is the following:

Theorem Let (M, g) be an n-dimensional complete simply-connected Riemannian manifold with sectional curvature $K \leq -1$ and Ricci curvature $Ric \geq -(1+\epsilon)^2(n-1)$.
(A) Assume that

$$\frac{1+\epsilon}{p} < \frac{k}{n-1} \quad \text{and} \quad \frac{k-1}{n-1} + \epsilon < \frac{1+\epsilon}{q},$$

then $H^k_{q,p}(M) \neq 0$.

(B) If furthermore

$$\frac{1+\epsilon}{p} < \frac{k}{n-1} \quad \text{and} \quad \frac{k-1}{n-1} + \epsilon < \min\left\{\frac{1+\epsilon}{q}, \frac{1+\epsilon}{p}\right\},$$

then $\overline{H}^k_{q,p}(M) \neq 0$.

For $p = q$ the result (B) was known by M.Gromov.

We also discuss a concept of Hölder-Poincaré duality for $L_{q,p}$-cohomology and its applications to vanishing of the reduced $L_{q,p}$-cohomology. A weak form of the dualitt (that we call "'an almost duality"') has applications to vanishing of $L_{q,p}$-cohomology and to existence of the Sobolev-Poincaré inequality for differential forms.

joint with M.Troyanov

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Geometry of Phase Portraits of Nonlinear Dissipative Dynamical Systems

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I. Questions of existence of periodic trajectories in phase portraits of dynamical systems appear in various domains of pure and applied mathematics: in Geometry (existence of closed geodesics on a Riemannian manifold, and estimates of their quantities), in Celestial Mechanics (3 bodies problem), in Ergodic Theory, in modeling of biochemical processes etc.

We study similar problems for nonlinear dissipative dynamical systems of chemical kinetics considered as models of gene networks functioning, see [4]. One simple class of these models is described as follows:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}) - \mathbf{X}, \qquad \text{or, in coordinates,} \qquad \frac{dx_i}{dt} = f_i(x_{i-1}) - x_i; \quad i = 1, \dots n. \tag{1}$$

All values of functions and variables are non-negative: $\mathbf{X} \in \mathbb{R}^n_+$, and $x_{i-1} = x_n$ for $i = 1$. The function f_i shows the rate of synthesis of the i-th substance (protein, RNA etc.) in gene network, the variable x_i is concentration of the i-th substance. Negative term in each equation describes natural degradation of corresponding substance. The system (1) is dissipative, $\mathrm{div}(\mathbf{F}(\mathbf{X}) - \mathbf{X}) \equiv -n$, so volume of each bounded domain in $\mathbb{R}^n_+$ decreases exponentially in time under action of the flow of this system. In contrast with Hamiltonian systems, we do not have here any conservation law, invariant tori etc.

II. Consider the case of negative feedback regulation in gene networks. Here all the functions f_i are monotonically decreasing, $f_i(x_{i-1}) \to 0$ for $x_{i-1} \to \infty$. Let $P_n = [0, f_1(0)] \times \ldots \times [0, f_n(0)] \subset \mathbb{R}^n_+$. All trajectories of the system (1) eventually enter P_n, so this is an invariant domain of this system. Usually in applications, the functions f_i have the form $f_i(w) = \dfrac{a_i}{b_i + w^{m_i}}$, so-called Hill functions, see [4,5]. All parameters a_i, b_i, and m_i here are positive. However, we use these representations in numerical experiments only.

Now let $n = 2k+1$. Odd-dimensional system (1) has a unique stationary point $\mathbf{X}_* = \mathbf{F}(\mathbf{X}_*)$. We call this point *hyperbolic*, if linearization matrix M of the system (1) at this point has eigenvalues with positive real parts, with negative real parts, and does not have purely imaginary eigenvalues. (All these eigenvalues can be expressed explicitly.) Under these assumptions, we construct a non-convex polyhedral invariant domain $Q \subset P_{2k+1}$ of the system (1). This polyhedron Q is composed of $(4k+2)$ triangle prisms, and the stationary point $\mathbf{X}_*$ is its "central vertex". For the cases $k = 1, 2$, detailed descriptions of these polyhedra are given in [2,3], respectively. We show that all trajectories of the system (1) which are contained in Q pass through each of these prisms periodically and do not approach the stationary point. Hence, we obtain:

Theorem 1. *If $n = 2k+1$, and the stationary point $\mathbf{X}_*$ is hyperbolic, then the invariant domain Q contains at least one cycle of the system* (1).

Some sufficient conditions for existence of stable cycles in such invariant domains Q were obtained as well, see [3].

III. We also study questions of non-uniqueness of these periodic trajectories. Consider the case of symmetric system (1), i.e., let all the functions f_i coincide: $f_i \equiv f_j$ for all i and j. If dimension of this system is not a prime number, $2k+1 = p \cdot q$, where $p \neq 1 \neq q \neq p$, then its phase portrait contains p-dimensional and q-dimensional invariant linear subspaces L^p, L^q of the system (1), and Theorem 1 can be applied to each of these subspaces. The subspace L^p is determined by linear equations $x_i - x_j = 0$ for $i \equiv j \pmod{p}$, the subspace L^q is determined similarly, see [1]. For example, this theorem implies the following

Proposition 1. *If $2k+1 = 15$, and $\mathbf{X}_*$ is hyperbolic for restrictions of symmetric 15-dimensional system* (1) *to each of its invariant subspaces L^3, L^5, then this system has at least 3 different cycles. Two of them, C_3 and C_5 are contained in L^3 and in L^5, respectively, and do not intersect the polyhedron Q, see* [1]*; the third cycle C_{15} is contained in Q, see Theorem* 1.

If conditions of stability of that third cycle are satisfied, then the cycles C_3, C_5 are stable within the subspaces L^3, L^5, respectively. Numerical experiments show that in contrast with C_{15}, these two cycles are not stable in the ambient phase portrait of the 15-dimensional system (1).

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More rich collections of cycles appear in symmetric systems (1) for $2k+1 = 81, 105$, etc., if the conditions of Theorem 1 are satisfied. So, Proposition 1 extends to higher-dimensional cases in an obvious way.

On the other hand, even in the case of asymmetric dynamical systems (1), it follows from the Grobman-Hartman theorem that if $\mathbf{X}_*$ is hyperbolic, then each pair of "positive" complex conjugate eigenvalues $\lambda_j, \bar{\lambda}_j$ of the matrix M with $\mathbb{R}e\lambda_j > 0$ corresponds to a 2-dimensional unstable invariant manifold S_j of the system (1). Trajectories of this system which are contained in S_j are repelled from the stationary point $\mathbf{X}_*$ but they still remain in the bounded domain P_{2k+1}.

Actually, the existence of all these invariant manifolds S_j is proved just for some small neighborhood W of $\mathbf{X}_*$, as in the theory of Andronov–Hopf bifurcations. Numerical experiments show that even in the cases of prime dimensions $2k+1 = 11$, and 13, for different pairs of these "positive" eigenvalues $\lambda_j, \bar{\lambda}_j$, these manifolds S_j can be extended "rather far" from W, and different manifolds S_j contain different cycles in phase portraits of corresponding dynamical systems (1). For high dimensions $2k+1$ these cycles are located near the boundaries of the parallelepipeds P_{2k+1}. At present time, mathematical proofs of these statements is one of our main tasks.

IV. Much more complicated phase portraits appear for even-dimensional dynamical systems of the type (1) and in the cases when some of the functions f_i describe other types of regulatory mechanisms. For example, if functioning of some part of a gene network is regulated by simple combinations of positive and negative feedbacks, then corresponding functions $f_i(w)$ are unimodal, each of them grows monotonically (positive feedback) below some threshold value w_i^0, and for $w > w_i^0$ it decreases monotonically as above (negative feedback).

Usually in applications, see [4,5], such unimodal functions have the form $f_i(w) = \dfrac{a_i \cdot w}{b_i + w^{m_i}}$ (so-called Glass–Mackey functions), Ricker functions $f(w) = a_i \cdot w \cdot \exp(-b_i \cdot w)$, logistic function, etc. In most of these cases phase portraits of dynamical systems contain several stationary points, and in some cases it is possible to construct polyhedral invariant domains, analogous to Q, near the stationary points with appropriate topological indices of the velocity vector field $\mathbf{F}(\mathbf{X}) - \mathbf{X}$. Now, Theorem 1 can be applied to each of these domains, as it was done in [2] for 3-dimensional dynamical systems of the type (1), where some of the functions f_i are unimodal and the others decrease monotonically, as in **II**. Corresponding conditions of existence of stable cycles can be derived here as well. On the other hand, for some values of parameters a_i, b_i, etc. behavior of trajectories in higher-dimensional phase portraits of these systems becomes chaotic, see [4]. Prediction of this phenomena is also one of our tasks.

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Poincaré's polyhedron theorem for compact 4-manifolds

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Poincaré's polyhedron theorem (PPT) is a common tool for constructing manifolds carrying a prescribed geometry. It has a long history and plenty of versions (see, for instance, [5]). While elaborating a version of PPT for the construction of fibre bundles over surfaces (see [2] and [3]), we realized that, the more local in nature, the more verifiable in practice are the conditions providing tessellation.

Carrying on this idea, we prove the following theorems.

Theorem. Let M be an oriented 3-dimensional complete Riemannian manifold with simply-connected components and let $P \subset M$ be a compact piecewise smooth polyhedron equipped with face-pairing isometries that send the polyhedron interior into its exterior. Suppose that the sum of angles at a single point along an arbitrary (geometric) cycle of edges equals 2π and that the polyhedron is locally connected after removing its vertices. Then P is a fundamental polyhedron, the cycle relations are defining relations, and M is therefore tessellated by the copies of P.

Theorem. Let M be an oriented 4-dimensional complete Riemannian manifold with simply-connected components and let $P \subset M$ be a compact piecewise smooth polyhedron equipped with face-pairing isometries that send the polyhedron interior into its exterior. Suppose that the sum of angles at a single point along an arbitrary (geometric) cycle of codimension 2 faces equals 2π and that the polyhedron is locally connected after removing its codimension 3 faces. Suppose also that the stabilizer of every codimension 3 face in the group generated by the face-pairing isometries induces a finite group of isometries of the face. Then P is a fundamental polyhedron, the cycle relations are defining relations, and M is therefore tessellated by the copies of P.

The theorems are surprisingly 'plane-like' (i.e., there are no solid angle conditions at points in faces of codimension ≥ 3) and impose conditions of the alleged local nature. When it comes to applications, these conditions are actually necessary for tessellation. Curiously, the proof of the theorems requires studying discrete *groupoids* in detail.

It turns out that we are not the first to revolve around these ideas. Studying the tiling of a space by polyhedra in [1], A. D. Alexandrov does not work in terms of groups of isometries. It seems he was understanding that the appropriate symmetries in this case do not constitute a group. (Back in 1954, it was natural for A. D. Alexandrov to deal only with the constant curvature case.) Much later, W. P. Thurston [6, p. 127, 11th line from the bottom] also pointed out that, in the 3-dimensional constant curvature case, no solid angle condition at vertices is necessary.

The above theorems have a wide range of applications and can be generalized to the case of higher dimension and other geometric structures. Our particular interest lies in dimension 4: the theorem is planned for constructing compact $\mathbb{C}$-surfaces satisfying $c_1^2 = 3c_2$.

This is a joint work [4] with Sasha Anan′in and Júlio C. C. da Silva.

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Tits geometry and positive curvature

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Regular-Aleksandrov polyhedra

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The geometrical Problems of the Time Machine Construction

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A.D. Alexandrov considered as one of important scientific tasks finding of the Time machine construction. Below we offer such construction.

