On Killing fields of constant length on Riemannian manifolds

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Killing vector fields of constant length

This talk is devoted to some recent results on Killing vector fields of constant length on Riemannian manifolds, that are obtained mainly in our joint with Prof. V.N. Berestovskii papers [1, 2, 3].

Recall that a smooth vector field X on a Riemannian manifold (M, g) is called Killing if $L_X g = 0$. A vector field X on a (complete) Riamannian manifold (M, g) is Killing if and only if it generates a flow on M, that consists of isometries of (M, g).

The main object of our talk are Killing vector fields of constant length. The following proposition is evident.

Proposition 1. **A Killing vector field X on a Riemannian manifold (M, g) has constant length if and only if every integral curve of the field X is a geodesic in (M, g).**

Clifford-Wolf translations

It should be noted, that Killing vector fields of constant length are closely related with Clifford-Wolf translations on Riemannian manifolds [1].

Recall that Clifford-Wolf translation in a Riemannian manifold (M, g) is an isometry moving all points in M one and the same distance. Notice that Clifford-Wolf translations are often called Clifford translations.

Clifford-Wolf translations naturally appear in the investigation of homogeneous Riemannian coverings of homogeneous Riemannian manifolds. Let us indicate yet another construction of such transformations.

Suppose that some isometry group G acts transitively on a Rimannian manifold M. Then any central element s of this group is a Clifford-Wolf translation. Indeed, if x and y are some points of manifold M, then there is $g \in G$ such that $g(x) = y$. Thus

$$\rho(x, s(x)) = \rho(g(x), g(s(x))) = \rho(g(x), s(g(x))) = \rho(y, s(y)).$$

In particular, if the center Z of the group G is not discrete, then every one-parameter subgroup in Z is a one-parameter group of Clifford-Wolf translations on (M, g).

Notice that several classical Riemannian manifolds possess a one-parameter group of Clifford-Wolf translations. For instance, one knows that among irreducible compact simply connected symmetric spaces only odd-dimensional spheres, spaces $SU(2m)/Sp(m)$, $m > 1$, and simple compact Lie groups, supplied with some bi-invariant Riemannian metric, admit one-parameter groups of Clifford-Wolf translations [11].

There exists a connection between Killing vector fields of constant length and Clifford-Wolf translations in a Riemannian manifold (M, g). The following proposition is almost evident.

Proposition 2. Suppose that a one-parameter isometry group $\gamma(t)$ on (M, g), generated by a Killing vector field X, consists of Clifford-Wolf translations. Then X has constant length.

Proposition 2 can be partially inverted. More exactly, we have

Proposition 3 (Berestovskii-Nikonorov [2]). Suppose a Riemannian manifold (M, g) has the injectivity radius, bounded from below by some positive constant (in particularly, this condition is satisfied for arbitrary compact or homogeneous manifold), and X is a Killing vector field on (M, g) of constant length. Then isometries $\gamma(t)$ from the one-parameter motion group, generated by the vector field X, are Clifford-Wolf translations if t is close enough to 0.

Note that this proposition could not be generalized to all values of t. We will discuss it a little later.

Clifford-Wolf translation on symmetric spaces

In the papers [1, 2] by V.N. Berestovskii and Yu.G. Nikonorov, nontrivial Killing vector fields of constant length and corresponding flows on smooth complete Riemannian manifolds are studied. In particular, it is proved that such a flow on a symmetric space is free or is induced by a free isometric action of the circle S^1. For more details, let us consider the following result.

Theorem 1 (Berestovskii-Nikonorov [2]). Let M be a symmetric Riemannian space, X is a Killing vector field of constant length on M. Then the 1-parameter isometry group $\gamma(t)$, $t \in \mathbb{R}$, of the space M, generated by the field X, consists of Clifford-Wolf translations. Moreover, if the space M has positive sectional curvature, then the flow $\gamma(t)$, $t \in \mathbb{R}$, admits a factorization up to a free isometric action of the circle S^1 on M.

In the proof of the above theorem, the authors have used an interesting characterization of Clifford-Wolf translation on symmetric spaces, obtained by V. Ozols in [9].

