Topological properties of Busemann G-spaces

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Herbert Busemann (1905 - 1994) was a German-American mathematician specializing in convex and differential geometry.

Werner Fenchel, Aleksander Danilovič Aleksandrov
Herbert Busemann and Borge Jessen (1954)

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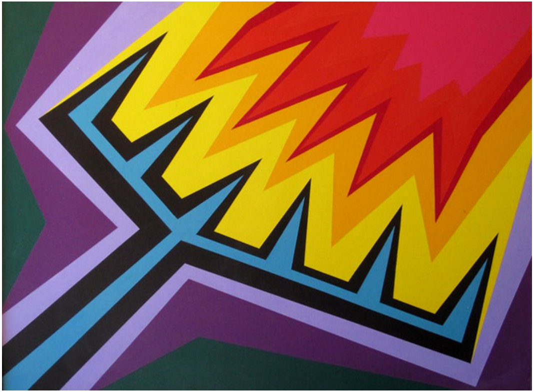
After retiring from USC as a professor emeritus in 1970, Busemann spent the rest of his life painting in Santa Ynez, California. Fittingly, his paintings were also geometrical and executed with mathematical precision.



Conflict (1972)

Beginning in 1942, Herbert Busemann developed the notion of a G-space as a way of putting a Riemannian like geometry on a metric space.

Definition: A *Busemann G-space* is a metric space that satisfies four basic axioms on a metric space.

These axioms imply the existence of geodesics, local uniqueness of geodesics, and local extension properties.

These axioms also infer homogeneity and a cone structure for small metric balls.

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Definition: Let (X, d) be a metric space. X is said to be a *Busemann G-space* provided it satisfies the following axioms:

Menger Convexity: Given distinct points $x, y \in X$, there is a point $z \in X - \{x, y\}$ such that $d(x, z) + d(z, y) = d(x, y)$.

Finite Compactness: Every d-bounded infinite set has an accumulation point.

Local Extendibility: For every point $w \in X$, there is a radius $\rho_w > 0$, such that for any pair of distinct points $x, y \in B(w, \rho_w)$, there is a point $z \in \text{int}\, B(w, \rho_w) - \{x, y\}$ such that $d(x, y) + d(y, z) = d(x, z)$.

Uniqueness of the Extension: Given distinct points $x, y \in X$, if there are points $z_1, z_2 \in X$ for which both

$$d(x, y) + d(y, z_i) = d(x, z_i) \quad \text{for } i = 1, 2,$$

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Facts: From these basic properties, a rich structure on a G-space can be derived. Let (X, d) be a G-space and let $x \in X$. Then (X, d) satisfies the following properties:

Complete Inner Metric: (X, d) is a locally compact complete metric space.

Existence of Geodesics: Any two points in X are joined by a geodesic.

Local Uniqueness of Joins: There is a radius $r_x > 0$ such that any two points $y, z \in B_{r_x}(x)$ in the closed ball can be joined by a unique segment in X.

Local Cones: There is a radius $\epsilon_x > 0$ for which the closed metric ball $B_{\epsilon_x}(x)$ is homeomorphic to the cone over its boundary.

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Busemann Conjecture (1955): Every ($n \geq 3$)-dimensional Busemann G-space is a topological n-manifold.

Krakus Theorem (1968): Busemann G-spaces of dimension $n = 3$ are topological 3-manifolds.

Already in the 1950's Busemann predicted: "Although this conjecture is probably true for any G-space, the proof seems quite inaccessible in the present state of topology." His prediction was correct – the proof of the case $n = 4$ required the theory of 4-manifolds, developed almost three decades later.

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Example: The boundary of a convex region in $\mathbb{R}^n$ has nonnegative Alexandrov curvature.

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The general case of Busemann's Conjecture of $n \geq 5$ remains unsolved. However, there is a proof, due to Berestovskii, of the special case when the Busemann G-space (X, d) has the *Aleksandrov curvature* $\leq K$, which means that geodesic triangles in X are at most as "fat"as corresponding triangles in a surface S_K of constant curvature K, i.e. the length of a bisector of the triangle in X is at most the length of the corresponding bisector of the corresponding triangle in S_K.

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We now pass to *geometric topology*, because the Busemann Conjecture is closely related to another classical (and also still unproven conjecture) stated in the 1960's by Bing and Borsuk. We begin by some preliminaries concerning the so-called Recognition Problem for Topological Manifolds.

