Ricci curvature in Finsler geometry and applications

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Plan of talk

Preliminaries for Finsler manifolds

Definition of weighted Ricci curvature

Geometric & analytic applications

Partly joint with Karl-Theodor Sturm (Univ. Bonn).
A $C^\infty$-Finsler manifold will be a pair $(M, F)$ of a connected $C^\infty$-manifold $M$ and $F : TM \longrightarrow [0, \infty)$ s.t.

1. $F$ is $C^\infty$ on $TM \setminus \{0\}$,
2. $F(cv) = cF(v)$ for all $v \in TM$ and $c > 0$,
3. For any $v \in TM \setminus \{0\}$, the $n \times n$-symmetric matrix 

$$g_{ij}(v) := \frac{1}{2} \frac{\partial^2 (F^2)}{\partial v^i \partial v^j}, \text{ where } v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i},$$

is positive-definite (strong convexity).
For each $v \in T_xM \setminus \{0\}$, $g_{ij}(v)$ defines the inner product $g_v$ of $T_xM$ by ($n = \dim M$)

$$g_v \left( \sum_{i=1}^{n} a_i \frac{\partial}{\partial x^i}, \sum_{j=1}^{n} b_j \frac{\partial}{\partial x^j} \right) := \sum_{i,j=1}^{n} a_i b_j g_{ij}(v).$$

This approximates the (Minkowski) norm $F|_{T_xM}$ in the direction $v$ in the following sense.
The unit sphere of $g_v$ tangents to the unit sphere of $F|_{T_xM}$ at $v/F(v)$ up to the second order.
A Finsler structure $F$ naturally induces

- the *distance* $d(x,y)$ as the infimum of the lengths of curves from $x$ to $y$ (possibly $d(x,y) \neq d(y,x)$),
- *geodesics* as constant-speed, locally shortest curves w.r.t. $d$,
- the *forward completeness* as the extendability of any geodesic $\eta : [0, \varepsilon] \longrightarrow M$ to $\bar{\eta} : [0, \infty) \longrightarrow M$. 
Easy to see: a lower or upper curvature bound in the sense of Alexandrov implies that all tangent spaces are inner product spaces. So it is Riemannian. Ricci curvature comparison is possible! But how? The Ricci curvature is defined by using a connection. However, there is no canonical measure like the volume measure in Riemannian geometry. Thus we start with an arbitrary measure $m$ on $M$ and modify the Ricci curvature according to $m$. 

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The Ricci curvature is defined by using a connection. However, there is no canonical measure like the volume measure in Riemannian geometry.
Thus we start with an arbitrary measure \( m \) on \( M \) and modify the Ricci curvature according to \( m \).
Instead of giving the precise definition, we explain Z. Shen’s interpretation of the Finsler-Ricci curvature $\text{Ric}(v)$ of a unit vector $v \in UM = F^{-1}(1)$:

- Extend $v$ to a $C^\infty$-vector field $V$ in such a way that every integral curve is geodesic (always possible).

Then $\text{Ric}(v)$ coincides with the Ricci curvature $\text{Ric}_V(v)$ of $v$ w.r.t. the Riemannian structure $g_V$. 

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Weighted version

We fix an arbitrary positive $C^\infty$-measure $m$ on $M$ and modify $\text{Ric}(v)$ as follows (for $v$, $V$ as above):

Decompose $m$ as $m = e^{-\psi} \text{vol} V$, where $\text{vol} V$ is the Riemannian volume measure of $g$. Let $\eta$ be the geodesic with $\dot{\eta}(0) = v$. For $N \in (n, \infty)$ ($n = \dim M$), define $\text{Ric}_N(v) : = \text{Ric}(v) + (\psi \circ \eta)'(0) - (\psi \circ \eta)''(0)_{2N-n}$, $\text{Ric}_N(cv) : = c^2 \text{Ric}_N(v)$ for $c > 0$. 