It should be noted that the (first assertion of) Theorem 1 could be generalized even to locally symmetric spaces. it is not difficult to give examples of locally symmetric M with Killing vector fields of constant length X such that

1) all integral curves of X are closed;

2) there are integral curves of different lengths.

Clear, that such a Killing field X does not generate a group $\gamma(t)$ $(t \in \mathbb{R})$ that consists of Clifford-Wolf translations.

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Clifford-Wolf homogeneous Riemannian manifolds

An inner metric space (M, ρ) is called Clifford-Wolf homogeneous if for every two points y, z in M there exists a Clifford-Wolf translation of the space (M, ρ) moving y to z. This notion has been introduced in the paper of V.N. Berestovskii and C. Plaut [4].

It should be noted, that a composition of two Clifford-Wolf translations is not necessarily a Clifford-Wolf translation too. This generates some additional difficulties in the study.

Obviously, every Euclidean space $\mathbb{E}^n$ is Clifford-Wolf homogeneous. Since $\mathbb{E}^n$ can be treated as a (commutative) additive vector group with bi-invariant inner product, the following example can be considered as a generalization.

Let G be a Lie group supplied with a bi-invariant Riemannian metric, that generates the inner metric ρ. In this case both the group of left shifts $L(G)$ and the group of right shifts $R(G)$ consist of Clifford-Wolf isometries of (G, ρ). Indeed, both $L(G)$ and $R(G)$ act transitively on G, and every $a \in L(G)$ commutes with every $b \in R(G)$.

For example, let us show that $L_a \in L(G)$ is a Clifford-Wolf translation for every $a \in G$ (the case $R_a \in R(G)$ could be considered by analogy). If x and y are some points of G, then there is $g \in G$ such that $R_g(x) = xg = y$. Thus

$$\rho(x, L_a(x)) = \rho(x, ax) = \rho(R_g(x), R_g(ax)) =$$

$$\rho(xg, axg) = \rho(y, ay) = \rho(y, L_a(y)).$$

Therefore, (G, ρ) is Clifford-Wolf homogeneous.

Another well known class of Clifford-Wolf homogeneous manifolds consists of round odd-dimensional spheres.
Let us show that every odd-dimensional round sphere S^{2n-1} is Clifford-Wolf homogeneous. Indeed,

$$S^{2n-1} = \left\{ \xi = (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{k=1}^{n} |z_k|^2 = 1 \right\}.$$

Then the formula $\gamma(t)(\xi) = e^{it}\xi$ defines a one-parameter group of Clifford-Wolf translations on S^{2n-1} with all orbits as geodesic circles. Now, since S^{2n-1} is homogeneous and isotropic (two-point homogeneous), any its geodesic circle is an orbit of a one-parameter group of Clifford-Wolf translations, and so S^{2n-1} is Clifford-Wolf homogeneous. Note that S^1 and S^3 can be treated as the Lie groups $SO(2)$ and $SU(2)$ with bi-invariant Riemannian metrics.

Note also that any direct metric product of Clifford-Wolf homogeneous Riemannian manifolds is Clifford-Wolf homogeneous itself. On the other hand, the condition for a Riemannian manifold to be Clifford-Wolf homogeneous, is quite strong. Therefore, one should hope to get a complete classification of such manifolds. We have the following

Theorem 2 (Berestovskii-Nikonorov [3]). **Every simply connected Clifford-Wolf homogeneous Riemannian manifold is a direct metric product of an Euclidean space, odd-dimensional spheres of constant curvature and simply connected compact simple Lie groups supplied with bi-invariant Riemannian metrics (some of these factors may absent).**

The following theorem is the most important technical tool in the study of Clifford-Wolf homogeneous Riemannian manifolds.

Theorem 3 (Berestovskii-Nikonorov [3]). **For any Killing field of constant length Z on a Riemannian manifold (M,g) the equality**

$$(\nabla_Z R)(\cdot, Z)Z = 0$$

holds at every point of M, where R is the curvature tensor of (M,g).

