Ricci curvature in Finsler geometry and applications

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§Plan of talk

- Preliminaries for Finsler manifolds
- Definition of weighted Ricci curvature
- Geometric & analytic applications

Partly joint with Karl-Theodor Sturm (Univ. Bonn).

§Finsler manifolds

A C^∞-*Finsler manifold* will be a pair (M, F) of a connected C^∞-manifold M and $F : TM \longrightarrow [0, \infty)$ s.t.

(1) F is C^∞ on $TM \setminus \{0\}$,

(2) $F(cv) = cF(v)$ for all $v \in TM$ and $c > 0$,

(3) For any $v \in TM \setminus \{0\}$, the $n \times n$-symmetric matrix

$$g_{ij}(v) := \frac{1}{2}\frac{\partial^2(F^2)}{\partial v^i \partial v^j}, \quad \text{where } v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i},$$

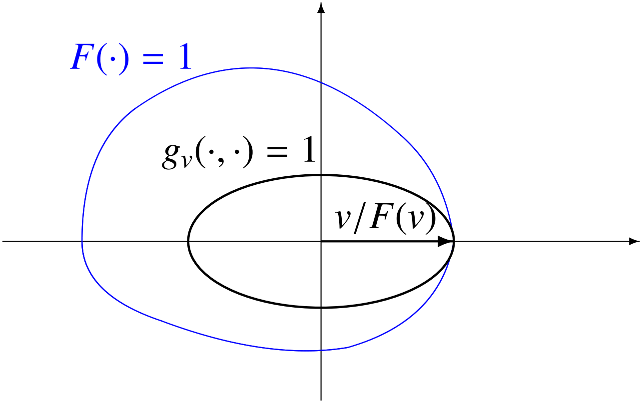
is positive-definite (*strong convexity*).

Strong convexity

For each $v \in T_xM \setminus \{0\}$, $g_{ij}(v)$ defines the inner product g_v of T_xM by ($n = \dim M$)

$$g_v\Big(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x^i}, \sum_{j=1}^{n} b_j \frac{\partial}{\partial x^j}\Big) := \sum_{i,j=1}^{n} a_i b_j g_{ij}(v).$$

This approximates the (Minkowski) norm $F|_{T_xM}$ in the direction v in the following sense.



The unit sphere of g_v tangents to the unit sphere of $F|_{T_xM}$ at $v/F(v)$ up to the second order.

Metric structure of (M, F)

A Finsler structure F naturally induces

- the *distance* $d(x, y)$ as the infimum of the lengths of curves from x to y (possibly $d(x, y) \neq d(y, x)$),
- *geodesics* as constant-speed, locally shortest curves w.r.t. d,
- the *forward completeness* as the extendability of any geodesic $\eta : [0, \varepsilon] \longrightarrow M$ to $\bar{\eta} : [0, \infty) \longrightarrow M$.

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Ricci curvature comparison is possible! But how?

The Ricci curvature is defined by using a connection. However, there is no canonical measure like the volume measure in Riemannian geometry.

Thus we start with an arbitrary measure m on M and modify the Ricci curvature according to m.

§Weighted Ricci curvature

Instead of giving the precise definition, we explain Z. Shen's interpretation of the *Finsler-Ricci curvature* $\mathrm{Ric}(v)$ of a unit vector $v \in UM = F^{-1}(1)$:

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Extend v to a C^∞-vector field V in such a way that every integral curve is geodesic (always possible).

Then $\mathrm{Ric}(v)$ coincides with the Ricci curvature $\mathrm{Ric}^V(v)$ of v w.r.t. the Riemannian structure g_V.

Weighted version

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We fix an arbitrary positive C^∞-measure m on M and modify $\mathrm{Ric}(v)$ as follows (for v, V as above):

Decompose m as $m = e^{-\psi} \mathrm{vol}_V$, where vol_V is the Riemannian volume measure of g_V, and let η be the geodesic with $\dot{\eta}(0) = v$.

For $N \in (n, \infty)$ ($n = \dim M$), define

$$\mathrm{Ric}_N(v) := \mathrm{Ric}(v) + (\psi \circ \eta)''(0) - \frac{(\psi \circ \eta)'(0)^2}{N - n},$$

$$\mathrm{Ric}_N(cv) := c^2 \mathrm{Ric}_N(v) \qquad \text{for } c > 0.$$

As the limits,

$$\mathrm{Ric}_\infty(v) := \mathrm{Ric}(v) + (\psi \circ \eta)''(0) \quad \textit{(Bakry-Émery tensor)},$$

$$\mathrm{Ric}_n(v) := \begin{cases} \mathrm{Ric}(v) + (\psi \circ \eta)''(0) & \text{if } (\psi \circ \eta)'(0) = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

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- $\mathrm{Ric}_\infty \geq \mathrm{Ric}_N \geq \mathrm{Ric}_n$ by definition.
- $(\psi \circ \eta)'(0)$ coincides with Shen's $\mathbf{S}$-*curvature* $\mathbf{S}(v)$.
- $\mathbf{S} \equiv 0$ does not hold for any m in some spaces (O. 2011), so there is no nice reference measure.