The Time Machine Construction is 4-dimensional wormhole which connects two events (at present and at past) after the transformation of space-time M^4 into a resilient (or dense) leaf in 5-dimensional Hyperspace M^5.

Let $< M^4, g_{\alpha\beta} >$ be a leaf of an orientable foliation $\mathcal{F}$ of codimension 1 in the 5-dimensional Lorentz manifold $< M^5, g^{(5)}_{AB} >$, $g = g^{(5)} |_{M^4}$, $A, B = 0, 1, 2, 3, 5$. Foliation $\mathcal{F}$ is defined by the differential 1-form $\gamma = \gamma_A dx^A$. If the Godbillon-Vey class $GV(\mathcal{F}) \neq 0$ then the foliation $\mathcal{F}$ has a resilient leaves.

We suppose that real global space-time M^4 is a resilient one, i.e. is a resilient leaf of some foliation $\mathcal{F}$. Hence there exists an arbitrarily small neighborhood $U_a \subset M^5$ of the event $a \in M^4$ such that $U_a \cap M^4$ consists of at least two connected components U_a^1 and U_a^2.

Remove the 4-dimensional balls $B_a \subset U_a^1, B_b \subset U_a^2$, where an event $b \in U_a^2$, and join the boundaries of formed two holes by means of 4-dimensional cylinder. As result we have a 4-wormhole C, which is a Time machine if b belongs to the past of event a. The past of a is lying arbitrarily nearly. The distant Past is more accessible than the near Past. A movement along 5-th coordinate (in the direction γ^A) gives the infinite piercing of space-time M^4 at the points of Past and Future. It is the property of a resilient leaf.

If σ is the characteristic 2-dimensional section of the 3-dimensional domain D_0 that one contains the 4-wormhole than we have for the mean value of energy density jump which one is required for creation of 4-wormhole the following formula:

$$< \delta\varepsilon > \sim \frac{c^4}{4\pi G} \frac{1}{\sigma},$$

where c is the light velocity, G is the gravitational constant.

If foliation $\mathcal{F}$ has no a resilient leaf we transform $\mathcal{F}$ into foliation $\mathcal{F}'$ with resilient leaves with the help of non-integrable deformation $\mathcal{F}_t$, $t \in [0, 1]$, $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_1 = \mathcal{F}'$.

The value of energy density jump that one need for this deformation $\mathcal{F} \to \mathcal{F}'$ (with $g^{(5)}_{AB} \to (g')^{(5)}_{AB}$) is equal to

$$\delta[\varepsilon] \sim \frac{\pi c^4}{G} \left[\frac{l(\xi')}{vol'(M^5)} [-2\beta_1'(M^5) + \beta_2'(M^5)] - \frac{l(\xi)}{vol(M^5)} [-2\beta_1(M^5) + \beta_2(M^5)] \right],$$

where $\beta_i(M^5)$ are the Betti's numbers, $l(\xi)$ is the trajectory length of some vector field ξ on M^5.

We can declare that our local power actions in space-time are capable to reconstruct its placement in Hyperspace.

In a dense leaf there is a possibility to make transition in the past, having left in Hyperspace and having passed rather small distance. It is a question: in what moment and from what point of dense leaf such trip is possible? But possibility of such travel exists. If all leaves of the foliation are dense, i.e. the foliation is minimal one, than the travel to the past is possible from of any leaf.

We see that construction of the Time Machine requires the solutions a number of geometrical problems of the foliation theory.

Spherical Growth Series of the Free Product of Groups $\mathbb{Z}_m \times \mathbb{Z}_n$

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Some property of curvature of almost contact 3-structure manifolds

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Inequalities in additive N-isometries on linear N-normed Banach spaces

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Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an *isometry* if f satisfies

$$d_Y\big(f(x), f(y)\big) = d_X(x, y)$$

for all $x, y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively. For some fixed number $r > 0$, suppose that f preserves distance r; i.e., for all x, y in X with $d_X(x, y) = r$, we have $d_Y\big(f(x), f(y)\big) = r$. Then r is called a *conservative* (or *preserved*) distance for the mapping f. Aleksandrov [1] posed the following problem:

Aleksandrov problem: Examine whether the existence of a single conservative distance for some mapping T implies that T is an isometry.

We prove the Hyers-Ulam stability of additive N-isometries on linear N-normed Banach spaces.

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Coarse Ricci curvature on metric measure spaces

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Ollivier defined a coarse Ricci curvature on a metric space with a random walk.

Definition ([3]). Let $(X, d, \{m_x\}_{x\in X})$ be a metric space with a random walk. We define the *coarse Ricci curvature* $\kappa(x, y)$ *along* xy by

$$\kappa(x,y) := 1 - \frac{W_1(m_x, m_y)}{d(x,y)}.$$

He proved an asymptotic estimate for coarse Ricci curvatures of a Riemannian manifold with respect to r-step random walks as $r \to 0$, where r-step random walk m_x is the volume measure normalized and restricted on the ball centered at x of radius r. In this talk, we show an estimate for a coarse Ricci curvature with respect to an r-step random walk for the space satisfying the Bishop-Gromov inequality $[BG_{K,N}]$.

Definition. For two real numbers K and $N > 1$, we define a function $s_{K,N} : [0, \infty) \to \mathbb{R}$ by

$$s_{K,N}(t) := \begin{cases} \sqrt{(N-1)/K}\sin(t\sqrt{K/(N-1)}) & \text{if } K > 0, \\ t & \text{if } K = 0, \\ \sqrt{(N-1)/-K}\sinh(t\sqrt{-K/(N-1)}) & \text{if } K < 0. \end{cases}$$

A metric measure space (X, d, ν) satisfies the *Bishop-Gromov inequality* $[BG_{K,N}]$ if

$$\frac{\nu\left(B_R(x)\right)}{\nu\left(B_r(x)\right)} \le \frac{\int_0^R s_{K,N}(t)^{N-1}\,dt}{\int_0^r s_{K,N}(t)^{N-1}\,dt}$$

holds for any $x \in X$ and for any $0 < r < R \le \pi\sqrt{(N-1)/\max\{K, 0\}}$ with the convention $1/0 = \infty$.

Theorem A ([1]). *Let (X, d, ν) be a geodesic metric space with a Borel measure. Suppose X satisfies the Bishop-Gromov inequality $[BG_{K,N}]$. Then*

$$\inf_{x,y\in X} \kappa(x,y) \ge 1 - 2r\frac{s_{K,N}(r)^{N-1}}{\int_0^r s_{K,N}(t)^{N-1}\,dt}$$

holds for r-step random walk.

As a corollary, we obtain that the space satisfying the curvature-dimension condition has a lower bound of coarse Ricci curvature (see [1]).

We also investigate the L^1-Wasserstein space over a space which has a lower bound of a coarse Ricci curvature.

Theorem B ([2]). *Let $(X, d, \{m_x\}_{x\in X})$ be a metric space with a random walk. Assume the coarse Ricci curvature on $(X, d, \{m_x\}_{x\in X})$ is bounded below by κ_0. Then there exists a random walk $\{\tilde{m}_\mu\}_{\mu\in\mathcal{P}_1(X)}$ on $\mathcal{P}_1(X)$ such that the coarse Ricci curvature on $(\mathcal{P}_1(X), W_1, \{\tilde{m}_\mu\}_{\mu\in\mathcal{P}_1(X)})$ is bounded below by κ_0.*

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The slow surface in a problem of chemical kinetics

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We consider a singularly perturbed system of ordinary differential equations

$$\varepsilon \dot{z} = Z(z, t, \varepsilon), \tag{1}$$

where $z \in \mathbb{R}^{m+n},\ t \in \mathbb{R},\ 0 < \varepsilon \ll 1$, and the vector-function Z is sufficiently smooth in all variables [1].

Basing on the method of integral manifolds [2], we carried out qualitative analysis of a system of the kind (1) with finding the number and the form of the leaves of the integral manifolds, the number of the stationary states of slow subsystems on the stable leaves of the integral submanifolds and their classification, the conditions of multiplicity and oscillations of various types, in particular canard solutions.

The construction of slow integral manifolds is a difficult problem. One of the ways to construct the leaves of the slow surface is the choice of a convinient parametrization.

Recall the notions of a curve and its parametrizations following [3]. Given a metric space M with metric ρ, by a *parameterized curve* or a *path* in M we mean a continuous mapping $x : [a, b] \to M$. A curve in a metric space is defined as an equivalence class of parameterized curves or paths in this space. First, we introduce a parametrization of a simple arc K as a continuous one-to-one mapping of an interval $[a, b]$, where $a < b$, onto K. Then the notion of a simple arc is extended to normal paths. The set of all normal parameterized curves in the space splits into classes of equivalent paths. Each class is called a *curve* in M. If K is a curve in M, i. e., a class of equivalent normal parameterized curves in M, then the elements of this class are called the *parametrizations of* K. We give the example of the curve parametrization.

We consider the system of differential equations corresponding to the catalytic reaction

$$\begin{aligned}
&\dot{x}_1 = 2b_1x_7^2 - b_2x_6x_1 - b_8x_1x_2,\\
&\dot{x}_2 = b_4x_7 - b_5x_2 - b_8x_1x_2 - b_9x_2x_3 - b_{12}x_2x_4,\\
&\dot{x}_3 = b_2x_6x_1 - 2b_3x_3^2 - b_6x_3 + b_7x_5 - b_9x_2x_3 + 2b_{10}x_4x_5 + b_{12}x_2x_4,\\
&\dot{x}_4 = 2b_3x_3^2 - b_{10}x_4x_5 - b_{12}x_2x_4,\\
&\dot{x}_5 = b_6x_3 - b_7x_5 - b_{10}x_4x_5 - b_{11}x_5,
\end{aligned} \tag{2}$$

where $x_6 = 1 - x_3 - x_4 - x_5,\ x_7 = 1 - x_1 - x_2 - x_3 - x_4 - x_5,\ 0 \le x_i \le 1,$ $\sum_{j=1}^{5} x_j \le 1, i = 1, \dots, 5.$

We use the following hierarchy of parameters in the model under study

$$b_{10} > b_8 \gg b_7 > b_1, b_2, b_3, b_4, b_6, b_{11}, b_{12} \gg b_5, b_9.$$

The change of variables

$$s = x_4 - x_5,\ u = x_1 - x_2,\ v = x_3 + x_4 + x_5$$

leads (2) to the singularly perturbed system of equations of the form

$$\dot{x} = f(x, y, t, \varepsilon),\ \varepsilon \dot{y} = g(x, y, t, \varepsilon), \tag{3}$$

where $x = (s, u, v)$ are the slow variables and $y = (x_2, x_5)$ are the fast, $\varepsilon = 1/b_{10}$ is a small positive parameter, $f = (f_1, f_2, f_3)$, $g = (\varkappa f_4/b_8,\ f_5/b_{10})$, $\varkappa = b_8/b_{10},\ f_i,\ i = 1, 2, \dots, 5$, are the right-hand sides of equations (2).

The slow surface is defined by $g(x, y, t, 0) = 0$.

Using the hierarchy of parameters and applying twice the method of integral manifolds to (3) obtained from the five-dimensial system (2), we reduce the analysis of the system to the consideration of the slow susbsystem, consisting in this case of a single equation, on the one-dimensional integral manifold. The use of the method is possible because the system has small parameters.