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We fix an arbitrary positive $C^\infty$-measure $m$ on $M$ and modify $\text{Ric}(v)$ as follows (for $v$, $V$ as above):

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For $N \in (n, \infty)$ ($n = \dim M$), define

$$\text{Ric}_N(v) := \text{Ric}(v) + (\psi \circ \eta)''(0) - \frac{(\psi \circ \eta)'(0)^2}{N - n},$$

$$\text{Ric}_N(cv) := c^2 \text{Ric}_N(v) \quad \text{for } c > 0.$$
As the limits, 

\[ \text{Ric}_\infty(v) := \text{Ric}(v) + (\psi \circ \eta)''(0) \]  

(Bakry-Émery tensor), 

\[ \text{Ric}_n(v) := \begin{cases} 
\text{Ric}(v) + (\psi \circ \eta)''(0) & \text{if } (\psi \circ \eta)'(0) = 0, \\
-\infty & \text{otherwise.} 
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\end{align*}
\]

- \(\text{Ric}_\infty \geq \text{Ric}_N \geq \text{Ric}_n\) by definition.
- \((\psi \circ \eta)'(0)\) coincides with Shen’s S-curvature \(S(v)\).
- \(S \equiv 0\) does not hold for any \(m\) in some spaces (O. 2011), so there is no nice reference measure.
The following theorem generalizes the corresponding theorem in the (weighted) Riemannian case by Lott, Renesse, Sturm and Villani.

**Theorem (O. 2009)**

Let \((M, F, m)\) be forward complete, \(N \in [n, \infty]\), \(K \in \mathbb{R}\). Then the lower bound \(\text{Ric}_N \geq K\) (i.e., \(\text{Ric}_N(v) \geq KF(v)^2\)) is equivalent to the *curvature-dimension condition* \(\text{CD}(K, N)\).
CD$(K, N)$ is a convexity condition of an entropy function on the space $\mathcal{P}(M)$ of probability measures on $M$ (minimal geodesics in $\mathcal{P}(M) = \text{‘optimal transports’}.$)
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Geometric image of CD($K, N$)

\[
\begin{array}{ccc}
\mu_0 & \mu_{1/2} & \mu_1 \\
 K > 0 & & \\
\mu_0 & \mu_{1/2} & \mu_1 \\
 K < 0 & &
\end{array}
\]

($K > 0$ case: less concentrated = less entropy at $\mu_{1/2}$)
Metric measure spaces satisfying $\text{CD}(K, N)$ behave like spaces with $\text{Ric} \geq K \& \text{dim} \leq N$ (Sturm, Lott-Villani). By the general theory of $\text{CD}(K, N)$, we have:
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- **Bishop-Gromov volume comparison for** $N < \infty$;
- **Talagrand inequality, log-Sobolev inequality, global Poincaré inequality & normal concentration of measures for** $K > 0$ & $N = \infty$;
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- **Bishop-Gromov volume comparison** for $N < \infty$;
- **Talagrand inequality, log-Sobolev inequality, global Poincaré inequality & normal concentration of measures** for $K > 0 \& N = \infty$;
- **Bonnet-Myers diameter bound, Lichnerowicz inequality** for $K > 0 \& N < \infty$. 
We introduce:

- the *gradient vector* $\nabla u(x) \in T_x M$ as the Legendre transform of the derivative $Du(x) \in T^*_x M$

$$ (F^*(Du) = F(\nabla u), Du[\nabla u] = F^*(Du)^2), $$
We introduce:

- the gradient vector $\nabla u(x) \in T_xM$ as the Legendre transform of the derivative $Du(x) \in T^*_xM$
  
  $F^*(Du) = F(\nabla u), \quad Du[\nabla u] = F^*(Du)^2$,

- the nonlinear Laplacian $\Delta u := \text{div}_m(\nabla u)$ in the distributional sense that

$$\int_M \phi \Delta u \, dm = -\int_M D\phi(\nabla u) \, dm \quad \forall \phi \in C^\infty_c(M).$$
It is not difficult to see (for forward complete $M$):

**Laplacian comparison theorem (O.-Sturm 2009)**

If $\text{Ric}_N \geq K$ for some $K < 0$ and $N \in [n, \infty)$, then $u(x) := d(z, x)$ for fixed $z \in M$ satisfies

$$\Delta u(x) \leq \sqrt{-(N-1)K} \coth \left( \sqrt{\frac{-K}{N-1}} d(z, x) \right)$$

point-wise on $M \setminus (\{z\} \cup \text{Cut}(z))$, and in the weak sense on $M \setminus \{z\}$. The RHS will be $(N-1)/d(z, x)$ if $K = 0$; $\sqrt{(N-1)K} \cot(\sqrt{K/(N-1)} d(z, x))$ if $K > 0$.

