Hyperpolar Actions on Noncompact Symmetric Spaces

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Polar representations

- H compact connected Lie group acting on
  V real vector space with H-invariant inner product
- $\pi : H \to O(V)$ representation
- $v \in V$, $\Sigma_v \subset V$ cross-section of action at v
- Σ_v minimal $\iff$ $\dim H \cdot v$ maximal

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Dadok 1985: Polar representations on $\mathbb{R}^n$ are orbit equivalent to isotropy representations of Riemannian symmetric spaces

Polar actions

M connected Riemannian manifold, $H \subset I(M)$ connected subgroup

Definition. The action of H on M is **polar** if there exists a connected closed submanifold Σ of M such that

- $\forall p \in M : \Sigma \cap H \cdot p \neq \emptyset$
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Problem. Classification of hyperpolar/polar actions on Riemannian symmetric spaces

S^n and $\mathbb{R}H^n$: apply Dadok's result

Compact symmetric spaces

Podestà, Thorbergsson 1999: Classification of *polar* actions on projective spaces

Kollross 2002: Classification of *hyperpolar* actions on irreducible Riemannian symmetric spaces of compact type and rank ≥ 2

Every *polar* action on an irreducible Riemannian symmetric spaces of compact type and rank ≥ 2 is hyperpolar

- **Podestà-Thorbergsson 2002**: SO_{n+2}/SO_nSO_2, $n \geq 3$
- **Biliotti-Gori 2005**: $SU_{n+k}/S(U_nU_k)$, $n \geq k \geq 2$
- **Biliotti 2006**: Hermitian symmetric spaces
- **Kollross 2007**: Simple isometry group
- **Kollross 2009**: G_2, F_4, E_6, E_7, E_8
- **Lytchak 2011**: Cohomogeneity is ≥ 3
- **Kollross-Lytchak 2011**: Cohomogeneity is 2

Compact vs noncompact

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- Concept of *duality* between symmetric spaces of compact type and of noncompact type is useful only for special situations, e.g. actions by algebraic reductive subgroups (**Kollross 2011**)
- In the compact case one can restrict to actions of compact groups (well understood!), whereas in the noncompact case one needs to consider noncompact groups (not well understood!)

Current state of affairs

	regular foliation	singular foliation
cohom 1	explicit classification	general construction
hyperpolar	explicit classification	?
polar	?	?
	$\mathbb{C}H^n$: classification	$\mathbb{C}H^n$: classification

Joint work with

- José Carlos Díaz-Ramos (Santiago de Compostela)
- Hiroshi Tamaru (Hiroshima)

The general setting

- $M = G/K$ connected irreducible Riemannian symmetric space of noncompact type
  G noncompact semisimple real Lie group
  K maximal compact subgroup of G
  $o \in M$ with $K \cdot o = o$
- H connected closed subgroup of G acting on M polarly

Parabolic subalgebras (I)

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition
- $\mathfrak{a}$ maximal abelian subspace of $\mathfrak{p}$
- restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$$

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- Φ subset of Λ, $\Sigma_\Phi = \Sigma \cap \mathrm{span}\{\Phi\}$
- $\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha \right)$, $\mathfrak{n}_\Phi = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma^+_\Phi} \mathfrak{g}_\alpha$
  $\mathfrak{l}_\Phi$ reductive subalgebra, $\mathfrak{n}_\Phi$ nilpotent subalgebra
- $\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$ **parabolic subalgebra** (**Chevalley decomposition**)

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- Every parabolic subalgebra of $\mathfrak{g}$ is conjugate to $\mathfrak{q}_\Phi$ for some subset $\Phi \subset \Lambda$

Parabolic subalgebras (II)

- $\mathfrak{l}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi$ with $\mathfrak{a}_\Phi$ split component of $\mathfrak{l}_\Phi$
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- $M_\Phi \cdot o = B_\Phi$ semisimple symmetric space with rank equal to $|\Phi|$, totally geodesic in M, **boundary component** of M with respect to maximal Satake compactification
- $A_\Phi \cdot o = \mathbb{E}^{r-|\Phi|}$ Euclidean space, totally geodesic in M
- $L_\Phi \cdot o = F_\Phi = B_\Phi \times \mathbb{E}^{r-|\Phi|}$ totally geodesic in M
- $M = B_\Phi \times \mathbb{E}^{r-|\Phi|} \times N_\Phi$ (**horospherical decomposition**)