We have the system with the small parameter, v is a slow variable, u and s are fast variables; $\dot{v} = f_2(u, v, s)$ is the slow subsystem on the slow curve described by the system of two equations

$$\begin{aligned} &ka(1+u-v)^2 - (1+u-v) - bus = 0, \\ &a(v-s)^2 - (v-s) + bus = 0, \end{aligned} \tag{4}$$

where $a = \frac{2b_3}{b_6}$, $b = \frac{b_{12}}{b_6}$, $k = \frac{b_1}{b_3}$.

The slow surface is a one-dimensional closed curve. For its calculation, it is convinient to use the following parametrisation, introducing the parameter $\theta = us$. Then, putting

$$\alpha = \sqrt{1 - 4ab\theta}, \quad \beta = \sqrt{1 + 4abk\theta}$$

and solving the equations (4) with respect to $(1 + u - v)$ and $(v - s)$, obtain

$$\begin{aligned} &2ka(1+u-v) = 1 \pm \beta, \\ &2a(v-s) = 1 \pm \alpha. \end{aligned}$$

On the leaf $S_2(u \leq 0, s \geq 0)$, the quantity θ satisfies the inequality $-1/4abk \leq \theta \leq 0$.

We described the construction of this curve in [4]. Note that the curve may be disconnected set. There is a possibility of the existence of canard solutions in (2).

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Classical and Quantum Dynamics on Orbifolds

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In this talk, we will describe some results on the spectral theory of the Laplace-Beltrami operator on a compact Riemannian orbifold and its connections with dynamics of the geodesic flow.

Let (X, g_X) be a compact Riemannian orbifold, $\dim X = n$, and let Δ_X be the associated Laplace-Beltrami operator on X. The spectrum of Δ_X is discrete, and there is an orthonormal basis of eigenfunctions $\{\psi_j\}$ with corresponding eigenvalues $\{\lambda_j^2\}$: $\Delta_X \psi_j = \lambda_j^2 \psi_j, 0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$. Let f_t be the geodesic flow of the metric g_X on the cosphere bundle $S^*X = \{(x,\xi) \in T^*X : |\xi| = 1\}$. For any zero order pseudodifferential operator $A \in \Psi^0(X)$, we denote by $\sigma_A \in C^\infty(T^*X \setminus 0)$ its principal symbol. Define a functional ω on $\Psi^0(X)$ by

$$\omega(A) = \frac{1}{vol(S^*X)} \int_{S^*X} \sigma_A \, d\mu, \quad A \in \Psi^0(X),$$

where $d\mu$ is the Liouville measure on S^*X. In [2], we extend to orbifolds the classical result on quantum ergodicity due to Shnirelman, Colin de Verdière and Zelditch.

Theorem 1 *If the flow f_t is ergodic on $(S^*X, d\mu)$ then there is a subsequence $\{\psi_{j_k}\}$ of density one such that for any $A \in \Psi^0(X)$*

$$\lim_{k\to\infty} \langle A\psi_{j_k}, \psi_{j_k}\rangle = \omega(A).$$

For the proof of this theorem, we need two results, which maybe of independent interest.

The first result is the local Weyl law for elliptic operators on orbifolds [2]. Let us introduce the generalized eigenvalue distribution function of Δ_X, setting for any $A \in \Psi^0(X)$

$$N_A(\lambda) = \sum_{\{j:\lambda_j\leq\lambda\}} \langle A\psi_j, \psi_j\rangle, \quad \lambda \in \mathbb{R}.$$

Theorem 2 *For any $A \in \Psi^0(X)$, we have as $\lambda \to +\infty$*

$$N_A(\lambda) = \frac{1}{(2\pi)^n} \left(\int_{S^*X} \sigma_A \, d\mu \right) \lambda^n + O(\lambda^{n-1}).$$

As a consequence of Theorem 2, we obtain the Weyl law for the eigenvalue distribution function $N(\lambda) = \#\{j : \lambda_j \leq \lambda\}$ of Δ_X, first proved by Farsi, 2001: as $\lambda \to +\infty$, we have

$$N(\lambda) = \frac{1}{(2\pi)^n} vol(S^*X)\lambda^n + O(\lambda^{n-1}).$$

The second result is the Egorov theorem for orbifolds [1], which relates the evolution of pseudodifferential operators on a compact Riemannian orbifold (quantum observables) determined by the first order elliptic operator $P = \sqrt{\Delta_X}$ with the corresponding evolution of classical observables given by the action of the geodesic flow f_t on the space of symbols.

Theorem 3 *For any pseudodifferential operator A of order 0 on X with the principal symbol $\sigma_A \in C^\infty(T^*X \setminus 0)$, the operator $A(t) = e^{it\sqrt{\Delta_X}} A e^{-it\sqrt{\Delta_X}}, t \in \mathbb{R}$, is a pseudodifferential operator of order 0 on X. Moreover, its principal symbol $\sigma_{A(t)} \in C^\infty(T^*X \setminus 0)$ is given by*

$$\sigma_{A(t)}(x,\xi) = \sigma_A(f_t(x,\xi)), \quad (x,\xi) \in T^*X \setminus 0.$$

To provide examples of ergodic geodesic flows, we prove in [2] the orbifold version of a classical result by Anosov.

Theorem 4 *The geodesic flow on a compact Riemannian orbifold of negative sectional curvature is ergodic.*

As an example of a compact Riemannian orbifold of negative sectional curvature, one can consider the hyperbolic orbifold $\mathbb{H}^n/\Gamma$, which is the quotient of the n-dimensional hyperbolic space $\mathbb{H}^n$ by a cocompact discrete group Γ of orientation-preserving isometries of $\mathbb{H}^n$.

Finally, we will discuss noncommutative geometry of orbifolds and noncommutative versions of the results mentioned above [1].

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Invariant Decomposable Almost Complex Structures on Homogeneous Spaces

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Let M be a Riemannian homogeneous space of dimension $2\,n$. G is a connected Lie group that transitive acts on M, H is a isotropy subgroup for element $o \in M$, and g is Ad_H-invariant Riemannian metric on G. Then Lie algebra $\mathfrak{g}$ of Lie group G is a direct sum of isotropy subalgebra $\mathfrak{h}$ and subspace $\mathfrak{p}$ that is orthogonal complementary of $\mathfrak{h}$ with respect to metric g. The metric g produces the G-invariant metric on M, and is identified with it. It is known that $\mathfrak{p}$ can be decomposed into orthogonal sum of minimal irreducible Ad_H-invariant subspaces. Subspace $\mathfrak{p}$ is isomorphic to the tangent space T_oM, and any left-invariant smooth endomorphisms field J on G so that $J^2 = -\,id\,|_{\mathfrak{p}}$ determines the G-invariant almost complex structure on M. This almost complex structure is called an *Invariant Almost Complex Structure on Homogeneous Space M.*

Definition 1. *An almost complex structure J is called an decomposable if exists an continuous distribution of proper tangent subspaces invariant under the action of J, and for any such distribution exists continuous complementary distribution that also is invariant under the action of J.*

It is clearly that on homogeneous space M an decomposable almost complex structure provides decomposition $\mathfrak{p}$ into direct sum of minimal subspaces irreducible with respect to action of this decomposible structure. The number of these minimal subspaces is called an *Decomposable Almost Complex Structure Index.* Denote by i_J an Decomposable Almost Complex Structure Index. Then $2 \le i_J \le n$, where $2\,n = \dim M$.

For an decomposable almost complex structure having index 2 or more may be proved next useful result:

Theorem 1. *Let $M \cong G/H$ be a homogeneous space, and J be an invariant decomposable almost complex structure on M. Let group G has nontrivial center, A and B are subspaces in $\mathfrak{p}$ invariant under the action of J so that $\mathfrak{p} = A \oplus B$. Let A belongs to center of Lie algebra $\mathfrak{g}$, and Nijenhuis tensor is equal to zero on B. Then almost complex structure J is integrable on M.*

Remark. If homogeneous space M is decomposed into direct product of two-dimensional subspaces, then M always admits an integrable decomposable almost structure having maximal index n.

The next theorem describes intersection of classes of orthogonal and decomposable almost complex structures.

Theorem 2. *An orthogonal invariant almost complex structure can be decomposable if and only if orthogonal complementary of isotropy subalgebra comtains proper subspace invariant under the action of this orthogonal structure.*

To describe relationship between Ad_H-invariant subspaces in $\mathfrak{p}$ and subspaces invariant under the action of decomposable almost complex structure, it is needed to determine two special classes of homogeneous spaces.

Definition 2. *An Riemannian homogeneous space $M \cong G/H$ is called an Regular-reducible if for any Ad_H-invariant subspace in $\mathfrak{p} = \mathfrak{h}^{\perp}$ exists Ad_H-invariant orthogonal complementary having the same dimension. And it is called an Irregular-reducible if $\mathfrak{p}$ contains even one Ad_H-invariant subspace whose orthogonal complementary has not the same dimension. An irregular-reducible homogeneous space is called an Strictly Irregular-reducible if all Ad_H-invariant subspace in $\mathfrak{p}$ have different dimension.*

An invariant decomposable almost complex structure is said to be Isotropic Invariant if any subspace invariant under the action of this structure is Ad_H-invariant. Now we can state the next result:

Theorem 3. *Let $M \cong G/H$ be an irregular-reducible homogeneous space equipped with G-invariant Riemannian metric. Let also any Ad_H-invariant subspace in orthogonal complementary $\mathfrak{p}$ of isotropy subalgebra $\mathfrak{h}$ has even dimension. Then any left-invariant almost complex structure on group G, mapping $\mathfrak{p}$ to itself and keeping decomposition of $\mathfrak{p}$ into orthogonal sum of Ad_H-irreducible terms, determines an G-invariant decomposable almost complex structure on M. Along with it, the Ad_H-irreducible subspaces are invariant under the action of this almost complex structure. If additionally, M is an strictly irregular-reducible homogeneous space, and almost complex structure is an isotropic invariant, then invariant under* *the action of almost complex structure irreducible subspaces coincide with Ad_H-irreducible subspaces, and a decomposable almost complex structure index is equal to the number of Ad_H-irreducible subspaces in $\mathfrak{p}$.*

Let Ω be an invariant symplectic form on homogeneous space M. We can choose basic invariant 1-form $\theta^1, \ldots, \theta^{2n}$ on M so that

$$\Omega = \theta^1 \wedge \theta^2 + \ldots + \theta^{2n-1} \wedge^{2n} .$$

Choosing the dual basis of invariant vector fields $e_1, \ldots, e_{2n}$ we can give the set of two-dimensional subspaces

$$\{e_1, e_2\}, \ldots, \{e_{2n-1}, e_{2n}\}.$$

It is possible to show that any invariant maximal index n decomposable almost complex structure, having these subspace as minimal invariant subspaces, has the next view with respect to above basis:

$$\begin{pmatrix} A_1 & 0 & \ldots & 0 \\ \vdots & \ldots & \ldots & \vdots \\ 0 & \ldots & 0 & A_n \end{pmatrix},$$

where

$$A_k = \begin{pmatrix} a_k & b_k \\ -\frac{a_k^2+1}{b_k} & -a_k \end{pmatrix}, \ b_k < 0.$$

Denote this decomposable almost complex structure by $J_z : z = (a_1 + ib_1, \ldots, a_n + ib_n)|_{b_k<0}, k = 1, 2, \ldots, n, i = \sqrt{-1}$.