§Application I: Curvature-dimension condition

The following theorem generalizes the corresponding theorem in the (weighted) Riemannian case by Lott, Renesse, Sturm and Villani.

Theorem (O. 2009)

Let (M, F, m) be forward complete, $N \in [n, \infty]$, $K \in \mathbb{R}$. Then the lower bound $\mathrm{Ric}_N \geq K$ (i.e., $\mathrm{Ric}_N(v) \geq KF(v)^2$) is equivalent to the *curvature-dimension condition* $\mathrm{CD}(K, N)$.

$\mathrm{CD}(K,N)$ is a convexity condition of an entropy function on the space $\mathcal{P}(M)$ of probability measures on M (minimal geodesics in $\mathcal{P}(M)$ = 'optimal transports').

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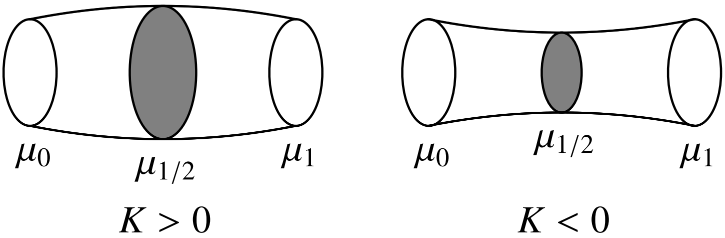
Geometric image of $\mathrm{CD}(K, N)$



($K > 0$ case: less concentrated = less entropy at $\mu_{1/2}$)

Metric measure spaces satisfying $\mathrm{CD}(K,N)$ behave like spaces with $\mathrm{Ric} \ge K$ & $\dim \le N$ (Sturm, Lott-Villani).
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- *Talagrand inequality*, *log-Sobolev inequality*, *global Poincaré inequality* & *normal concentration of measures* for $K > 0$ & $N = \infty$;
- *Bonnet-Myers diameter bound*, *Lichnerowicz inequality* for $K > 0$ & $N < \infty$.

§Application II: Nonlinear Laplacian

We introduce:

- the *gradient vector* $\nabla u(x) \in T_x M$ as the Legendre transform of the derivative $Du(x) \in T_x^* M$ $(F^*(Du) = F(\nabla u), Du[\nabla u] = F^*(Du)^2)$,

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- the *nonlinear Laplacian* $\Delta u := \mathrm{div}_m(\nabla u)$ in the distributional sense that

$$\int_M \phi \Delta u \, dm = -\int_M D\phi(\nabla u) \, dm \quad \forall \phi \in C_c^\infty(M).$$

It is not difficult to see (for forward complete M):

Laplacian comparison theorem (O.-Sturm 2009)

If $\mathrm{Ric}_N \geq K$ for some $K < 0$ and $N \in [n, \infty)$, then $u(x) := d(z, x)$ for fixed $z \in M$ satisfies

$$\Delta u(x) \leq \sqrt{-(N-1)K} \coth\left(\sqrt{\frac{-K}{N-1}} d(z, x) \right)$$

point-wise on $M \setminus (\{z\} \cup \mathrm{Cut}(z))$, and in the weak sense on $M \setminus \{z\}$. The RHS will be $(N-1)/d(z, x)$ if $K = 0$; $\sqrt{(N-1)K} \cot(\sqrt{K/(N-1)} d(z, x))$ if $K > 0$.

(Extended to general metric measure spaces with $\mathrm{CD}(K, N)$ recently by Gigli.)

§Application III: Bochner-Weitzenböck formula

We further need:

- $\Delta^{\nabla u} f := \mathrm{div}_m(\nabla^{\nabla u} f)$: the Laplacian linearized via $g_{\nabla u}$ (Note: $\boldsymbol{\Delta} u = \Delta^{\nabla u} u$),

§Application III: Bochner-Weitzenböck formula

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- $\Delta^{\nabla u} f := \mathrm{div}_m(\nabla^{\nabla u} f)$: the Laplacian linearized via $g_{\nabla u}$ (Note: $\boldsymbol{\Delta} u = \Delta^{\nabla u} u$),
- $\nabla^2 u := D^{\nabla u}(\nabla u) : TM \longrightarrow TM$: the 'Hessian' of u,
- $\|\nabla^2 u\|_{HS(\nabla u)}$: the Hilbert-Schmidt norm of $\nabla^2 u$ w.r.t. $g_{\nabla u}$.

