The CMP admits a natural extension to hedgehogs

- **The classical Minkowski problem (CMP):**

  Existence, uniqueness and regularity of a closed convex hypersurface of $\mathbb{R}^{n+1}$ whose Gauss curvature is prescribed as a positive function on $S^n$.

- **Central role in:**
  - the theory of convex bodies.
  - the theory of elliptic Monge-Ampère equations.

- **The CMP admits a natural extension to hedgehogs.**

  Hedgehogs $= \text{Minkowski differences of convex bodies (or hypersurfaces)}$

- **A way for exploring Monge-Ampère equations of mixed type.**
Hedgehogs as differences of convex bodies

Let \((\mathcal{K}^{n+1}, +, .)\) be the set of convex bodies of \(\mathbb{R}^{n+1}\) equipped with Minkowski addition and multiplication by nonnegative real numbers:

\[
\mathcal{K} + \mathcal{L} = \{x + y | x \in \mathcal{K}, y \in \mathcal{L}\};
\]
\[
\lambda \mathcal{K} = \{\lambda x | x \in \mathcal{K}\}.
\]

\((\mathcal{K}^{n+1}, +, .)\) is not a linear space: no subtraction in \(\mathcal{K}^{n+1}\).

Formal differences of convex bodies of \(\mathbb{R}^{n+1}\) do constitute a linear space \((\mathcal{H}^{n+1}, +, .)\).

Any formal difference \(\mathcal{K} - \mathcal{L}\) of two convex bodies \(\mathcal{K}, \mathcal{L} \in \mathcal{K}^{n+1}\) has a nice geometrical representation in \(\mathbb{R}^{n+1}\), (Y.M.², Canad. J. Math 2006).
Case of convex bodies with positive Gauss curvature

- Subtracting two convex hypersurfaces (with positive Gauss curvature) by subtracting the points corresponding to a same outer unit normal to obtain a (possibly singular and self-intersecting) hypersurface:

Figure: Hedgehogs as differences of convex bodies of class $C^2_+$
Interest of hedgehogs

- To study convex bodies or hypersurfaces by decomposition into a sum of hedgehogs.

**Ex:** **Study of a conjectured characterization of the sphere**

Idea: \( S = S \left( 0_{\mathbb{R}^3}; r \right) + \left( S - S \left( 0_{\mathbb{R}^3}; r \right) \right) \) and study of \( (S - S \left( 0_{\mathbb{R}^3}; r \right)) \).

- To geometrize analytical problems by considering functions as support functions.

**Ex:** **Geometrical proof of the Sturm-Hurwitz theorem**
Support functions

Every \( K \in \mathcal{K}^{n+1} \) is determined by its support function

\[
h_K : \mathbb{S}^n \longrightarrow \mathbb{R} \\
u \longmapsto \sup \{ \langle x, u \rangle : x \in K \}.
\]

A closed convex hypersurface of class \( C^2_+ \) is determined by its support function \( h \in C^2(\mathbb{S}^n; \mathbb{R}) \) as the envelope \( \mathcal{H}_h \subset \mathbb{R}^{n+1} \) of the hyperplanes \( \langle x, u \rangle = h(u) \).
The natural parametrization of the envelope $\mathcal{H}_h$ of the hyperplanes with equation $\langle x, u \rangle = h(u)$ assigns to each $u \in S^n$, the unique solution of the system

\[
\begin{align*}
\langle x, u \rangle &= h(u) \\
\langle x, u \rangle &= dh_u(u)
\end{align*}
\]

that is $x_h(u) = h(u)u + (\nabla h)(u)$. In fact, $\mathcal{H}_h = x_h(S^n)$ is defined for any $h \in C^2(S^n; \mathbb{R})$. It is called hedgehog with support function $h$.

At each regular point $x_h(u) \in \mathcal{H}_h$, $u$ is normal to $\mathcal{H}_h$. 
Gauss curvature

- The singularities of $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ are the very points where the Gauss curvature $\kappa_h(u) = 1/ \det [T_{px}h]$ is infinite.

- The curvature function $R_h := 1/\kappa_h$ is well-defined and continuous on $\mathbb{S}^n$, so that the Minkowski Problem arises for hedgehogs.

- A calculation gives: $R_h(u) = \det [H_{ij}(u) + h(u) \delta_{ij}]$, where $(H_{ij}(u))$ is the Hessian of $h$ at $u$ with respect to an orthonormal frame on $\mathbb{S}^n$.

\[ \text{Case } n = 2 \]

- The curvature function of $\mathcal{H}_h \subset \mathbb{R}^3$ is given by

\[ 1/\kappa_h = h^2 + h\Delta_2 h + \Delta_{22} h \]

($\Delta_2$ is the Laplacian and $\Delta_{22}$ the Monge-Ampère operator, i.e. the sum and the product of the eigenvalues of Hess $h$).