Definition: Let Y be a metric space. Y is said to be an *absolute neighborhood retract (ANR)* provided for every closed embedding $e : Y \to Z$ of Y into a metric space Z, there is an open neighborhood U of the image $e(Y)$ which retracts to $e(Y)$. That is, there is a continuous surjection $r : U \to e(Y)$ with $r(x) = x$ for all $x \in e(Y)$.

Fact: Let Y be a finite-dimensional, locally contractible separable metric space. Then Y is an ANR.

Definition: A topological space X is said to be *homogeneous* if for any two points $x_1, x_2 \in X$, there is a homeomorphism of X onto itself taking x_1 to x_2.

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Bredon Theorem (1967): If X is an n-dimensional homogeneous ENR ($n \in \mathbb{N}$) and for some (and, hence all) points $x \in X$, the groups $H_k(X, X - \{x\}; \mathbb{Z})$ are finitely generated, then X is a $\mathbb{Z}$-homology n-manifold.

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Definition: An n-dimensional topological space X is called a *generalized n-manifold* ($n \in \mathbb{N}$) if X is an ENR and a $\mathbb{Z}$-homology n-manifold.

Fact: Every generalized ($n \leq 2$)-manifold is a topological n-manifold. However, for every $n \geq 3$ there exist *totally singular* generalized n-manifolds X.

Definition: A proper onto map $f : M \to X$ is said to be *cell-like* if for every point $x \in X$, the point-inverse $f^{-1}(x)$ contracts in any neighborhood of itself.

The following classical result was proved for $n \leq 2$ by Wilder, for $n = 3$ by Armentrout, for $n = 4$ by Quinn and for $n \geq 5$ by Siebenmann.

Cell-like Approximation Theorem: Every cell-like map between topological manifolds is a near-homeomorphism.

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Resolution Conjecture (1978): Every generalized ($n \geq 3$)-manifold has a resolution.

In dimension 3, the Resolution Conjecture implies the Poincaré Conjecture. In dimensions ≥ 6 it turns out to be false:

Bryant-Ferry-Mio-Weinberger Theorem (1996): There exist non-resolvable generalized n-manifolds, for every $n \geq 6$.

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Cannon Manifold Recognition Problem (1978): Is every resolvable generalized ($n \geq 5$)-manifold with the DDP a topological n-manifold?

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Remark: If the Bryant-Ferry-Mio-Weinberger Conjecture is true, then the Bing Borsuk conjecture is false for $n \geq 7$.

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In 1991 Repovš, Skopenkov and Ščepin proved the *smooth version* of the Bing-Borsuk Conjecture.

Definition: A subset $K \subset \mathbb{R}^n$ is said to be C^1–*homogeneous* if for every pair of points $x, y \in K$ there exist neighborhoods $O_x, O_y \subset \mathbb{R}^n$ of x and y, respectively, and a C^1–*diffeomorphism*

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i.e. h and h^{-1} have continuous first derivatives.

Repovš-Skopenkov-Ščepin Theorem (1991): Let K be a locally compact (possibly nonclosed) subset of $\mathbb{R}^n$. Then K is C^1–homogeneous if and only if K is a C^1–submanifold of $\mathbb{R}^n$.

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Remark: This theorem clearly does not work for all *homeomorphisms*, a counterexample is the *Antoine Necklace* – a wild Cantor set in $\mathbb{R}^3$ which is clearly *homogeneously* (but not C^1–*homogeneously* embedded in $\mathbb{R}^3$.

In fact, it does not even work for *Lipschitz* homeomorphisms, i.e. the maps for which $d(f(x), f(y)) < \lambda\, d(x, y)$, for all $x, y \in X$.

Malešič-Repovš Theorem (1999): There exists a Lipschitz homogeneous wild Cantor set in $\mathbb{R}^3$.

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Malešič-Repovš Theorem (1999): There exists a Lipschitz homogeneous wild Cantor set in $\mathbb{R}^3$.

Garity-Repovš-Željko Theorem (2005): There exist uncountably many rigid Lipschitz homogeneous wild Cantor sets in $\mathbb{R}^3$.