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- $M = B_\Phi \times \mathbb{E}^{r-|\Phi|} \times N_\Phi$ (**horospherical decomposition**)
- The action of N_Φ on M is polar
- The action of N_Φ on M is hyperpolar $\Longleftrightarrow \Phi = \emptyset$

Examples of hyperpolar foliations

- V linear subspace of $\mathbb{E}^m$
  $\Longrightarrow \mathcal{F}^m_V = \{p + V \mid p \in \mathbb{E}^m\}$ homogeneous hyperpolar foliation of $\mathbb{E}^m$

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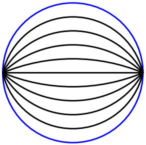
- $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, $M = G/K = \mathbb{F}H^n$
  $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{g}_\alpha \ominus \ell) \oplus \mathfrak{g}_{2\alpha}$, ℓ line in $\mathfrak{g}_\alpha$
  $\implies \mathcal{F}_{\mathbb{F}}^n$ homogeneous codimension one foliation of $\mathbb{F}H^n$ with unique minimal leaf



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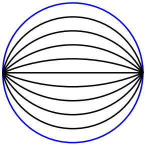

- $\mathcal{F}_{\mathbb{F}_1}^{n_1} \times \cdots \times \mathcal{F}_{\mathbb{F}_k}^{n_k} \times \mathcal{F}_V^m$ homogeneous hyperpolar foliation of $\mathbb{F}_1 H^{n_1} \times \cdot \times \mathbb{F}_k H^{n_k} \times \mathbb{E}^m$

Examples of hyperpolar foliations (II)

- $M = G/K$ symmetric space of noncompact type
- Φ orthogonal set of simple roots, $k = |\Phi|$
- $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ Langlands decomposition of parabolic subalgebra $\mathfrak{q}_\Phi$ of $\mathfrak{g}$

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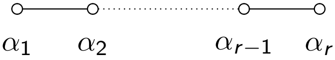
- $F_\Phi \cong \underbrace{\mathbb{F}_1 H^{n_1} \times \cdots \times \mathbb{F}_k H^{n_k}}_{M_\Phi \cdot o} \times \underbrace{\mathbb{E}^{r-k}}_{A_\Phi \cdot o}$
- $\mathcal{F}^{n_1}_{\mathbb{F}_1} \times \cdots \times \mathcal{F}^{n_k}_{\mathbb{F}_k} \times \mathcal{F}^{r-k}_V$ homogeneous hyperpolar foliation of F_Φ
- $\mathcal{F}_{\Phi,V} = \mathcal{F}^{n_1}_{\mathbb{F}_1} \times \cdots \times \mathcal{F}^{n_k}_{\mathbb{F}_k} \times \mathcal{F}^{r-k}_V \times N_\Phi$ homogeneous hyperpolar foliation of $M = F_\Phi \times N_\Phi$
- $\mathcal{F}_{\emptyset,\{0\}}$ horocycle foliation of M

Classification of homogeneous hyperpolar foliations

Berndt-DiazRamos-Tamaru 2010: *Let M be a symmetric space of noncompact type. Every homogeneous hyperpolar foliation on M is isometrically congruent to $\mathcal{F}_{\Phi,V}$ for some orthogonal set Φ of simple roots and some linear subspace $V \subset \mathbb{E}^{r-|\Phi|}$.*

The symmetric space $SL_{r+1}(\mathbb{R})/SO_{r+1}$

- Dynkin diagram



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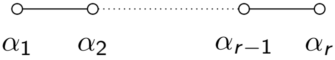

- $\Phi \subset \Lambda = \{\alpha_1, \ldots, \alpha_r\}$ orthogonal, $k = |\Phi|$
- horospherical decomposition:
  $SL_{r+1}(\mathbb{R})/SO_{r+1} \cong \underbrace{\mathbb{R}H^2 \times \ldots \times \mathbb{R}H^2}_{k \text{ factors}} \times \mathbb{E}^{r-k} \times N_\Phi$
- N_Φ corresponds to the set of all upper block diagonal matrices with certain 2×2 and 1×1 diagonal blocks, diagonal entries are 1

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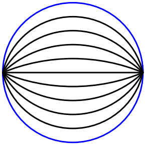

- On $\mathbb{E}^{r-k}$ select a foliation by parallel affine subspaces
- On N_Φ select the foliation with one leaf N_Φ