The invariant decomposable almost complex structure J_z holds on symplectic form Ω while all available values of z, i.e. $\Omega \circ J_z = \Omega$. It makes possible to give the n-parameter family of almost Kähler metrics

$$g_z : g_z(X, Y) = \Omega(X, J_z Y), \ \forall \ X, Y \in \mathfrak{p}.$$

Let ∇ be the Levi-Civita connection for metric g_z. Since in invariant case ∇J_z is a tensor of type $(2, 1)$, we can determine $\|\nabla J_z\|$ as the Euclidian norm of tensor ∇J_z. Denote by Σ the surface in $\mathbb{C}^n$ that vanishes the function $\|\nabla J_z\|$. It is known that condition $\nabla J_z = 0$ implies the integrability of almost complex structure J_z. We obtain that surface Σ parameterizes the Kähler metrics associated with symplectic form Ω. By this way, each invariant symplectic form on real homogeneous space determines the remarkable surface in $\mathbb{C}^n$ which can be viewed as surface of Kähler metrics in the associated metrics space.

Perfect prismatoids and the conjecture concerning with face numbers of centrally symmetric polytopes

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We construct a class of centrally symmetric polytopes – perfect prismatoids and prove some its properties related to the famous conjecture concerning with face numbers of centrally symmetric polytopes. Also it is proved that any Hanner polytope is a perfect prismatoid and any perfect prismatoid is affinely equivalent to some 0/1-polytope.

A polytope $P \subset \mathbb{R}^d$ is centrally symmetric (or cs, for short) if $P = -P$. We say that faces F and F' of a cs polytope P are *antipodal* if $F = -F'$. Let f_i be a number of i-faces of a cs polytope P. Define $f(P)$ by the total number of non-empty faces of P.

In 1989 Kalai stated the conjecture known as the 3^d-conjecture that every cs d-polytope has at least 3^d non-empty faces. In dimension $d = 3$ the conjecture follows from Euler's relation. In $d = 4$ the conjecture was proved by Ziegler, Werner, Sanyal in 2007. For simplicial and simple d-polytopes the conjecture follows from Stanley's results obtained in 1987.

An important class that attains the bound is the class of Hanner polytopes. It is unknown whether there exist other polytopes with exactly 3^d non-empty faces. To define Hanner polytopes we introduce the concept of a cross.

Let $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ be subspaces of $\mathbb{R}^{d_1+d_2}$ and $\mathbb{R}^{d_1} \cap \mathbb{R}^{d_2} = \{O\}$. Let a d_1-dimensional polytope $P_1 \subset \mathbb{R}^{d_1}$ and a d_2-dimensional polytope $P_2 \subset \mathbb{R}^{d_2}$ be centrally symmetric polytopes with the common centre of symmetry O. Then $P = conv\{P_1, P_2\}$. is called a cross of P_1 and P_2. The cross of P_1 and P_2 is denoted by $P_1 \boxtimes P_2$.

Hanner polytopes are defined recursively: every centrally symmetric 1-polytope is a Hanner polytope. For dimensions $d \geqslant 2$, a d-polytope is a Hanner polytope if it is the direct product or the cross of two lower dimensional Hanner polytopes.

A centrally symmetric polytope P is called *prismatoid with bases F and F'* if $P = conv(F \cup F')$ where F and F' are antipodal facets of F. A centrally symmetric convex polytope P is called a *perfect prismatoid* if $P = conv(F \cup F')$ for any pair of its antipodal facets F and F'. This definition is equivalent to the following one: every pair of antipodal facets contains all vertices of P.

In case $d = 3$ the class of perfect prismatoids and the class of Hanner polytopes coincides. (They contain only the cube and octahedron). We prove that in $d > 4$ the class of perfect prismatoids contains the class of Hanner polytopes but does not coincide with it.

The dual P^* of a perfect prismatoid P, the direct product $P_1 \times P_2$ and cross $P_1 \boxtimes P_2$ of two perfect prismatoids P_1 and P_2 are perfect prismatoids. So every Hanner polytope is a perfect prismatoid. For $d \leqslant 4$ any perfect prismatoid is a Hanner polytope. But there exist perfect d-prismatoids that are not Hanner polytopes in $d \geqslant 5$.

On non-linear spectral gap for symmetric Markov chains with coarse Ricci curvature

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I will talk about the positivity of non-linear spectral gap for energy functional over L^2-maps into complete separable CAT(0)-space (more generally, complete separable 2-uniformly convex space satisfying some geometric conditions) in terms of symmetric Markov chains, equivalently global Poincaré inequality for L^2-maps into such spaces, provided its Ollivire's coarse Ricci curvature is bounded below by a positive constant. I will also provide a generalization of inequalities for spectral gaps between linear and non-linear one in terms of Izeki-Nayatani invariant $\delta(Y)$ for CAT(0)-space Y extending the result by Izeki-Kondo-Nayatani for finite graphs. This is a joint work with E. Kokubo, who was my student in master course last year.

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Convex Hull of a Poisson Point Process in the Clifford Torusk

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Convex Hull of a Poisson Point Process in the Clifford Torus

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Modern stochastic geometry appeared in 1960s as a branch of integral geometry. However, stochastic questions in geometry are known to appear much earlier. For example, the first famous problem connecting geometry and probability is a needle problem due to Buffon (1777). Other examples are geometrical expression of χ^2 statistical criterion by Student (1907) and results of Blaschke's integral geometry school (1930s).

The aim of stochastic geometry is constructing and studying various models of spatial structures. Among them there are discrete random point sets (so called *point processes*), boolean random sets, hyperplane processes and so on. Different topics of modern stochastic geometry are described in [1].

Considering random polytopes inscribed into a sphere $\mathbb{S}^d$ can be motivated in two ways. On the one hand, the combinatorics of these polytopes is exactly the combinatorics of a random Delaunay tessellation of $\mathbb{S}^d$. Indeed, let $A \subset \mathbb{S}^d$ be a generic finite point set, then the spherical Delaunay tessellation $\mathcal{D}(A)$ is combinatorially isomorphic to the polytope $\operatorname{conv} A$. Properties of random Delaunay tessellations have been extensively studied throughout recent decades. The first result of this kind is obtained in [2] and refers to the Delaunay-Poisson tessellation of $\mathbb{E}^3$.

On the other hand, a lot of research has been done on the topics concerning convex hulls of random point sets. The most usual models are distributions of points in a convex set (see [4], for example). However, there are numerous results for convex hulls of finite point sets chosen randomly on some surface, for instance, [5] and [6]. It will be demonstrated that there are methods applicable in both cases.

Convex hulls of finite subsets of the Clifford Torus have been studied by N. Dolbilin and M. Tanemura [3].

In the 4-dimensional Euclidean space $\mathbb{E}^4$ consider the two-dimensional Clifford torus

$$T^2 = \{(\cos\phi, \sin\phi, \cos\psi, \sin\psi) : -\pi < \phi, \psi \le \pi\}.$$

Clearly, T^2 is a submanifold of the three-dimensional sphere

$$S^3_{\sqrt{2}} = \{(\xi_1, \xi_2, \xi_3, \xi_4) : \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = 2\}.$$

The paper [3] defines the notion of a *regular* finite set $A \subset T^2$ and completely describes the hyperfaces of $\operatorname{conv} A$. Besides this, a computer simulation of a Poisson process $P_\lambda(\omega) \subset T^2$ has been performed. As a result, certain conjectures about the combinatorics of a random polytope $P(\lambda) = \operatorname{conv} P_\lambda(\omega)$ have been proposed. In particular, it was suggested that the mean valence of a vertex of $P(\lambda)$ has expectation of magnitude $O^*(\ln\lambda)$.

The following results hold for $P(\lambda)$.

Theorem 1. *For $i = 1, 2, 3$ one has $\mathsf{E}\, f_i(P(\lambda)) = O^*(\lambda \ln \lambda)$ as $\lambda \to \infty$.*

Theorem 2. *The expectation of the mean valence of a vertex of $P(\lambda)$ has asymptotics $\mathsf{E}\,\bar{v}(P(\lambda)) = O^*(\ln\lambda)$ as $\lambda \to \infty$.*

Remark. The statement of theorem 2 is exactly the conjecture posed by Dolbilin and Tanemura.

All the proofs are conducted in essentially the same way using the notion of a *cap*, which is an intersection $C = T^2 \cap H$, where H is a closed half-space. First one obtains an integral expression for $\mathsf{E}\, f_i(P(\lambda))$ or $\mathsf{E}\,\bar{v}(P(\lambda))$. After that the integral can be estimated either by reparametrisation of caps (which is simply a variable substitution under the integral), or using the *cap covering technique* similar to one described in [4].

The cap covering technique allows to estimate the variance of $f_3(P(\lambda))$.

Theorem 3. $\operatorname{Var} f_3(P(\lambda)) \ll \lambda \ln^2 \lambda$.

Theorem 3 immediately implies *the law of large numbers* for $f_3(P(\lambda))$.

Theorem 4.

$$\mathsf{P}\left(\left|\frac{f_3(P(\lambda)) - \mathsf{E}\, f_3(P(\lambda))}{\mathsf{E}\, f_3(P(\lambda))}\right| > \varepsilon\right) \ll \frac{1}{\lambda\varepsilon^2}.$$

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On pyramids in a three-dimensional normed space

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Is it true that vertex set of every Euclidean tetrahedra can be isometrically embedded in a arbitrary three-dimensional normed space?

We prove this for tetrahedra of two special types: for every regular triangular pyramid and tetrahedra with ratio of lengthes of it's edges not less $(\sqrt{8/3}+1)/3 < 0.878$.

Rigidity result related to mapping class groups

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Mapping class groups (MCG's) of compact (oriented connected, possibly with punctures) surfaces of nonexceptional type are very mysterious. In some aspects, they behave like higher rank lattices, but in other aspects they also do like rank one lattices. The following result, well-known as the "*Farb–Kaimanovich–Masur superrigidity*" [3], [2], states a typical rank one phenomenon of MCG's:

> *Every group homomorphism from higher rank lattices* (*such as* $\mathrm{SL}(3,\mathbb{Z})$ *and cocompact lattices in* $\mathrm{SL}(3,\mathbb{R})$) *into MCG's has* **finite** *image.*

In this talk, we show a generalization of the superrigidity above, to the case where higher rank lattices are replaced with some (*non-arithmetic*) *matrix groups over general rings*. Our main examples of such groups are called the "**universal lattice**" and "**symplectic universal lattice**", that are, the special linear group and the symplectic group over commutative finitely generated polynomial rings over integers, (such as $\mathrm{SL}(3,\mathbb{Z}[x])$ and $\mathrm{Sp}(4,\mathbb{Z}[x,y])$). This result may be regarded as a generalization of the FKM superrigidity of higher rank lattices of the form of $\mathrm{SL}(n,\mathcal{O})$ and $\mathrm{Sp}(2n',\mathcal{O})$, where n at least 3, n' at least 2, and $\mathcal{O}$ is the ring of integers (or S-integers) of a number field.

Furthermore, we generalize the FKM superrigidity of *cocompact* lattices in $\mathrm{SL}_{n\geq 3}$ and $\mathrm{Sp}_{2n\geq 4}$. All these results are taken from our preprint [4].

These generalizations are proved with the aid of study of *quasi*-(1-)*cocycles* into **nontrivial** unitary representations (it is inspired by an alternative proof of the FKM theorem by Bestvina and Fujiwara [1] via study of quasi-homomorphisms). This study relates "*property* (TT)/T", which we introduce in [4] as a weakening of Monod's property (TT) (and a strengthening of Kazhdan's property (T)).

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Two-generated groups of Mobius transformations: geometry and sufficient conditions of the discreteness [1]

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Let $\mathbb{M}$ be the group of all Mobius (linear fractional) transformations of the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. An element $f \in \mathbb{M} - \{Id\}$ is called *loxodromic* if f is conjugate to $z \to \lambda z$, $|\lambda| > 1$; *elliptic* if f is conjugate to $z \to \lambda z$, $|\lambda| = 1$; *parabolic* if f is conjugate to $z \to z + 1$. It is well known that $\mathbb{M} \cong SL(2, \mathbb{C})/\{\pm Id\}$. A group $G < \mathbb{M}$ is said to be *discrete* if the identity is isolated in G.