We now discuss the groundbreaking work of Thurston, who proved the 4-dimensional case of the Busemann Conjecture:

Thurston Theorem (1996): Every Busemann G-space of dimension $n = 4$ is a topological 4-manifold.

Thurston Theorem (1996): Every Busemann G-space of dimension $n \geq 5$ is a generalized n-manifold.

Remarks: The fact that every finite-dimensional G-space is an ANR follows from local contractibility and local compactness. The fact that every finite-dimensional G-space is a homology $\mathbb{Z}$-manifold is proved by sheaf-theoretic methods.

Thurston Theorem (1996): Let (X, d) be a Busemann G-space, $\dim X = n < \infty$. Then for all $x \in X$ and sufficiently small $r > 0$:

1. $B_r(x)$ is a homology n-manifold with boundary $\partial B_r(x) = S_r(x)$.
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Definition: A space X is said to be a *codimension one manifold factor* if $X \times \mathbb{R}$ is a topological manifold.

Moore Conjecture: Every resolvable generalized manifold is a codimension one manifold factor.

Remark: Every Busemann G-space is a manifold if and only if small metric spheres are codimension one manifold factors. Equivalently in dimensions $n \geq 5$, every Busemann G-space X is a manifold if and only if X is resolvable and small metric spheres S in X satisfy the property that $S \times \mathbb{R}$ has DDP.

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Theorem: Each of the following general position properties of an ANR X characterizes $X \times \mathbb{R}$ having DDP:

- The disjoint arc-disk property (Daverman)
- The disjoint homotopies property (Edwards, Halverson)
- The plentiful 2-manifolds property (Halverson)
- The method of δ-fractured maps (Halverson)
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Facts:

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- Bryant-Ferry-Mio-Weinberger Conjecture $\Rightarrow$ The failure of Bing-Borsuk Conjecture
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Questions:

- Do all Busemann G-spaces have DDP (or equivalently, do all small metric spheres S in X have the property that $S \times \mathbb{R}$ has DDP)?
- Are all Busemann G-spaces resolvable?
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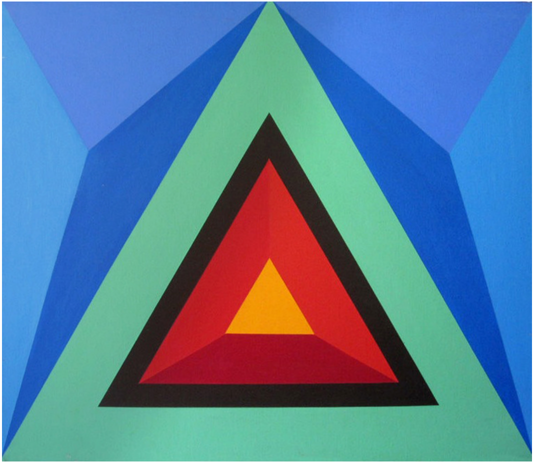
Recent Publications:

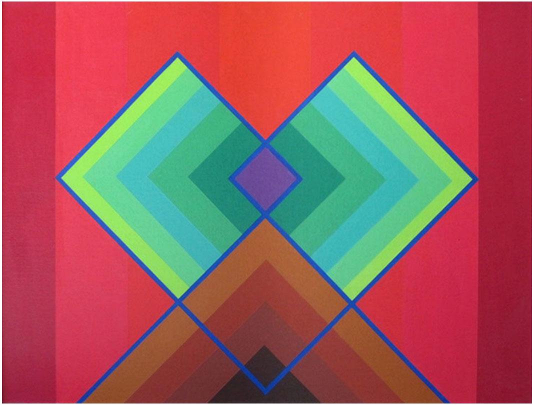
- D. Halverson and D. Repovš, *The Bing-Borsuk and the Busemann Conjectures*, Math. Comm. **13**:2 (2008), 163-184.
- D. M. Halverson and D. Repovš, *Detecting codimension one manifold factors with topographical techniques*, Topology Appl. **156**:17 (2009), 2870–2880.
- V. Berestovskiĭ, D. M. Halverson and D. Repovš, *Locally G-homogeneous Busemann G-spaces*, Diff. Geometry Appl. **29**:3 (2011), 299–318.

We conclude this talk with some more paintings by Herbert Busemann.

Untitled (circa 1970)



Pyramid (1976)



Squares on Red (1977)