(Extended to general metric measure spaces with CD($K, N$) recently by Gigli.)
We further need:

- $\Delta^{\nabla} u f := \text{div}_m(\nabla \nabla u f)$: the Laplacian linearized via $g \nabla u$ (Note: $\Delta u = \Delta^{\nabla} u u$),
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- $\Delta^{\nabla u} f := \text{div}_m(\nabla^{\nabla u} f)$: the Laplacian linearized via $g\nabla u$ (Note: $\Delta u = \Delta^{\nabla u} u$),

- $\nabla^2 u := D^{\nabla u}(\nabla u) : TM \to TM$: the ‘Hessian’ of $u$,

- $\|\nabla^2 u\|_{HS(\nabla u)}$: the Hilbert-Schmidt norm of $\nabla^2 u$ w.r.t. $g\nabla u$. 
**Bochner-Weitzenböck formula (O.-Sturm 2011)**

For $u \in C^\infty(M)$,

$$
\Delta \nabla u \left( \frac{F(\nabla u)^2}{2} \right) - D(\Delta u)(\nabla u) = \text{Ric}_\infty(\nabla u) + \|\nabla^2 u\|_{HS(\nabla u)}^2
$$

point-wise on $\{\nabla u \neq 0\}$, and in the weak sense on $M$.

There arise some technical difficulties on $\{\nabla u = 0\}$ since the Legendre transform is not differentiable on $\{0\} \subset TM$. 
By a standard argument, we moreover obtain:

**Bochner inequality for \( N < \infty \) (O.-Sturm 2011)**

For \( u \in C^\infty(M) \) and \( N \in [n, \infty) \),

\[
\Delta \nabla^u \left( \frac{F(\nabla u)^2}{2} \right) - D(\Delta u)(\nabla u) \geq \text{Ric}_N(\nabla u) + \frac{(\Delta u)^2}{N}
\]

point-wise on \( \{\nabla u \neq 0\} \), and in the weak sense on \( M \).
(a) What happens when $F$ is replaced with $g \nabla u$?
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$\Delta^{\nabla u} u = \Delta u$, but $\text{Ric}_{\nabla u}(\nabla u) \neq \text{Ric}_{\nabla u}(\nabla u)$ (unless all integral curves of $\nabla u$ are geodesic). Therefore the above formula is not the Bochner formula for the Riemannian structure $g\nabla u$. 

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(b) Applications include the Bakry-Émery gradient estimate ($N = \infty$), the Li-Yau gradient estimate and the Harnack inequality ($N < \infty$) for the heat flow associated with our nonlinear Laplacian.
Finally, we consider a generalization of the Cheeger-Gromoll splitting theorem.
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Let $(M, F, m)$ be both forward & backward complete and

- $\text{Ric}_N \geq 0$ for some $N \in [n, \infty]$,
- $\sup \psi < \infty$ if $N = \infty$ ($\psi : UM \to \mathbb{R}$ is the weight function as in the definition of $\text{Ric}_N$),
- there is a **straight line** $\eta : \mathbb{R} \to M$ (i.e., $d(\eta(s), \eta(t)) = t - s, \forall s < t$).
Define

\[ f_t(x) := t - d(x, \eta(t)), \quad b_\eta(x) := \lim_{t \to \infty} f_t(x) \quad \text{for } x \in M. \]

\[ b_\eta = 0 \quad f_t = 0 \]
Lemma

The Laplacian comparison theorem implies that $b_\eta$ is \textit{subharmonic} (i.e., $\Delta b_\eta \geq 0$ in the distributional sense).
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Lemma
The maximum principle shows that $b_\eta$ is harmonic ($\Delta b_\eta \equiv 0$). Moreover, $F(\nabla b_\eta) \equiv 1$ and $b_\eta$ is $C^\infty$.

(The harmonicity means that $b_\eta$ is a static solution to the heat equation. Then the non-vanishing of $\nabla b_\eta$ shows that $b_\eta$ is $C^\infty$.)
Applying the Bochner inequality to $b_\eta$, the weighted Riemannian manifold $(M, g\nabla b_\eta, m)$ splits isometrically.
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**Diffeomorphic splitting for general Finsler manifolds** (O. 2012)

$(M, m)$ splits off $\mathbb{R}$ as

$$M = M' \times \mathbb{R} \text{ diffeomorphically, } m = m|_{M'} \times L^1,$$

where $M' := b_\eta^{-1}(0)$, $L^1$ is the Lebesgue measure on $\mathbb{R}$. 
Remarks

- We know only a little about the structure of $M'$ in general, so this procedure cannot be iterated.
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- We can go further only in the special case of *Berwald spaces* ($\Rightarrow$ all tangent spaces are isometric each other).

In this case, $M'$ is totally geodesic and isometric to $b^{-1}_\eta(t)$ for all $t \in \mathbb{R}$. Thus the splitting can be iterated, and we can also obtain the Betti number estimate à la Cheeger-Gromoll.