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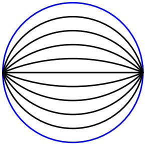


- On $\mathbb{E}^{r-k}$ select a foliation by parallel affine subspaces
- On N_Φ select the foliation with one leaf N_Φ
- The product foliation is hyperpolar, and every homogeneous hyperpolar foliation of $SL_{r+1}(\mathbb{R})/SO_{r+1}$ arises in this way

Cohomogeneity one actions

Berndt-Tamaru 2012: *Let $M = G/K$ be a Riemannian symmetric space of noncompact type. Assume that H acts on M with cohomogeneity one. Then one of the following statements holds:*

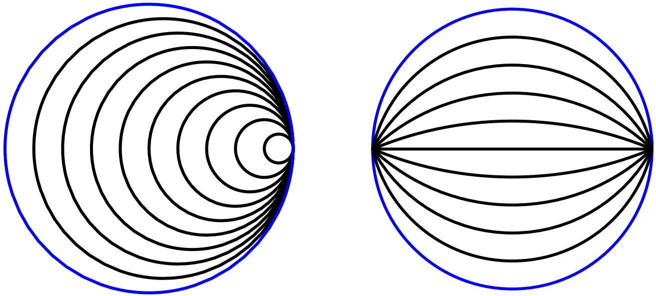
1. *The orbits of H form a Riemannian foliation on M*
2. *There is a totally geodesic singular orbit*
3. *The action of H is orbit equivalent to the canonical extension of a cohomogeneity one action on a boundary component of M*
4. *The action of H is orbit equivalent to one which is obtained by the "nilpotent construction"*

Homogeneous foliations of codimension one

- M symmetric space of noncompact type
- $\mathcal{M}_F$ = set of all homogeneous codimension one foliations on M up to isometric congruence
- r = rank of M
- $\mathrm{Aut}(\Delta) \in \{I, \mathbb{Z}_2, \mathfrak{S}_3\}$ automorphism group of the Dynkin diagram Δ associated to M

$$\boxed{\mathcal{M}_F \cong (\mathbb{R}P^{r-1} \cup \{1, \ldots, r\})/\mathrm{Aut}(\Delta)}$$

The two foliations on hyperbolic spaces



- horosphere foliation
- foliation with exactly one minimal leaf S
  - $\underline{M = \mathbb{R}H^n}$: $S = \mathbb{R}H^{n-1}$ totally geodesic
  - $\underline{M = \mathbb{C}H^n}$: $S =$ ruled real hypersurface associated to a horocycle in a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$

Duality and triality

$\mathcal{M}_F$ depends only on the rank and on possible duality or triality principles on the symmetric space

Example: $\underline{r = 8,\ \mathrm{Aut}(\Delta) = I}$

$$\boxed{\mathcal{M}_F = \mathbb{R}P^7 \cup \{1, \dots, 8\}}$$

for the symmetric spaces

$$SO_{17}^{\mathbb{C}}/SO_{17}\ ,\ Sp_8^{\mathbb{R}}/U_8\ ,\ Sp_8^{\mathbb{C}}/Sp_8\ ,\ SO_{16}^{\mathbb{H}}/U_{16}\ ,\ SO_{17}^{\mathbb{H}}/U_{17}$$

$$E_8^8/SO_{16}\ ,\ E_8^{\mathbb{C}}/E_8$$

and for the hyperbolic Grassmannians

$$G_8^*(\mathbb{R}^{n+16})\ (n \geq 1)\ ,\ G_8^*(\mathbb{C}^{n+16})\ (n \geq 0)\ ,\ G_8^*(\mathbb{H}^{n+16})\ (n \geq 0)$$

Totally geodesic singular orbit

F is a totally geodesic singular orbit of a cohomogeneity one action on $M \iff$

- *F reflective* (**Leung 1974**) *and* $\operatorname{rank} F^{\perp} = 1$, *or*
- *F is one of the following totally geodesic non-reflective submanifolds:*

F	M	$\dim F$	$\dim M$
$\mathbb{C}H^2$	G_2^2/SO_4	4	8
$SL_3(\mathbb{R})/SO_3$	G_2^2/SO_4	5	8
G_2^2/SO_4	$SO^o_{3,4}/SO_3SO_4$	8	12
$SL_3(\mathbb{C})/SU_3$	$G_2^{\mathbb{C}}/G_2$	8	14
$G_2^{\mathbb{C}}/G_2$	$SO_7^{\mathbb{C}}/SO_7$	14	21