- The type of the equation $h^2 + h\Delta_2 h + \Delta_{22} h = 1/\kappa$ is given by $\text{sgn}[1/\kappa]$. So, the PB leads to PDE’s of mixed type for non-convex hedgehogs.
Key results on the CMP

- Major contributions by Minkowski, Alexandrov, Nirenberg, Pogorelov, Cheng-Yau and others.
- **Existence of a weak solution:**

**Theorem (Minkowski - 1903)**

If \( \kappa \in C(S^n; \mathbb{R}) \) is positive and such that

\[
\int_{S^n} \frac{u}{\kappa(u)} \, d\sigma(u) = 0
\]

then \( \kappa \) is the Gauss curvature of a unique (up to translation) closed convex hypersurface \( \mathcal{H}_h \) of \( \mathbb{R}^{n+1} \).

- **Strong result:**

**Theorem (Pogorelov - 1975, Cheng and Yau - 1976)**

If \( \kappa \in C^m(S^n; \mathbb{R}) \), with \( m \geq 3 \), then: \( \forall \alpha \in [0, 1[ \), \( h \in C^{m+1,\alpha}(S^n; \mathbb{R}) \).
Existence problem

Existence of a $C^2$-solution:

What are necessary and sufficient conditions for $R \in C(S^n; \mathbb{R})$ to be the curvature function of some hedgehog $\mathcal{H} = \mathcal{K} - \mathcal{L}$?

- Integral condition (1) $\int_{S^n} uR(u) \, d\sigma(u) = 0$ is still necessary (but of course not sufficient: consider $-1$).
- Equations with no solution (Y.M.², Adv. in Math. 2001):

  For every $v \in S^2$, $R(u) = 1 - 2 \langle u, v \rangle^2$ satisfies (1) and changes sign cleanly on $S^2$ but is not a curvature function:

  there is no $h \in C^2(S^2; \mathbb{R})$ such that $R_h = R$.

- Can the curvature function of a hedgehog $\mathcal{H}_h$ be nonpositive on $S^2$?

This problem is equivalent to the following conjecture:
Conjecture \((C)\): If \( S \subset \mathbb{R}^3 \) is a closed convex surface of class \( C^2_+ \) such that

\[(k_1 - c)(k_2 - c) \leq 0, \]

with \( c = \text{cst} \), then \( S \) must be a sphere of radius \( 1/c \).

\((C)\) is equivalent to \((H)\):

\((H)\) If \( H_h \subset \mathbb{R}^3 \) is a hedgehog such that \( R_h \leq 0 \), then \( H_h \) is a point.

Counter-example to \((H)\) \( (Y.M.^2, \text{C. R. Acad. Sci. Paris 2001}) \).
**Uniqueness problem**

**Uniqueness of a $C^2$-solution:**

Let $R \in C(S^n; \mathbb{R})$ be the curvature function of some hedgehog $H_h$. What are necessary and sufficient conditions on $R$ for $H_h$ to be uniquely determined by $R$ (up to parallel translations and identifying $h$ with $-h$)?

In the convex case, the uniqueness comes from the equality condition in a well-known Minkowski’s inequality. This inequality cannot be extended to hedgehogs and uniqueness is lost.

**Question.** Does there exist any pair of noncongruent analytic hedgehogs with the same curvature function?
Results relative to the uniqueness

Let $H_3$ be the linear space of $C^2$-hedgehogs defined up to a translation in $\mathbb{R}^3$.

**Theorem** (Y.M.², Central European J. Math. 2012). Let $\mathcal{H}$ and $\mathcal{H}'$ be $C^2$-hedgehogs that are linearly independent in $H_3$. If some linear combination of $\mathcal{H}$ and $\mathcal{H}'$ is of class $C^2_+$, then $\mathcal{H}$ and $\mathcal{H}'$ have distinct curvature functions.

Our second result relies on the extension to hedgehogs of the notion of mixed curvature function.

**Theorem** (Y.M.², Central European J. Math. 2012). Let $\mathcal{H}$ and $\mathcal{H}'$ be analytic (resp. projective $C^2$) hedgehogs of $\mathbb{R}^3$ that are linearly independent in $H_3$. If the mixed curvature function of $\mathcal{H}$ and $\mathcal{H}'$ does not change sign, then $\mathcal{H}$ and $\mathcal{H}'$ have distinct curvature functions.
Example of a uniqueness result

The following result relies on the decomposition of hedgehogs into centered and projective parts.

**Theorem** (Y.M. 2, Central European J. Math. 2012). Let $\mathcal{H}$ and $\mathcal{H}'$ be $C^2$-hedgehogs that are linearly independent in $H_3$ and the centered parts of which are non-trivial and proportional to one and the same convex surface of class $C^2_+$. Then $\mathcal{H}$ and $\mathcal{H}'$ have distinct curvature functions.

**Corollary.** Two $C^2$-hedgehogs of nonzero constant width that are linearly independent in $H_3$ must have distinct curvature function.

**Consequence.** The Monge-Ampère equation $h^2 + h\Delta_2 h + \Delta_{22} h = R, \quad R \in C\left(S^2;\mathbb{R}\right)$, cannot admit more than one solution of the form $f + r$, where $f \in C^2\left(S^2;\mathbb{R}\right)$ is antisymmetric and $r$ is a nonzero constant.

(Solutions are identified if they are opposite or if they differ by the restriction to $S^2$ of a linear form on $\mathbb{R}^3$)
Thank you very much

Thank you very much for your attention!

Figure: European hedgehog