Riemannian manifolds with the Killing property

In the paper [5], J.E. D'Atri and H.K. Nickerson studied Riemannian manifolds with the Killing property:

A Riemannian manifold (M, g) is said to have the Killing property if, in some neighborhood of each point of M, there exists an orthonormal frame $\{X_1, \dots, X_n\}$ such that each X_i, $i = 1, \dots, n$, is a Killing vector field. Such a frame is called a Killing frame.

It is easy to see that every Lie group supplied with a bi-invariant Riemannian metric has the (global) Killing property. It could be shown that the same property has 7-dimensional round spheres S^7.

In [5], J.E. D'Atri and H.K. Nickerson tried but failed to classify simply connected complete Riemannian manifolds with the Killing property. The following theorem solves this question completely.

Theorem 4 (Berestovskii-Nikonorov [3]). A simply connected complete Riemannian manifold (M,g) has the Killing property if and only if it is isometric to a direct metric product of Euclidean space, compact simply connected simple Lie groups with bi-invariant metrics, and round spheres S^7 (some mentioned factors could be absent).

The main idea is the following: If a simply connected complete Riemannian manifold has the Killing property, then it is Clifford-Wolf homogeneous.

Riemannian geodesic orbit manifold

Finally, we clarify connections between Killing fields of constant length on a Rimannian geodesic orbit manifold (M, g) and the structure of its full isometry group. A Riemannian manifold (M, g) is called a manifold with homogeneous geodesics or a geodesic orbit manifold, if any geodesic γ of M is an orbit of a one-parameter subgroup of the full isometry group of (M, g). This terminology was introduced in [7] by O. Kowalski and L. Vanhecke, who initiated a systematic study of such spaces.
Geodesic orbit Riemannian manifold may be considered as a natural generalization of symmetric spaces, classified by É. Cartan.

Indeed, a simply connected symmetric space can be defined as a Riemannian manifold (M, g) such that any geodesic $\gamma \subset M$ is an orbit of one-parameter group $g(t)$ of transvections, that is one-parameter group of isometries which preserves γ and induces the parallel transport along γ. If we remove the assumption that $g(t)$ induces the parallel transport, we get the notion of a geodesic orbit manifold.
The class of geodesic orbit Riemannian manifolds is much larger then the class of symmetric spaces. For example, any homogeneous space $M = G/H$ of a compact Lie group G admits a metric g^M such that (M, g^M) is a a geodesic orbit Riemannian manifold.

It is sufficient to take the metric g^M which is induced with a bi-invariant Riemannian metric g on the Lie group G such that $(G, g) \to (M = G/H, g^M)$ is a Riemannian submersion with totally geodesic fibres. Such homogeneous space $(M = G/H, g^M)$ is called a normal homogeneous space. More generally, any naturally reductive manifold is geodesic orbit. Recall, that a Riemannian manifold (M, g^M) is called naturally reductive if it admits a transitive Lie group G of isometries with a bi-invariant pseudo-Riemannian metric g, which induces the metric g^M on $M = G/H$. An important class of geodesic orbit Riemannian manifolds consists of weakly symmetric spaces, introduced by A. Selberg [10]. Recall, that a Riemannian manifold (M, g) is a weakly symmetric space, if any two points $p, q \in M$ can be interchanged by an isometry of (M, g).

The study of geodesic orbit manifolds develops actively in the last decade. But we have no time to survey all more or less recent results on this subject. We consider only one result related to Killing vector fields of constant length.

The Lie algebra of the full isometry group of a Riemannian manifold (M, g) is naturally identified with the Lie algebra of Killing fields $\mathfrak{g}$ on (M, g). The following result is useful for a study of geodesic orbit Rimannian manifolds.

Theorem 5 (Nikonorov [8]). Let (M,g) be a geodesic orbit Rimannian manifold, $\mathfrak{g}$ is its Lie algebra of Killing fields. Suppose that $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$. Then any $X \in \mathfrak{a}$ has constant length on (M,g).

On the ground of Theorem 5 one can give (see [8]) a new proof of one result of C. Gordon:

Theorem 6 (Gordon [6]). Every Riemannian geodesic orbit manifold of nonpositive Ricci curvature is a symmetric space.

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Thank you for your time and attention!

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