Canonical extension

Basic example:
Extension of SO_2-action on $\mathbb{R}^2$ to $(SO_2 \times \mathbb{R})$-action on $\mathbb{R}^3$

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- $\mathfrak{h} = \mathfrak{h}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ subalgebra of $\mathfrak{q}_\Phi$

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H acts on M with cohomogeneity one

Rank reduction - Such a cohomogeneity one action can be constructed by a CANONICAL EXTENSION OF A COHOMOGENEITY ONE ACTION ON A BOUNDARY COMPONENT

Nilpotent construction

Skip the construction here, too technical

Only two examples known which arise through this construction method and none of the others:

- Cohomogeneity one action on G_2^2/SO_4 with 6-dimensional singular orbit
- Cohomogeneity one action on $G_2^{\mathbb{C}}/G_2$ with 12-dimensional singular orbit

Nilpotent construction

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $\Phi_j = \Lambda \setminus \{\alpha_j\}$: Put $\mathfrak{q}_j = \mathfrak{q}_{\Phi_j}$, $\mathfrak{n}_j = \mathfrak{n}_{\Phi_j}$, etcetera
- $\mathfrak{n}_j^\nu = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_j^+, \alpha(H^j) = \nu} \mathfrak{g}_\alpha$
- $\mathfrak{n}_j = \bigoplus_{\nu > 0} \mathfrak{n}_j^\nu$ gradation generated by $\mathfrak{n}_j^1$

Assume that

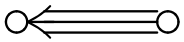
- $\mathfrak{v} \subset \mathfrak{n}_j^1$; define $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ subalgebra of $\mathfrak{n}_j$
- $N^o_{L_j}(\mathfrak{n}_{j,\mathfrak{v}}) = \theta N^o_{L_j}(\mathfrak{v})$ acts transitively on $F_j = B_j \times \mathbb{E}$
- $N^o_{L_j \cap K}(\mathfrak{v})$ acts transitively on the unit sphere in $\mathfrak{v}$ if $\dim \mathfrak{v} \geq 2$

Then

$$H_{j,\mathfrak{v}} = N^o_{L_j}(\mathfrak{n}_{j,\mathfrak{v}}) N_{j,\mathfrak{v}} \text{ acts on } M \text{ with cohomogeneity one}$$

Nilpotent construction - An example (I)

- $M = G_2^2/SO_4$, $\dim M = 8$, $\mathrm{rank}\, M = 2$
- root system Σ is of type G_2:



α_1 α_2

- $\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$
- $\Lambda = \{\alpha_1, \alpha_2\}$
- $\Phi_1 = \Lambda \setminus \{\alpha_1\} = \{\alpha_2\}$

Nilpotent construction - An example (II)

- $\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$
- $\mathfrak{n}_1^1 = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \cong \mathbb{R}^2$
- $\mathfrak{n}_1^2 = \mathfrak{g}_{2\alpha_1+\alpha_2} \cong \mathbb{R}$
- $\mathfrak{n}_1^3 = \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2} \cong \mathbb{R}^2$
- $\mathfrak{n}_1 = \mathfrak{n}_1^1 \oplus \mathfrak{n}_1^2 \oplus \mathfrak{n}_1^3$
- $\mathfrak{l}_1 = \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_2} \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R}$
- $\mathfrak{k}_1 = \mathfrak{k}_{\alpha_2} \cong \mathfrak{so}_2$
- $\mathfrak{h}_{1,\mathfrak{n}_1^1} = \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2}$
- $H_{1,\mathfrak{n}_1^1}$ acts on G_2^2/SO_4 with cohomogeneity one and singular orbit $H_{1,\mathfrak{n}_1^1} \cdot o$ with codimension 2

Polar foliations of complex hyperbolic spaces

- $\mathbb{C}H^n = SU_{n,1}/S(U_n U_1) = G/K$
- $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ restricted root space decomposition
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ Iwasawa decomposition, $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$
- $\mathbb{C}H^n = AN$ solvable Lie group with left-invariant metric
- $V = \{0\}$ or $V = \mathfrak{a}$; $\mathfrak{w} \subset \mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ real subspace
- $\mathfrak{s}_{V,\mathfrak{w}} = (\mathfrak{a} \ominus V) \oplus (\mathfrak{n} \ominus \mathfrak{w})$ subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$
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Berndt-DiazRamos 2012:

- The orbits of $S_{V,\mathfrak{w}}$ form a homogeneous polar foliation of $\mathbb{C}H^n$
- Every homogeneous polar foliation of $\mathbb{C}H^n$ is holomorphically congruent to one of these foliations

Proof relies on following result (**Gorodski 2004** for compact case):

Let $M = G/K$ be a Riemannian symmetric space of noncompact type and H be a connected closed subgroup of G whose orbits form a regular foliation $\mathcal{F}$ of M. Consider the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and define

$$\mathfrak{h}_{\mathfrak{p}}^{\perp} = \{\, \xi \in \mathfrak{p} : \langle \xi, Y \rangle = 0 \text{ for all } Y \in \mathfrak{h} \,\}.$$

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Then the action of H on M is polar if and only if

- $\mathfrak{h}_{\mathfrak{p}}^{\perp}$ is a Lie triple system in $\mathfrak{p}$, and
- $\mathfrak{h}$ is orthogonal to the subalgebra $[\mathfrak{h}_{\mathfrak{p}}^{\perp}, \mathfrak{h}_{\mathfrak{p}}^{\perp}] \oplus \mathfrak{h}_{\mathfrak{p}}^{\perp}$ of $\mathfrak{g}$.

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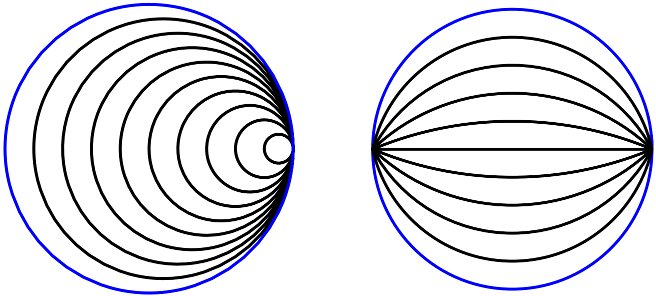
In this case, let $H_{\mathfrak{p}}^{\perp}$ be the connected subgroup of G with Lie algebra $[\mathfrak{h}_{\mathfrak{p}}^{\perp}, \mathfrak{h}_{\mathfrak{p}}^{\perp}] \oplus \mathfrak{h}_{\mathfrak{p}}^{\perp}$. Then the orbit $\Sigma = H_{\mathfrak{p}}^{\perp} \cdot o$ is a section of the H-action on M.

The case of codimension one



- horosphere foliation
- foliation with exactly one minimal leaf S = ruled real hypersurface associated to a horocycle in a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$

Polar actions on $\mathbb{C}H^2$

- N horosphere in $\mathbb{C}H^2$; $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$; N is a 3-dim Heisenberg group
- S ruled real hypersurface in $\mathbb{C}H^2$ generated by a horocycle in $\mathbb{R}H^2 \subset \mathbb{C}H^2$; $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{g}_\alpha^{\mathbb{R}} \oplus \mathfrak{g}_{2\alpha}$
- $N \cap S$ is a Euclidean plane $\mathbb{E}^2$ embedded in N as a minimal surface and in $\mathbb{C}H^2$ as a real surface with nonzero constant mean curvature; $\mathfrak{n} \cap \mathfrak{s} = \mathfrak{g}_\alpha^{\mathbb{R}} \oplus \mathfrak{g}_{2\alpha}$

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Berndt-DiazRamos 2012: Every polar action on $\mathbb{C}H^2$ is orbit equivalent to the action of the invariance group of one of the following geometric objects in $\mathbb{C}H^2$:

- Cohom 1: $\{o\}$, $\mathbb{C}H^1$, $\mathbb{R}H^2$, N, S
- Cohom 2: $\{o\} \subset \mathbb{C}H^1$ (full flag), $\mathbb{R}H^1$, horocycle in $\mathbb{C}H^1$, $\mathbb{E}^2$

Outline of proof

- Possible cohomogeneity is 1 or 2
- Cohomogeneity 1: known by earlier work
- Assume cohomogeneity 2
- 0-dimensional orbit: group is compact and action has a fixed point, only possibility is $S(U_1 U_1 U_1)$
- 1-dimensional orbit, no fixed point: Lie-theoretical arguments, technical
- regular foliation: known by earlier work