Let $Isom^+(\mathbb{H}^3)$ be the group of all orientation-preserving isometries of hyperbolic 3-space; then $\mathbb{M} \cong Isom^+(\mathbb{H}^3)$. A group $G < \mathbb{M}$ is called *elementary* if there exists a finite G-orbit in $\mathbb{H}^3 \cup \overline{\mathbb{C}}$. Otherwise, it is *nonelementary*. All elementary discrete groups are described. Jorgensen showed [2] that a nonelementary group $G < \mathbb{M}$ is discrete if and only if every its two-generated subgroup is discrete. Thus, the problem of discreteness for subgroups of $\mathbb{M}$ is reduced to the problem of discreteness for two-generated subgroups of $\mathbb{M}$.

Gehring, Maclachlan, Martin [1] and Rasskazov [3] found the sufficient conditions that a subgroup of $\mathbb{M}$ with two elliptic generators is nonelementary and discrete. We obtain the sufficient conditions that a subgroup of $\mathbb{M}$ with two nonelliptic generators is nonelementary and discrete (theorems 1–5).

Let $f \in \mathbb{M} - \{Id\}$ be nonparabolic. The hyperbolic line joining two fixed points of f in $\overline{\mathbb{C}}$ is called the *axis* of f and is denoted by ℓ_f. If f is elliptic then f is a rotation about ℓ_f. If f is loxodromic then f is the composition of the translation along ℓ_f by an amount $\tau(f) > 0$ and a rotation about ℓ_f. Let us denote $\alpha(f) = \arcsin\left(\dfrac{1}{\cosh(\tau(f)/2)}\right)$.

Let $f, g \in \mathbb{M} - \{Id\}$ be nonparabolic, $\delta(f, g)$ be the hyperbolic distance between ℓ_f and ℓ_g, and $\theta(f, g)$ be the dihedral angle between the hyperbolic planes containing each axes and the common perpendicular respectively.

Theorem 1. Let $f, g \in \mathbb{M}$ be loxodromic; if

$$\text{either} \quad \alpha(f) + \alpha(g) \le \theta(f, g)$$

$$\text{or} \quad \alpha(f) + \alpha(g) > \theta(f, g) \quad \text{and} \quad \cosh\delta(f, g) \ge \frac{\cosh\dfrac{\tau(f)}{2}\cosh\dfrac{\tau(g)}{2}\cos\theta(f, g) + 1}{\sinh\dfrac{\tau(f)}{2}\sinh\dfrac{\tau(g)}{2}},$$

then the group $\langle f, g\rangle$ is discrete, nonelementary and $\langle f, g\rangle = \langle f\rangle * \langle g\rangle$.

Theorem 2. Let $f \in \mathbb{M}$ be loxodromic, $g \in \mathbb{M}$ be elliptic of order $n \ge 2$, and

$$\sinh\delta(f, g) \ge \frac{\cosh\dfrac{\tau(f)}{2}\cos\dfrac{\pi}{n}\sin\theta(f, g) + 1}{\sinh\dfrac{\tau(f)}{2}\sin\dfrac{\pi}{n}};$$

then the group $\langle f, g\rangle$ is discrete, nonelementary and $\langle f, g\rangle = \langle f\rangle * \langle g\rangle$.

Let $f \in \mathbb{M}$ be parabolic, and $g \in \mathbb{M} - \{Id\}$ be nonparabolic. After a preliminary conjugation we may assume that f has the form $f(z) = z + \sigma e^{i\psi}$, where $\sigma = \sigma(f, g) > 0$ and $\psi = \psi(f, g) \in (-\frac{\pi}{2}; \frac{\pi}{2}]$, and that the fixed points of f are $\pm 1 \in \overline{\mathbb{C}}$.

Theorem 3. Let $f \in \mathbb{M}$ be parabolic, and $g \in \mathbb{M}$ be loxodromic; if

$$\text{either} \quad \sigma(f, g) \ge 2 \cdot \frac{\cosh\dfrac{\tau(g)}{2}\cos\psi(f, g) + 1}{\sinh\dfrac{\tau(g)}{2}}$$

$$\text{or} \quad \alpha(g) \le |\psi(f, g)| < \frac{\pi}{2} \quad \text{and} \quad \sigma(f, g) \ge 2 \cdot \frac{1}{\sinh\dfrac{\tau(g)}{2}},$$

then the group $\langle f, g\rangle$ is discrete, nonelementary and $\langle f, g\rangle = \langle f\rangle * \langle g\rangle$.

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Theorem 4. Let $f \in \mathbb{M}$ be parabolic, $g \in \mathbb{M}$ be elliptic of order $n \geq 2$, and

$$\sigma(f,g) \geq 2 \cdot \frac{\cos\dfrac{\pi}{n}|\sin\psi(f,g)| + 1}{\sin\dfrac{\pi}{n}};$$

then the group $\langle f, g\rangle$ is discrete, nonelementary and $\langle f, g\rangle = \langle f\rangle * \langle g\rangle$.

Let $f, g \in \mathbb{M}$ be parabolic. After a preliminary conjugation we may assume that f has the form $f(z) = z + \xi e^{i\eta}$, where $\xi = \xi(f,g) > 0$ and $\eta = \eta(f,g) \in (-\frac{\pi}{2}; \frac{\pi}{2}]$, and that g has the form $g(z) = \frac{z}{z+1}$.

Theorem 5. Let $f, g \in \mathbb{M}$ be parabolic, and

$$\xi(f,g) \geq 2 \cdot (\cos\eta(f,g) + 1);$$

then the group $\langle f, g\rangle$ is discrete, nonelementary and $\langle f, g\rangle = \langle f\rangle * \langle g\rangle$.

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Conformal circles and the geometry of the 3rd order ODEs systems

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Collapsing three-dimensional Alexandrov spaces

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We report that the topologies of collapsing three-dimensional Alexandrov spaces without boundaries are determined. The talk is based on a joint work with Takao Yamaguchi [MY].

An Alexandrov space is a geodesic space having a notion of curvature bound. Such a space is a generalization of Riemannian manifold and is naturally appeared in converging and collapsing phenomenon of Riemannian manifolds. We recall the definition of Alexandrov spaces. We say that a complete metric space M is an *Alexandrov space of curvature bounded from below by* -1 if it is a geodesic space and any geodesic triangle $\triangle$ in M is not thiner than a comparison triangle $\tilde{\triangle}$ in the hyperbolic plane $\mathbb{H}^2$. Here, $\tilde{\triangle}$ denotes the triangle consisting of edges of the same length as $\triangle$.

For $n \in \mathbb{N}$ and $D > 0$, let us denotes by $\mathfrak{Alex}(n, D)$ the set of all isometry classes of n-dimensional Alexandrov spaces of curvature ≥ -1 and diameter $\leq D$. Let $\{M_i\}_{i=1,2,\dots} \subset \mathfrak{Alex}(n, D)$ be a sequence of n-dimensional Alexandrov spaces. Due to Gromov's compactness theorem, there exists a subsequence $\{M_{n(i)}\}$ of $\{M_i\}$ converging to an $X \in \mathfrak{Alex}(k, D)$ in the Gromov-Hausdorff topology, for some $k \leq n$. From now on, we consider a converging sequence $\{M_i\} \subset \mathfrak{Alex}(n, D)$. We say that $\{M_i\} \subset \mathfrak{Alex}(n, D)$ converging to X is *collapsing* if $\dim X < \dim M_i = n$. According to Perelman's stability theorem, if $\{M_i\}$ converges and does not collapse to X, i.e. $\dim X = \dim M_i$, then M_i is homeomorphic to X for large i. A purpose of collapsing theory is finding an answer to the following problem.

Problem. When a sequence $\{M_i\} \subset \mathfrak{Alex}(n, D)$ of Alexandrov spaces collapses to an Alexandrov space X, is there any geometric relation between M_i and X for large i?

Recently, we determine the topologies of collapsing three-dimensional Alexandrov spaces without boundaries.

Main Result [MY]. Let $\{M_i\} \subset \mathfrak{Alex}(3, D)$ be a sequence of three-dimensional Alexandrov spaces of curvature ≥ -1 and of diameter $\leq D$. Suppose that all M_i have no boundary. We assume that $\{M_i\}$ collapses to some Alexandrov space $X \in \mathfrak{Alex}(k, D)$ with $k \leq 2$. Then, without exceptional cases, we determine the topology of M_i in terms of the topology of X for large i.

This result is a generalization of the works of Shioya-Yamaguchi [SY] and Fukaya-Yamaguchi [FY]. They determined the topologies of collapsing three-dimensional closed Riemannian manifolds having uniformly a lower curvature bound and an upper diameter bound, up to Poincaré conjecture.

In the talk, we concentrate to introduce the following result in our main result.

Theorem [MY]. Let $\{M_i\} \subset \mathfrak{Alex}(3, D)$ and each M_i has no boundary. Suppose that $\{M_i\}$ collapses to an Alexandrov surface $X \in \mathfrak{Alex}(2, D)$ and X has no boundary. Then, M_i is a generalized Seifert fibered space over X for large i.

Here, a *generalized Seifert fibered space* is a generalized notion of usual Seifert fibered space. It is a "singular circle bundle" over a surface which may have a singular fiber homeomorphic to an interval. In other words, it is considered as a circle bundle over a surface in the orbifold-sense.

If time permits, we will talk another result in our main result.

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On knots and links in lens spaces

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We study some aspects of knots and links in lens spaces. Namely, if we consider lens spaces as a quotient of the unit ball B^3 with suitable identification of boundary points, then we can project the links on the equatorial disk of B^3, obtaining a regular diagram for them. The main results are: a complete finite set of Reidemeister type moves establishing equivalence, up to ambient isotopy; a Wirtinger type presentation for the fundamental group of the complement of the link; a diagrammatic method giving the torsion of the first homology group of the link. We also compute Alexander polynomial and twisted Alexander polynomials of this class of links, and we show their correlation with Reidemeister torsion. Furthermore, we prove that the twisted Alexander polynomials of a local link vanish and its Alexander polynomial gives information about the order of the torsion of the link homology.

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Planar loops with prescribed curvature: existence, multiplicity and uniqueness results

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Let $k : \mathbb{C} \to \mathbb{R}$ be a smooth given function. A k-loop is a closed curve u in $\mathbb{C}$ having prescribed curvature $k(p)$ at every point $p \in u$. We use variational methods to provide sufficient conditions for the existence of k-loops. Then we show that a breaking symmetry phenomenon may produce multiple k-loops, in particular when k is radially symmetric and somewhere increasing.

If $k > 0$ is radially symmetric and non increasing we prove that any embedded k-loop is a circle, that is, round circles are the only convex loops in $\mathbb{C}$ whose curvature is a non increasing function of the Euclidean distance from a fixed point. Our proof uses the Osserman construction for the four vertex theorem, which is based on an essential way on the Alexandrov moving planes idea. Several uniqueness (up to homothety) results are available for similar geometrical problems, starting from the pioneering paper [1] by Alexandrov.

Our uniqueness result is sharp, as there exist radially increasing curvatures $k > 0$ which have embedded k-loops that are not circles.

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On Killing fields of constant length on Riemannian manifolds

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This talk is devoted to some recent results on Killing vector fields of constant length on Riemannian manifolds, that are obtained mainly in our joint with Prof. V.N. Berestovskii papers [1-3].

