Infinitesimal rigidity of convex surfaces: Variational methods and duality

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Alexandrov 100 Conference
Isometric deformations of polyhedral surfaces

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There exist flexible polyhedra (Connelly).
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Move vertices linearly: $p_i(t) := p_i + tq_i$.
An infinitesimal deformation $(q_i)$ is called isometric, if

$$\frac{d}{dt} \bigg|_{t=0} \| p_i(t) - p_j(t) \| = 0$$

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There are lots of infinitesimally flexible polyhedra.
Smooth case

Similar definitions in the smooth case. **Infinitesimal deformation** of $M \subset \mathbb{R}^3$ is a vector field $\xi : M \to \mathbb{R}^3$. 

![Diagram of a vector field on a surface $M$.]
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$$\frac{d\phi^*_t(\text{can}_{\mathbb{R}^3})}{dt} = 0.$$ 

Cohn-Vossen: Examples of infinitesimally flexible $C^2$-surfaces.
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It is not known whether a higher order smoothness implies infinitesimal rigidity.
Convex surfaces are infinitesimally rigid

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We present a common variational approach to these theorems.
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The first idea is easier to explain on the example of isometric embedding theorems.
Isometric embedding theorems

Theorem (Weyl, Levy, Pogorelov, Nirenberg)

\( g \) a Riemannian metric on \( S^2 \), with everywhere positive Gauss curvature.

Then there exists a unique isometric embedding \( (S^2, g) \rightarrow \mathbb{R}^3 \).
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An equivalent reformulation:

Metric $g$ on $\mathbb{S}^2 = \partial \mathbb{B}^3$ can be extended to a flat metric $\tilde{g}$ on $\mathbb{B}^3$. 
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Isometric embedding = geometrization with boundary conditions
Infinitesimal rigidity revisited

An infinitesimal isometric deformation of a Riemannian metric $\tilde{g}$ is a field $h$ of symmetric bilinear forms.

\[ \tilde{g}_t := \tilde{g} + th \]

is a Riemannian metric for small $t$

$h$ is curvature-preserving def $\iff$ $\sec t = o(t)$

\((N, \tilde{g})\)

inf. rigid def $\iff$ every curvature-preserving deformation (vanishing on the boundary) is trivial

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Compact hyperbolic manifolds of dim $\geq 3$ are infinitesimally rigid.
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\kappa_i := 2\pi - \omega_i
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\[P \text{ infinitesimally rigid } \iff \dim \ker \left( \frac{\partial \kappa_i}{\partial r_j} \right) = 3\]
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How to study \(\left( \frac{\partial \kappa_i}{\partial r_j} \right)\)?
The discrete Hilbert-Einstein functional

$$\text{HE}(r) := \sum_i r_i \kappa_i + \sum_{ij} \ell_{ij} \lambda_{ij}$$
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Thus we are investigating the rank of the Hessian \( \left( \frac{\partial^2 \text{HE}}{\partial r_i \partial r_j} \right) = \left( \frac{\partial \kappa_i}{\partial r_j} \right) \)
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**Theorem**

\( \left( \frac{\partial^2 \text{HE}}{\partial r_i \partial r_j} \right) \) has the signature \((+, 0, 0, 0, -, \ldots, -)\)

**Corollary**

*Convex polyhedra in \( \mathbb{R}^3 \) are infinitesimally rigid.*
Duality with the Minkowski theorem

Theorem (Minkowski)

\( \nu_1, \ldots, \nu_n \) unit vectors spanning \( \mathbb{R}^3 \),

Then there exists a unique convex polyhedron \( Q \subset \mathbb{R}^3 \) with outer unit normals \( (\nu_i) \) and face areas \( (F_i) \).

Can formulate an infinitesimal rigidity statement:

If the faces of a convex polyhedron \( Q \subset \mathbb{R}^3 \) are parallelly translated so that \( F_i(t) = F_i + o(t) \), then this is a parallel translation of the whole polyhedron (in the first order).

Equivalently: \( \dim \ker (\partial F_i / \partial h_j) = 3 \) support numbers \( (h_i) \mapsto \text{face areas} (F_i) \).
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\( \nu_1, \ldots, \nu_n \) unit vectors spanning \( \mathbb{R}^3 \), \( F_1, \ldots, F_n > 0 \) such that

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support numbers \( (h_i) \leftrightarrow \) face areas \( (F_i) \)
Minkowski rigidity and the volume functional

Again, there is a functional Vol (just the usual volume) such that
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$$\left( \frac{\partial^2 \text{Vol}}{\partial h_i \partial h_j} \right) \text{ has the signature } (+, 0, 0, 0, -, \ldots, -)$$
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This is a bit stronger than the second Minkowski inequality

$$\text{Vol}(K, \ldots, K, L)^2 \geq \text{Vol}(K) \text{ Vol}(K, \ldots, L, L)$$

(interpreted as the negativity of a $2 \times 2$-determinant)
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A “mysterious” identity:

$$\frac{\partial^2 \text{HE}}{\partial r_i \partial r_j}(P) = \frac{\partial^2 \text{Vol}}{\partial h_i \partial h_j}(P^*)$$
And now, the smooth case

When deforming the metric on $\mathbb{B}^3$, it suffices to consider only warped products

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\tilde{g}_r = d\rho^2 + \rho^2 \left( \frac{g - dr \otimes dr}{r^2} \right),
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where $r : \mathbb{S}^2 \to \mathbb{R}^+$ is a smooth function.
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The curvature of $\tilde{g}_r$ is determined by a function $\sec : \mathbb{S}^2 \to \mathbb{R}$.
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Inf. rigidity $\iff$ if $\sec \cdot = 0$, then $r$ comes from moving the center around.
The Hilbert-Einstein functional

$$\text{HE}(\tilde{g}) = 2 \int_{B^3} \text{scal}\tilde{g} \, d\text{vol}\tilde{g} + \frac{1}{2} \int_{S^2} H \, d\text{area}$$
The Hilbert-Einstein functional

\[ HE(\tilde{g}) = 2 \int_{\mathbb{B}^3} \text{scal}_{\tilde{g}} \, d\text{vol}_{\tilde{g}} + \frac{1}{2} \int_{\mathbb{S}^2} H \, d\text{area} \]

\[ HE' = \int_M \dot{r} \frac{\text{sec}}{\cos \alpha} \, d\text{area}_g, \]

\[ 0 = \int_M \dot{r} \left( \frac{\text{sec}}{\cos \alpha} \right) \, d\text{area} = HE'' = \int_M 2h \det \dot{B} \, d\text{area} \leq 0 \]

where \( \text{sec}(x) = \text{sec}\tilde{g}(T_x M) \) is the sectional curvature in a tangent plane to \( M \)
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The shape does not change \( \Rightarrow \) the deformation is trivial.
Results and problems

- Inf. rigidity is dual to the “Minkowski” rigidity in two different ways. This makes part of the Darboux’ wreath of 12 surfaces.
- Koiso’78: infinitesimal rigidity of Einstein manifolds under certain restrictions on the curvature operator;
- Schlenker’06: infinitesimal rigidity of hyperbolic 3-manifolds with convex smooth boundary.
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- Koiso’78: infinitesimal rigidity of Einstein manifolds under certain restrictions on the curvature operator;
- Schlenker’06: infinitesimal rigidity of hyperbolic 3-manifolds with convex smooth boundary.
- hyperbolic 3-manifolds with convex polyhedral boundary?
- hyperbolic 3-manifolds with convex irregular boundary? (an approach to the Pleating Lamination Conjecture through the rigidity of the convex core)