Infinitesimal rigidity of convex surfaces: Variational methods and duality

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Isometric deformations of polyhedral surfaces

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There exist flexible polyhedra (Connelly).

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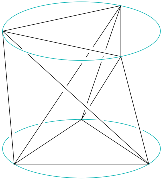
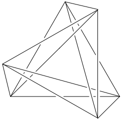
There are lots of infinitesimally flexible polyhedra.





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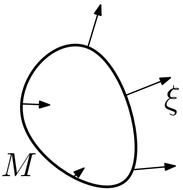
Similar definitions in the smooth case.
Infinitesimal deformation of $M \subset \mathbb{R}^3$
is a vector field $\xi \colon M \to \mathbb{R}^3$.



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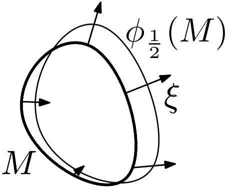

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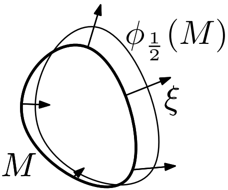

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It is not known whether a higher order smoothness implies infinitesimal rigidity.

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The first idea is easier to explain on the example of isometric embedding theorems.

Isometric embedding theorems

Theorem (Weyl, Levy, Pogorelov, Nirenberg)

g a Riemannian metric on $\mathbb{S}^2$, with everywhere positive Gauss curvature.
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Isometric embedding $=$ geometrization with boundary conditions

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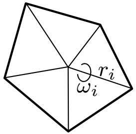

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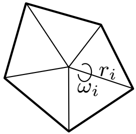

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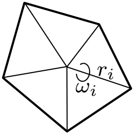

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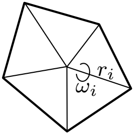

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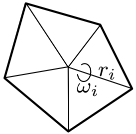

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How to study $\left(\frac{\partial \kappa_i}{\partial r_j}\right)$?

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Theorem

$\left(\frac{\partial^2\,\mathrm{HE}}{\partial r_i \partial r_j}\right)$ *has the signature* $(+, 0, 0, 0, -, \ldots, -)$

Corollary

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Duality with the Minkowski theorem

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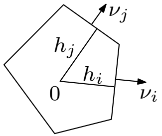
Equivalently: $\dim \ker \left(\frac{\partial F_i}{\partial h_j} \right) = 3$
support numbers $(h_i) \mapsto$ face areas (F_i)



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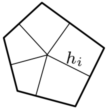
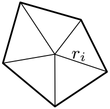
A "mysterious" identity:

$$\frac{\partial^2 \operatorname{HE}}{\partial r_i \partial r_j}(P) = \frac{\partial^2 \operatorname{Vol}}{\partial h_i \partial h_j}(P^*)$$

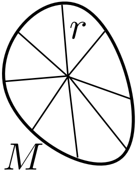




And now, the smooth case

When deforming the metric on $\mathbb{B}^3$, it suffices to consider only warped products

$$\widetilde{g}_r = d\rho^2 + \rho^2 \left(\frac{g - dr \otimes dr}{r^2} \right),$$



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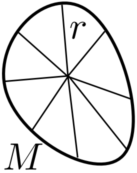

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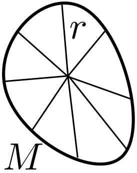

The curvature of $\widetilde{g}_r$ is determined by a function $\sec\colon \mathbb{S}^2 \to \mathbb{R}$.

Inf. rigidity $\Leftrightarrow$ if $\sec^{\cdot} = 0$, then $\dot{r}$ comes from moving the center around.

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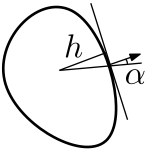

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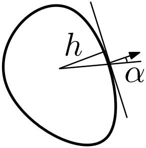

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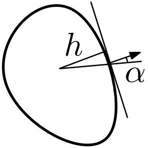
$\det \dot{B} = 0 \Rightarrow \dot{B} = 0$ (yes, there is no mistake! we use $(\det B)^{\cdot} = 0$)

The Hilbert-Einstein functional

$$\mathrm{HE}(\widetilde{g}) = 2\int_{\mathbb{B}^3} \mathrm{scal}_{\widetilde{g}}\, \mathrm{dvol}_{\widetilde{g}} + \frac{1}{2}\int_{\mathbb{S}^2} H\, \mathrm{darea}$$

$$\mathrm{HE}^{\cdot} = \int_M \dot{r}\, \frac{\sec}{\cos\alpha}\, \mathrm{darea}_g,$$



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The shape does not change $\Rightarrow$ the deformation is trivial.

Results and problems

- Inf. rigidity is dual to the "Minkowski" rigidity in two different ways. This makes part of the Darboux' wreath of 12 surfaces.
- Koiso'78: infinitesimal rigidity of Einstein manifolds under certain restrictions on the curvature operator;
- Schlenker'06: infinitesimal rigidity of hyperbolic 3-manifolds with convex smooth boundary.

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- Inf. rigidity is dual to the “Minkowski” rigidity in two different ways. This makes part of the Darboux’ wreath of 12 surfaces.
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- hyperbolic 3-manifolds with convex polyhedral boundary?
- hyperbolic 3-manifolds with convex irregular boundary? (an approach to the Pleating Lamination Conjecture through the rigidity of the convex core)

`arXiv:1105.5066, arXiv:1105.5067`