Bochner-Weitzenböck formula (O.-Sturm 2011)

For $u \in C^\infty(M)$,

$$\Delta^{\nabla u}\left(\frac{F(\nabla u)^2}{2}\right) - D(\Delta u)(\nabla u) = \mathrm{Ric}_\infty(\nabla u) + \|\nabla^2 u\|^2_{HS(\nabla u)}$$

point-wise on $\{\nabla u \neq 0\}$, and in the weak sense on M.

There arise some technical difficulties on $\{\nabla u = 0\}$ since the Legendre transform is not differentiable on $\{0\} \subset TM$.

By a standard argument, we moreover obtain:

Bochner inequality for $N < \infty$ (O.-Sturm 2011)

For $u \in C^\infty(M)$ and $N \in [n, \infty)$,

$$\Delta^{\nabla u}\left(\frac{F(\nabla u)^2}{2}\right) - D(\Delta u)(\nabla u) \ge \mathrm{Ric}_N(\nabla u) + \frac{(\Delta u)^2}{N}$$

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(b) Applications include the *Bakry-Émery gradient estimate* ($N = \infty$), the *Li-Yau gradient estimate* and the *Harnack inequality* ($N < \infty$) for the heat flow associated with our nonlinear Laplacian.

§Application IV: Splitting theorems

Finally, we consider a generalization of the *Cheeger-Gromoll splitting theorem*.

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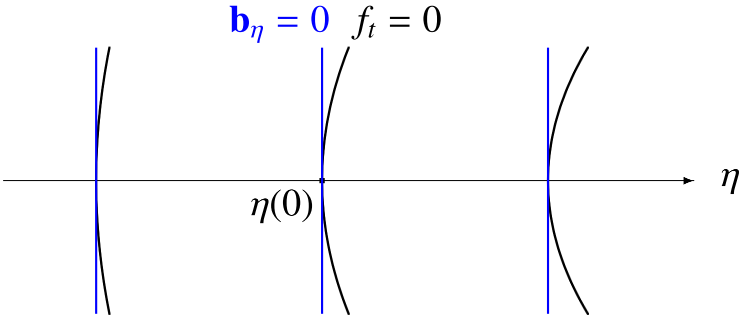
Let (M, F, m) be both forward & backward complete and

- $\mathrm{Ric}_N \geq 0$ for some $N \in [n, \infty]$,
- $\sup \psi < \infty$ if $N = \infty$ ($\psi : UM \longrightarrow \mathbb{R}$ is the weight function as in the definition of Ric_N),
- there is a straight line $\eta : \mathbb{R} \longrightarrow M$ (i.e., $d(\eta(s), \eta(t)) = t - s, \ \forall s < t$).

Analysis of Busemann functions

Define

$$f_t(x) := t - d(x, \eta(t)), \quad \mathbf{b}_\eta(x) := \lim_{t\to\infty} f_t(x) \quad \text{for } x \in M.$$



Lemma

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The maximum principle shows that $\mathbf{b}_\eta$ is *harmonic* ($\boldsymbol{\Delta}\mathbf{b}_\eta \equiv 0$). Moreover, $F(\boldsymbol{\nabla}\mathbf{b}_\eta) \equiv 1$ and $\mathbf{b}_\eta$ is C^∞.

(The harmonicity means that $\mathbf{b}_\eta$ is a static solution to the heat equation. Then the non-vanishing of $\boldsymbol{\nabla}\mathbf{b}_\eta$ shows that $\mathbf{b}_\eta$ is C^∞.)

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In particular,

Diffeomorphic splitting for general Finlser manifolds (O. 2012)

(M, m) splits off $\mathbb{R}$ as

$$M = M' \times \mathbb{R} \text{ diffeomorphically}, \quad m = m|_{M'} \times \mathbf{L}^1,$$

where $M' := \mathbf{b}_\eta^{-1}(0)$, $\mathbf{L}^1$ is the Lebesgue measure on $\mathbb{R}$.

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  In this case, M' is totally geodesic and isometric to $\mathbf{b}_\eta^{-1}(t)$ for all $t \in \mathbb{R}$. Thus the splitting can be iterated, and we can also obtain the *Betti number estimate à la Cheeger-Gromoll*.