Recall that a smooth vector field X on a Riemannian manifold (M, g) is called *Killing* if $L_X g = 0$. A vector field X on a (complete) Riamannian manifold (M, g) is Killing if and only if it generates a flow on M, that consists of isometries of (M, g).

It is easy to see, that a Killing vector field X on a Riemannian manifold (M, g) has *constant length* if and only if every integral curve of the field X is a geodesic in (M, g).

Recall that *Clifford-Wolf translation* in a Riemannian manifold (M, g) is an isometry moving all points in M one and the same distance. It should be noted, that Killing vector fields of constant length are closely related with Clifford-Wolf translations on Riemannian manifolds [1]. In particular, if a one-parameter isometry group $\gamma(t)$ on (M, g), generated by a Killing vector field X, consists of Clifford-Wolf translations, then X has constant length.

In the paper [2], nontrivial Killing vector fields of constant length and corresponding flows on smooth complete Riemannian manifolds are investigated. It is proved that such a flow on *symmetric space* is free or induced by a free isometric action of the circle S^1. Moreover, it is proved in [2], that a one-parameter isometry group $\gamma(t)$, generated by a Killing vector field X of constant length on a *symmetric space* (M, g), consists of Clifford-Wolf translations.

The following theorem plays a key role in the study of *Clifford-Wolf homogeneous Riemannian manifolds* [3].

Theorem 1 ([3]). *For any Killing field of constant length Z on a Riemannian manifold (M, g) the equality*

$$(\nabla_Z R)(\cdot, Z)Z = 0$$

holds at every point of M, where R is the curvature tensor of (M, g).

An inner metric space (M, ρ) is called *Clifford-Wolf homogeneous* if for every two points y, z in M there exists a Clifford-Wolf translation of the space (M, ρ) moving y to z. Obviously, every Euclidean space E^n is Clifford-Wolf homogeneous. Since E^n can be treated as a (commutative) additive vector group with bi-invariant scalar product, the following example can be considered as a generalization.

Let G be a Lie group supplied with a bi-invariant Riemannian metric ρ. In this case both the group of left shifts $L(G)$ and the group of right shifts $R(G)$ consist of Clifford-Wolf isometries of (G, ρ). Therefore, (G, ρ) is Clifford-Wolf homogeneous. With using of Theorem 1, the following remarkable result was obtained.

Theorem 2 ([3]). *Every simply connected Clifford-Wolf homogeneous Riemannian manifold is a direct metric product of an Euclidean space, odd-dimensional spheres of constant curvature and simply connected compact simple Lie groups supplied with bi-invariant Riemannian metrics (some of these factors may absent).*

Finally, we clarify connections between Killing fields of constant length on a Rimannian geodesic orbit manifold (M, g) and the structure of its full isometry group. A Riemannian manifold (M, g) is called *a manifold with homogeneous geodesics* or *a geodesic orbit manifold*, if any geodesic γ of M is an orbit of a one-parameter subgroup of the full isometry group of (M, g). This terminology was introduced in [4] by O. Kowalski and L. Vanhecke, who initiated a systematic study of such spaces.

Geodesic orbit Riemannian manifold may be considered as a natural generalization of symmetric spaces, classified by É. Cartan. Indeed, a simply connected symmetric space can be defined as a Riemannian manifold (M, g) such that any geodesic $\gamma \subset M$ is an orbit of one-parameter group g_t of transvections, that is one-parameter group of isometries which preserves γ and induces the parallel transport along γ. If we remove the assumption that g_t induces the parallel transport, we get the notion of a geodesic orbit manifold.

The class of geodesic orbit Riemannian manifolds is much larger then the class of symmetric spaces. For example, any homogeneous space $M = G/H$ of a compact Lie group G admits a metric g^M such that (M, g^M) is a a geodesic orbit Riemannian manifold. It is sufficient to take the metric g^M which is induced with a bi-invariant Riemannian metric g on the Lie group G such that $(G,g) \to (M = G/H, g^M)$ is a Riemannian submersion with totally geodesic fibres. Such homogeneous space $(M = G/H, g^M)$ is called *a normal homogeneous space*. More generally, any *naturally reductive manifold* is geodesic orbit. Recall, that a Riemannian manifold (M, g^M) is called *naturally reductive* if it admits a transitive Lie group G of isometries with a bi-invariant pseudo-Riemannian metric g, which induces the metric g^M on $M = G/H$. An important class of geodesic orbit Riemannian manifolds consists of weakly symmetric spaces, introduced by A. Selberg [5]. Recall, that a Riemannian manifold (M, g) is a *weakly symmetric space*, if any two points $p, q \in M$ can be interchanged by an isometry of (M, g).

The Lie algebra of *the full isometry group* of a Riemannian manifold (M, g) is naturally identified with the Lie algebra of Killing fields $\mathfrak{g}$ on (M, g). The following result is useful for a study of geodesic orbit Rimannian manifolds.

Theorem 3 ([6]). *Let (M, g) be a geodesic orbit Rimannian manifold, $\mathfrak{g}$ is its Lie algebra of Killing fields. Suppose that $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$. Then any $X \in \mathfrak{a}$ has constant length on (M, g).*

On the ground of Theorem 3 one can give (see [6]) a new proof of one result of C. Gordon [7]: Every Riemannian geodesic orbit manifold of nonpositive Ricci curvature is a symmetric space.

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Nontriviality of the Jones Polynomial

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Using a simple recurrence relation, we give a new method to compute the Jones polynomials of closed braids. We show that in any sequence of braids, there are a few trivial Jones polynomials.

On the existence of harmonic maps via exponentially harmonic maps

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In the present talk, I would like to introduce an approximation of harmonic maps via exponentially harmonic maps. More precisely, several convergences, as $\varepsilon \to 0$, of sequences of critical points of the *exponential energy functional*

$$\mathbb{E}_\varepsilon(u) := \int_M \frac{e^{\varepsilon |du|^2} - 1}{\varepsilon} \, d\mu_g$$

is discussed. Such a critical point is known to always exist in a given homotopy class of continuous maps between closed Riemannian manifolds and is known to have full regularity.

In the case that the target manifold has nonpositive sectional curvature, the following is obtained ([1]) and then the existence theorem (due to Eells-Sampson) of harmonic maps into a nonpositively curved manifold is reproved.

Theorem A. *Let (M, g) and (N, h) be compact and connected Riemannian manifolds without boundary. Assume that (N, h) has nonpositive sectional curvature. For every $\varepsilon > 0$, let $u_\varepsilon : (M, g) \to (N, h)$ be a smooth map which is a critical point of $\mathbb{E}_\varepsilon$ with respect to its smooth variation and has uniformly bounded energy*

$$\mathbb{E}_\varepsilon(u_\varepsilon) = \int_M \frac{e^{\varepsilon |du_\varepsilon|^2} - 1}{\varepsilon} \, d\mu_g \leq E_0$$

with a constant $E_0 > 0$. Then there exist a positive sequence $\varepsilon(k) \to 0$ as $k \to \infty$ and a harmonic map $u : (M, g) \to (N, h)$ such that the sequence $\{u_{\varepsilon(k)}\}_{k=1}^\infty$ converges uniformly with all derivatives to u:

$$u_{\varepsilon(k)} \to u \quad \text{in } C^\infty(M, N) \text{ as } k \to \infty.$$

If one tries to remove the curvature assumption on (N, h) in Theorem A, then the uniform convergence may fail and blow-up phenomena for $\{u_\varepsilon\}_{\varepsilon>0}$ may occur; their analysis is difficult in the case of $\dim M \geq 3$. In the case that the domain is a surface, the following result ([2]) which corresponds to the one due to Sacks-Uhlenbeck is obtained, and then the existence theorem (due to Lemaire, Schoen-Yau, Sacks-Uhlenbeck, et. al.) of harmonic maps into a manifold satisfying $\pi_2(N) = 0$ is reproved.

Theorem B. *Let (M, g) and (N, h) be compact and connected Riemannian manifolds without boundary. Assume that M is 2-dimensional. Let $\{u_\varepsilon\}_{\varepsilon>0}$ be as in Theorem A. Then there exist a positive sequence $\varepsilon(k) \to 0$ as $k \to \infty$, finitely many points $\{p_1, \ldots, p_l\} \subseteq M$ and a harmonic map $u : (M, g) \to (N, h)$ such that the sequence $\{u_{\varepsilon(k)}\}_{k=1}^\infty$ converges uniformly with all derivatives to u outside $\{p_1, \ldots, p_l\}$:*

$$u_{\varepsilon(k)} \to u \quad \text{in } C^\infty_{\mathrm{loc}}(M \setminus \{p_1, \ldots, p_l\}, N) \text{ as } k \to \infty.$$

If a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of critical points of $\mathbb{E}_\varepsilon$ from a higher-dimensional (≥ 3) manifold is considered, some existence results for higher-dimensional harmonic maps are expected, such as

"there exists a singular set $\Sigma \subseteq M$ whose $(\dim M - 2)$-dimensional Hausdorff measure is finite, out of which $\{u_\varepsilon\}_{\varepsilon>0}$ subconverges uniformly to a harmonic map".

Also, the time-evolution equations for exponentially harmonic maps are expected to have several applications. If time permits, I would also touch on them.

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Sectional Curvature Operator of the Conformally Flat Riemannian Manifolds

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Conformally flat Riemannian manifolds are studied by many authors (see [1,2]). The spectrum of a sectional curvature operator of the conformally flat Riemannian manifolds is considered in this paper. The following results are obtained.

i) A structure theorem about the spectrum of a sectional curvature operator of the conformally flat Riemannian manifolds is proved. In the case of the conformally flat homogeneous Riemannian spaces some calculating formulas for the spectrum are given .

ii) The spectrum of a sectional curvature operator of the 4-dimensional Lie groups with left-invariant (half)conformally flat Riemannian metrics is investigated.

iii) The case of the locally homogeneous Riemannian 3-manifolds is studied too. Full classification of the signatures for the spectrum of a sectional curvature operator is obtained and δ-pinch of the sectional curvature is investigated for the Lie groups with left-invariant Riemannian metrics.

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Aleksandrov problem in linear n-normed spaces

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Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an *isometry* if f satisfies

$$d_Y\big(f(x), f(y)\big) = d_X(x, y)$$

for all $x, y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively. For some fixed number $r > 0$, suppose that f preserves distance r; i.e., for all x, y in X with $d_X(x, y) = r$, we have $d_Y\big(f(x), f(y)\big) = r$. Then r is called a *conservative* (or *preserved*) distance for the mapping f. Aleksandrov [1] posed the following problem:

Aleksandrov problem: Examine whether the existence of a single conservative distance for some mapping T implies that T is an isometry.

The aim of this article is to generalize the Aleksandrov problem to the case of linear n-normed spaces.

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Extremal values of quermass integrals (Minkovski functionals) on the set of parallelepipeds with a given geodesic diameter

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The very natural functionals on a set of convex bodies in the k-dimensional Euclidean space are *quermass integrals (Minkovski functionals)* W_i, $i = 0, 1, ..., k$ (see, for example, [1, section 6.1.6]). Let A is a convex body, i. e. a closed and convex dot set, in 3-dimensional Euclidean space $\mathbb{E}^3$. For the convex body A the following quermass integrals (Minkovski functionals) is true: $W_0(A) = V(A)$ is the volume, $W_1(A) = F(A)/3$, $W_2(A) = M(A)/3$, $W_3(A) = \text{const} = 4\pi/3$, where $F(A)$ is the surface area, and $M(A)$ is the integral mean curvature .

Let's remind values of quermass integrals for a rectangular parallelepiped $P = ABCDA'B'C'D'$ in $\mathbb{E}^3$ with edge lengths $|AB| = a$, $|AD| = b$, $|AA'| = c$, where $0 \leq a \leq b \leq c$. Well-known formulas for volume $W_0(P) = V(P) = abc$ and the area of a surface $F(P) = 2(ab + ac + bc) = 3W_1(P)$ are supplemented by the formulas $W_3(P) = 4\pi/3$ and $M(P) = \pi(a + b + c) = 3W_2(P)$ [4].

For the parallelepiped P we will denote a surface of this parallelepiped by $\partial(P)$ (its boundary in a natural topology of 3-dimensional Euclidean space). Let $d(M, N)$ is a geodesic (intrinic) distance between points $M \in \partial(P)$ and $N \in \partial(P)$, i. e. minimal length of poligonal lines, connecting the points M and N, in $\partial(P)$.

By $D(P)$ we will denote a geodesic (intrinic, in other terminology) diameter of the parallelepiped P (more precisely, of the surfaces of a parallelepiped) is the maximal intrinic distance between pair of points on the surfaces of a parallelepiped.

An interesting problem of finding extremal values of quermass integrals (excepting a trivial case of a constant $W_3 = 4\pi/3$) for rectangular parallelepiped $P = ABCDA'B'C'D'$ with a given intrinsic diameter. For convenience we will also consider *degenerate parallelepipeds* with $a = 0$.

The maximal surface area was found in [3] Yu.G. Nikonorov and Yu.V. Nikonorova (a minimal surface area, obviously, is equal to 0 and it is attained exactly by a degenerate parallelepiped with $b = a = 0$), it is attained by a parallelepiped with the relation $a : b : c = 1 : 1 : \sqrt{2}$ for edge lengths. In particular, for any parallelepiped P inequality $ab + ac + bc \leq \frac{1+2\sqrt{2}}{6} \left(D(P)\right)^2$ is true.

The approximate computings made by the author for volume allow to assume that maximal volume is attained by a parallelepiped with the relation $a : b : c = 1 : 1 : \sqrt{2}$ for edge lengths, as well as in a case with the surface area (the minimal volume is equal to 0 and is attained in accuracy by degenerate parallelepipeds).

The main result of this research is the following:

Theorem *Among all rectangular parallelepipeds with given intrinsic diameter the maximal integral mean curvature is attained by a parallelepiped with the relation $a : b : c = 0 : 1 : 1$ for edge lengths, and the minimal integral mean curvature is attained by a parallelepiped with the relation $a : b : c = 0 : 0 : 1$ for edge lengths. In other words, for any rectangular parallelepiped P the following inequality holds:*

$$\pi D(P) \leq M(P) \leq \pi\sqrt{2}D(P),$$

where $D(P)$ is the geodesic diameter, and $M(P)$ is the integral mean curvature of the parallelepiped P.

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Nonlocal symmetries, pseudo-spherical surfaces and peakon equations

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In this talk I summarize some recent work on (nonlocal) differential equations appearing in the theory of equations of pseudo-spherical type and in some modern approaches to cosmology and string theory. Specifically, I present results on the existence and uniqueness of solutions to the so-called "modified" Camassa-Holm and Hunter-Saxton equations (which depend on inverses of second order differential operators), and I also show that the centerless Virasoro algebra appears as an algebra of (nonlocal) symmetries for the standard Camassa-Holm and Hunter-Saxton equations. Furthermore, I consider the generalized bosonic string equation (an equation formulated on compact Riemannian manifolds which depends on an exponential of a second order differential operator) and I explain how to prove existence of smooth solutions.

The Camassa-Holm and Hunter-Saxton equations appear in fluid theory, and their "modified" versions can be derived geometrically usign the theory of equations of pseudo-spherical type. The "bosonic" string equation is an euclidean version of an equation of interest for supersymmetric string theory.

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h-projective transformations on compact Kähler manifolds

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The basic geometric structure in h-projective geometry is the family of h-planar curves associated to a given Kähler metric. These curves were introduced in [3] and can be seen as generalisations of geodesics on Kähler manifolds. A diffeomorphism of the manifold is then called an h-projective transformation, if it preserves the set of h-planar curves. Since the h-projective transformations form a group, which contains the isometries of the given Kähler metric as a subgroup, it is natural to ask in which cases both groups are essentially different from each other.

The result which I want to present in my talk is a classical conjecture attributed to Yano and Obata and was obtained together with V. S. Matveev in the joint work [2]: the complex projective space with Fubini-Study metric is the only compact Kähler manifold (up to isomorphism and multiplication of the metric with a constant) in which the dimension of the group of h-projective transformations is bigger than the dimension of the group of isometries.

One part of the proof of this statement also uses results of the joint work [1].

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Flows of Metrics on a Fiber Bundle

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The talk is devoted to geometric analysis on closed manifolds endowed with two complementary distributions D and $D^{\perp}$. We investigate flows of metrics on codimension-one foliations (when D is tangent to the leaves), which yield hyperbolic and parabolic PDEs (see [1], [2]), examples represent flows on a surface of revolution, which yield convection-diffusion PDEs for the geodesic curvature of parallels (D-curves) and solitary solutions – non-linear waves. For distributions of arbitrary dimensions (when $D^{\perp}$ is tangent to the fibres of a fiber bundle), we introduce conformal flows of metrics restricted to D with the speed proportional to (**i**) the divergence of the mean curvature vector field H of D; (**ii**) the mixed scalar curvature $\mathrm{Sc}_{\mathrm{mix}}$ of the distributions. For (**i**), the flow is reduced to the heat flow of the 1-form dual to H, see [3]. For (**ii**), the vector field H satisfies the Burgers type PDE (on the fibers) with a unique stationary solution, see [4]. We prescribe in some cases the mean curvature vector field H and $\mathrm{Sc}_{\mathrm{mix}}$.

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The strong elliptic maximum principle for vector bundles and applications to minimal maps

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Based on works by E. Hopf [1], H.F. Weinberger [2], R. Hamilton [3, 4] and L.C. Evans [5], we state and prove the strong elliptic maximum principle for smooth sections in vector bundles over Riemannian manifolds and give some applications in Geometric Analysis. Moreover, we use this maximum principle to obtain various rigidity theorems and Bernstein type theorems for minimal graphs generated from maps between Riemannian manifolds. This is a joint work with Knut Smoczyk.

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Metric geometry of nonregular weighted Carnot–Carathéodory spaces and applications

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We investigate local and metric geometry of weighted Carnot-Carathéodory spaces which are a wide generalization of sub-Riemannian manifolds (see e.g. [1, 2, 3, 6, 7, 8, 9, 11, 12, 13, 18] and references therein) and naturally arise in nonlinear control theory [5], harmonic analysis [17], subelliptic equations [4] etc.

For such spaces the intrinsic Carnot-Carathéodory metric might not exist, and some other new effects take place, in particular we construct examples when different formal degree structures lead to different combinations of regular and nonregular (or singular) points on the space, though the algebraic structure induced by the commutator relations remains the same. These difficulties lead to necessity of developing new methods to study geometry of such spaces.

We describe the local algebraic structure of a C-C space, endowed with a certain quasimetric, first introduced in [12] for simplifying the computations, and compare local geometries of the initial C-C space and its tangent cone at some fixed (possibly nonregular) point. We prove analogs of such classical results as the Theorem on divergence of integral lines, the Local approximation theorem and the Tangent cone theorem w.r.t. the considered quasimetric. The obtained estimates can be applied to constructing motion planning algorithms for nonlinear control systems and to the theory of subelliptic equations.

The notion of the tangent cone to a quasimetric space, extending the Gromov's notion for metric spaces, was introduced and studied recently in [14]. Note that a straightforward generalization of the Gromov-Hausdorff convergence theory would make no sense for quasimetric spaces, since the Gromov-Haussdorff distance between any two quasimetric spaces would be equal to zero.

The main results of the present work are new even for the case of sub-Riemannian manifolds. Moreover, they yield new proofs of the Local approximation theorem and the Tangent cone theorem for the intrinsic C-C metric (if it exists) and w.r.t. the Gromov's convergence, proved for Hörmander vector fields in [2, 6, 9, 11]. The methods of proofs heavily rely on results of [10, 19] for the case of regular C-C spaces, and on methods of submersion of a C-C space into a regular one [13, 2, 4, 7, 9], as well as on obtaining new geometric properties of the considered quasimetric and of a similar quasimetric induced by nilpotentizations of the weighted vector fields defining the C-C space. In contrast to the proof of the Local approximation theorem in [2], we do not need special polynomial "privileged" coordinates and do not use Newton-type approximation methods.

This talk is based on the paper [15], which is to appear, and its preliminary short version [16].

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Geometry and Topology of some overdetermined elliptic problems

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Let $\Omega \subset \mathbb{R}^2$ a domain that support a solution to the overdetermined elliptic problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \langle \nabla u, \nu \rangle = \alpha & \text{on } \partial\Omega \end{cases} \tag{1}$$

where f has Lipschitz regularity, ν is the outward normal vector, and α is a constant. By a classical result due to J. Serrin, the only bounded domains where (1) can be solved are round balls, [4]. The proof of Serrin is based on the moving plane method, a method introduced some years before by A. D. Alexandrov to prove that the only compact constant mean curvature hypersurfaces embedded in $\mathbb{R}^n$ are the spheres, see [1].
The link between overdetermined elliptic problems and the theory of constant mean curvature hypersurfaces is very strong, and this is true also in the unbounded case, see for example [3].
In this talk we will show some geometric and topological properties for regular (unbounded) domains $\Omega \subset \mathbb{R}^2$ that support a solution to the overdetermined elliptic problem (1). Our principal ideas come from the theory of constant mean curvature surfaces in $\mathbb{R}^3$, and our principal tool is the Alexandrov moving plane method, adapted to overdetermined problems and used in the non trivial case of unbounded domains. In particular, we will show that if Ω has finite topology and there exists a positive constant R such that Ω does not contain balls of radius R, then any end of Ω stays at bounded distance from a straight line. As a corollary of this result, we will prove that under the hypothesis that $\mathbb{R}^2 \setminus \overline{\Omega}$ is connected and there exists a positive constant λ such that $f(t) \geq \lambda t$ for all $t \geq 0$, then Ω is a ball. Such result gives a partial answer to a conjecture of Berestycki-Caffarelli-Nirenberg in the plane.
The subject of this talk is based on a joint work with A. Ros, [2].

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Left invariant contact structures

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On a necessary flexibility condition of a nondegenerate suspension in Lobachevsky 3-space

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A *polyhedron* (more precisely, a *polyhedral surface*) is said to be *flexible* if its spatial shape can be changed continuously due to changes of its dihedral angles only, i.e., if every face remains congruent to itself during the flex.

A *suspension* is a polyhedron with two distinguished vertices (called north and south poles) which don't have a common edge, such that all remaining vertices of the polyhedron (called the vertices of the equator) are connected by edges with both poles, and the edges connecting vertices of the equator form a cycle.

In 1897 R. Bricard [1] described all flexible octahedra in Euclidean 3-space. The Bricards octahedra were the first examples of flexible polyhedra (with self-intersections). The Bricards octahedra are special cases of Euclidean flexible suspensions. In 1974 R. Connelly [2] proved that some combination of the lengths of all edges of the equator of a flexible suspension in Euclidean 3-space is equal to zero (each length is taken either positive or negative in this combination). The method applied by R. Connelly, is to reduce the problem to the study of an analytic function of complex variable in neighborhoods of its singular points.

In 2001 S. N. Mikhalev [3] reproved the above-mentioned result of R. Connelly by algebraic methods. Moreover, S. N. Mikhalev proved that for every spatial quadrilateral formed by edges of a flexible suspension and containing its both poles there is a combination of the lengths (taken either positive or negative) of the edges of the quadrilateral, which is equal to zero.

Applying R. Connelly's method to Lobachevsky 3-space we get:

Theorem. *Let $\mathcal{P}$ be a nondegenerate flexible suspension in Lobachevsky 3-space with the poles S and N, and with the vertices of the equator P_j, $j = 1, ..., V$. Then*

$$\sum_{j=1}^{V} \sigma_{j,j+1} |P_j P_{j+1}| = 0,$$

where $\sigma_{j,j+1} \in \{+1, -1\}$, $|P_j P_{j+1}|$ is the length of the edge $P_j P_{j+1}$, $j = 1, ..., V$, and by definition $P_V P_{V+1} \stackrel{\text{def}}{=} P_V P_1$, $\sigma_{V,V+1} \stackrel{\text{def}}{=} \sigma_{V,1}$.

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Quasi-isometric invariants of the fundamental group of an orthogonal graph-manifold

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A graph-manifold is called orthogonal, if all of its gluing maps are just permutations of coordinates. Kapovich and Leeb proved that the fundamental group of any 3-dimensional graph-manifold is quasi-isometric to the fundamental group of some flip graph-manifold, which is an orthogonal graph-manifold of dimension 3.

We will discuss certain quasi-isometric invariants of fundamental groups of such manifolds.

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Applications of the Intrinsic Flat Distance between Riemannian Manifolds

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Occasionally attracting compact sets

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Symmetric Polytopes with nonsymmetric faces

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We consider the closed convex symmetric polyhedra in three-dimensional Euclidean space with isolated asymmetric faces and polyhedra with isolated asymmetric zones.

We prove some theorems on the classification of the polyhedron.

Triangulations and spherical cone metrics

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Given a triangulation of a 3-dimensional pseudomanifold in which no edge has valence more than 5, we naturally get a spherical cone metric of curvature bounded below by 1 in the sense of Alexandrov, leading immediately to diameter and volume bounds. An enumeration shows there are exactly 4761 such triangulations of the 3-sphere. We investigate these geometrically, focusing first on the family with unbranched cone axes. Most of these are Seifert-fibered, arising as lifts of spherical cone metrics on "bad" 2D orbifolds.

Displacement convexity of generalized relative entropies

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The notion “displacement convexity” is the convexity of a functional on the space of probability measures equipped with a certain distance function called the Wasserstein distance function. We first generalize the relative entropy on the Wasserstein space over a weighted Riemannian manifold from the viewpoint of information geometry. Here the generalized relative entropy can be considered as the summation of an internal energy and a potential energy. We then classify generalized relative entropies by behavior of internal energies. The main theorem of this talk is that the displacement convexity of all the entropies in this class is equivalent to the combination of the nonnegative weighted Ricci curvature and the convexity of the potential function of the generalized relative entropies. As applications, we derive appropriate variants of the Talagrand, the HWI and the logarithmic Sobolev inequalities as well as the concentration of measures from the displacement convexity of the generalized relative entropy. We also analyze the gradient flow of this generalized relative entropy.

This is a joint work with Shin-ichi Ohta.

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Elementary proofs of a generalization of the pythagorean theorem and calculation of the dihedral angle of regular n-simplicies

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Almost Nilpotency of the fundamental groups of almost nonnegatively curved Alexandrov space

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On the filling radius of positively curved Alexandrov spaces

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In this talk, I will talk about the main result of [Yo] and its extension. Here, we are concerned with the metric invariant Fill Rad, the filling radius, of Alexandrov spaces with positive lower curvature bound. For the definition of the filling radius, we refer to the seminal paper [Gr] by M. Gromov.

In order to state the main theorem of [Yo], we need the following definition.

Definition ([Wi]) For any metric space $X = (X, d)$, we define its *spread*, denoted $\operatorname{Spread}(X)$, as the infimum of $R > 0$ for which there is a subset $Y \subset X$ of $\operatorname{Diam}(Y) \leq R$ such that $d(x, Y) \leq R$ for any $x \in X$.

Spread was used by M. Katz to find the exact value of the filling radius of the round sphere $\mathbb{S}^n$ of constant curvature 1;

$$2\operatorname{Fill Rad}(\mathbb{S}^n) = \operatorname{Spread}(\mathbb{S}^n) = \ell_n := \arccos\left(\frac{-1}{n+1}\right).$$

It turns out that ℓ_n is the spherical distance between vertices of a regular $(n+1)$-simplex inscribed in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, and the set $Y := \{p_1, \ldots, p_{n+2}\} \subset \mathbb{S}^n$ of its vertices gives $\operatorname{Spread}(\mathbb{S}^n) = \operatorname{Diam}(Y) = \ell_n$.

The main theorem of [Yo] is the following comparison theorem for the spread of finite-dimensional Alexandrov spaces of curvature ≥ 1.

Theorem ([Yo]) *For any n-dimensional Alexandrov space X of curvature ≥ 1, either* $\operatorname{Spread}(X) < \operatorname{Spread}(\mathbb{S}^n)$ *or X is isometric to the round sphere $\mathbb{S}^n$.*

This generalizes Wilhelm's theorem in [Wi] where he proved the above theorem for closed Riemannian manifolds of sectional curvature ≥ 1.

As in [Wi], the above theorem yields the following corollary.

Corollary ([Yo]) *For any n-dimensional Alexandrov space X of curvature ≥ 1 with $\partial X = \emptyset$, either* $\operatorname{Fill Rad}(X) < \operatorname{Fill Rad}(\mathbb{S}^n)$ *or X is isometric to the round sphere $\mathbb{S}^n$.*

Since any finite-dimensional Alexandrov space X with a lower curvature bound admits a fundamental homology class with $\mathbb{Z}_2$-coefficient, provided it has empty boundary $\partial X = \emptyset$, due to T. Yamaguchi and Grove–Petersen, and hence its filling radius $\operatorname{Fill Rad}(X)$ is defined.

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Few Alexandrov surfaces are Riemann

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Morse index of a cyclic polygon

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A *polygonal linkage* is a sequence of real positive numbers $L = (l_1, \dots, l_n)$, which can be realized as edge lengths of a planar polygon.

A *configuration* of a linkage L is a sequence of points on a plane $P = (p_1, p_2, \dots, p_n)$, $p_i = (x_i, y_i)$, such that $|p_i, p_{i+1}| = l_i$ (the numeration is cyclic, i.e. $p_{n+1} = p_1$). The first two vertices are fixed: $p_1 = (0, 0)$ $p_2 = (l_1, 0)$.

The set $\mathcal{M}(L)$ of all such configurations is *the moduli space of the polygonal linkage* L. Generically, it is a smooth manifold of dimension $n - 3$.

On the space $\mathcal{M}(L)$ we define the signed area function:

$$A(P) = (x_1 y_2 - x_2 y_1) + \ldots + (x_n y_1 - x_1 y_n).$$

Generically, function $A(P)$ is a Morse function. It was proven in [1], that the critical points of signed area function are exactly the cyclic configurations of L (a configuration is called cyclic if all its vertices lie on a circle).

For a cyclic configuration we use the following notations:

$m(P)$ is the Morse index of the function A in the point P.

O is the center of the circumscribed circle.

α_i is the half of the angle between the vectors $\overrightarrow{Op_i}$ and $\overrightarrow{Op_{i+1}}$. The angle is defined to be positive, orientation is not involved.

$$\varepsilon_i = \begin{cases} 1, & \text{if the center } O \text{ lies to the left of } p_i p_{i+1}; \\ -1, & \text{if the center } O \text{ lies to the right of } p_i p_{i+1}. \end{cases}$$

$e(P)$ is the number of positively oriented edges in P.

$\omega(P)$ is the winding number of P with respect to the center O.

$$\delta(P) = \sum_{i=1}^{n} \varepsilon_i \tan \alpha_i;$$

We prove a following explicit formula for the Morse index of the signed area function:

For a generic cyclic configuration P of a linkage L,

$$m(P) = \begin{cases} e(P) - 1 - 2\omega(P) & \text{if } \delta(P) > 0; \\ e(P) - 2 - 2\omega(P) & \text{otherwise.} \end{cases}$$

Further, we give full classification of all possible local maxima of signed area function:

A configuration P is a local maximum of the function A iff one of the following terms fulfill:

1. P is the convex positively oriented configuration.
2. P has only negatively oriented edges and all of them cross each other.
3. The following three terms are fulfilled:
   - every two negatively oriented edges of P cross each other;
   - no positively oriented edges of P cross each other;
   - no positively oriented edge of P cross negatively oriented edges of P.
4. The following four terms are fulfilled:
   - $\delta(P) > 0$;
   - every two negatively oriented edges of P cross each other;
   - no positively oriented edges of P cross each other;
   - only one positively oriented edge of P cross negatively oriented edges of P.

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The Structure of Conformal Foliations

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Recent results will be presented on the investigation of the structure of conformal foliations of codimension $q \geq 3$.

Let $(M, \mathcal{F})$ be an arbitrary smooth foliation. Remind that a subset of a manifold M is called a saturated whenever it is the union of some leaves of a foliation $(M, \mathcal{F})$. By definition, an attractor of a foliation $(M, \mathcal{F})$ is nonempty saturated subset $\mathcal{M}$, if there exists a saturated open neighborhood $Attr(\mathcal{M})$ such that the closure of every leaf from $Attr(\mathcal{M})$ includes $\mathcal{M}$. Here $Attr(\mathcal{M})$ is named as an attractor basin. If an addition $M = Attr(\mathcal{M})$, then the attractor $\mathcal{M}$ is called global.

The following theorem was proved without assumption that the foliation $(M, \mathcal{F})$ is complete [1]. The manifold M may be noncompact.

Theorem 1 *Every codimension $q \geq 3$ conformal foliation $(M, \mathcal{F})$ either is Riemannian or has an attractor that is a minimal set of $(M, \mathcal{F})$, and the restriction of the foliation to the attraction basin is a $(Conf(S^q), S^q)$-foliation.*

A foliation is said to be proper, if every of its leaves is an embedded submanifold of the foliated manifold. The leaf L of a foliation $(M, \mathcal{F})$ is called closed if L is a closed subset of M.

Corollary *Each proper non-Riemannian codimension $q \geq 3$ conformal foliation has a closed leaf that is an attractor.*

Moreover when the foliated manifold M is compact we have proved the following assertion [1].

Theorem 2 *Every conformal foliation $(M, \mathcal{F})$ on a compact manifold M either a Riemannian foliation or a $(Conf(S^q), S^q)$-foliation with a finite family of minimal sets. They all are attractors of this foliation.*

A sufficient condition for the existence of a global attractor of a conformal foliation has been found. The structure of the global attractors and foliations $(M, \mathcal{F})$ have been investigated [2].

Tarquini [3] and also Frances and Tarquini [4] posed the following question:

Is every codimension $q \geq 3$ conformal foliation on a compact manifold either a Riemannian foliation or a $(Conf(S^q), S^q)$-foliation?

Tarquini and later Tarquini with Frances gave positive answers to this question under some additional assumptions. Theorem 2 implies a positive answer to the Frances and Tarquini question in the general case.

Examples of conformal foliations with exceptional and exotic global attractions are constructed.

Applications of the previous results to problems of the local and global leaf stability in sense of Ehresmann and Reeb are considered.

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