Local Audibility of a Hyperbolic Metric

Vladimir Sharafutdinov
Sobolev Institute of Mathematics

A. D. Alexandrov 100
EIMI August 2012

Outline

1. Intro: Can one hear the shape of a drum?
2. Examples of isospectral manifolds
3. Finiteness and compactness results. Finiteness conjecture
4. Local audibility
5. Solenoidal tensor fields
6. Eigenvalue spectrum and Length spectrum
7. A compactness estimate implies the local audibility
8. Heat invariants
9. Derivation of the compactness estimate from heat invariants
10. Why cannot our approach be applied in the case of nonconstant negative curvature?

1. Introduction

(M, g) is a closed Riemannian manifold,

$$\Delta_g = -g^{ij}\nabla_i\nabla_j : C^\infty(M) \longrightarrow C^\infty(M)$$

$$\mathrm{Sp}\,(\Delta_g) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \to +\infty\}$$

To which extent are the geometry and topology of a Riemannian manifold determined by the eigenvalue spectrum of its Laplacian?
M. Kac [1966]: Can one hear the shape of a drum?
Osgood–Philips–Sarnak [1988]: a Riemannian manifold is said to be audible, if it is determined by its spectrum uniquely up to an isometry. H. Weyl [1911]:

$$\lambda_k \sim c_n \left(\frac{k}{\mathrm{Vol}\,(M)}\right)^{2/n} \quad \text{as } k \to \infty \quad (n = \dim M)$$

1. Introduction

(M, g) is a closed Riemannian manifold,

$$\Delta_g = -g^{ij}\nabla_i\nabla_j : C^\infty(M) \longrightarrow C^\infty(M)$$

$$\mathrm{Sp}\,(\Delta_g) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty\}$$

To which extent are the geometry and topology of a Riemannian manifold determined by the eigenvalue spectrum of its Laplacian?

M. Kac [1966]: Can one hear the shape of a drum?

Osgood–Philips–Sarnak [1988]: a Riemannian manifold is said to be audible, if it is determined by its spectrum uniquely up to an isometry. H. Weyl [1911]:

$$\lambda_k \sim c_n \left(\frac{k}{\mathrm{Vol}\,(M)}\right)^{2/n} \quad \text{as } k \to \infty \quad (n = \dim M)$$

1. Introduction

(M, g) is a closed Riemannian manifold,

$$\Delta_g = -g^{ij}\nabla_i\nabla_j : C^\infty(M) \longrightarrow C^\infty(M)$$

$$\mathrm{Sp}\,(\Delta_g) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty\}$$

To which extent are the geometry and topology of a Riemannian manifold determined by the eigenvalue spectrum of its Laplacian?

M. Kac [1966]: Can one hear the shape of a drum?

Osgood–Philips–Sarnak [1988]: a Riemannian manifold is said to be audible, if it is determined by its spectrum uniquely up to an isometry. H. Weyl [1911]:

$$\lambda_k \sim c_n \left(\frac{k}{\mathrm{Vol}\,(M)}\right)^{2/n} \quad \text{as } k \to \infty \quad (n = \dim M)$$

1. Introduction

(M, g) is a closed Riemannian manifold,

$$\Delta_g = -g^{ij}\nabla_i\nabla_j : C^\infty(M) \longrightarrow C^\infty(M)$$

$$\mathrm{Sp}\,(\Delta_g) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \to +\infty\}$$

To which extent are the geometry and topology of a Riemannian manifold determined by the eigenvalue spectrum of its Laplacian?

M. Kac [1966]: Can one hear the shape of a drum?

Osgood–Philips–Sarnak [1988]: a Riemannian manifold is said to be audible, if it is determined by its spectrum uniquely up to an isometry. H. Weyl [1911]:

$$\lambda_k \sim c_n \left(\frac{k}{\mathrm{Vol}\,(M)}\right)^{2/n} \quad \text{as } k \to \infty \quad (n = \dim M)$$

2. Examples of isospectral manifolds

Milnor [1964]: There exist two flat tori in the dimension 16 which are isospectral but not isometric.

Vignaras [1980], Sunada [1985]: Such examples exist in any dimension even within the class of hyperbolic manifolds (i.e., Riemannian manifolds of constant negative sectional curvature).

Gordon–Wilson [1984]: There exist examples of smooth families of isospectral but not isometric metrics.

Croke–Sh [1998]: But such examples are impossible within the class of negatively curved metrics.

3. Compactness results. Finiteness conjecture

McKean [1974]: Within the class of hyperbolic surfaces (i.e., two-dimensional Riemannian manifolds whose Gaussian curvature equals to -1), every isospectral family is finite if isometric surfaces are identified.

Osgood–Philips–Sarnak [1988]: Every isospectral set of metrics on a two-dimensional manifold is precompact in the C^∞-topology if isometric metrics are identified.
A similar compactness theorem for negatively curved 3-manifolds is proved by Brooks–Perry–Petersen [1992].

Conjecture

Every isospectral family of metrics of negative Gaussian curvature on a compact orientable surface of genius ≥ 2 is finite if isometric metrics are identified.

4. Local audibility

In virtue of the above-mentioned compactness theorem, the conjecture is equivalent to some local uniqueness statement. In connection with this, we introduce the following

Definition

A Riemannian manifold (M, g) is said to be locally audible if there exists a neighborhood V of the metric g in the C^∞-topology such that every metric belonging to V and isospectral to g is isometric to g.

The conjecture is equivalent to the local audibility of a two-dimensional manifold of negative Gaussian curvature. To our opinion, the question on the local audibility of some Riemannian metric is of independent interest regardless to the conjecture. The main result of the present article is the following

Theorem

A locally symmetric Riemannian manifold of negative sectional curvature is locally audible.

4. Local audibility

In virtue of the above-mentioned compactness theorem, the conjecture is equivalent to some local uniqueness statement. In connection with this, we introduce the following

Definition

A Riemannian manifold (M, g) is said to be locally audible if there exists a neighborhood V of the metric g in the C^∞-topology such that every metric belonging to V and isospectral to g is isometric to g.

The conjecture is equivalent to the local audibility of a two-dimensional manifold of negative Gaussian curvature. To our opinion, the question on the local audibility of some Riemannian metric is of independent interest regardless to the conjecture. The main result of the present article is the following

Theorem

A locally symmetric Riemannian manifold of negative sectional curvature is locally audible.

4. Local audibility

In virtue of the above-mentioned compactness theorem, the conjecture is equivalent to some local uniqueness statement. In connection with this, we introduce the following

Definition

A Riemannian manifold (M, g) is said to be locally audible if there exists a neighborhood V of the metric g in the C^∞-topology such that every metric belonging to V and isospectral to g is isometric to g.

The conjecture is equivalent to the local audibility of a two-dimensional manifold of negative Gaussian curvature. To our opinion, the question on the local audibility of some Riemannian metric is of independent interest regardless to the conjecture. The main result of the present article is the following

Theorem

A locally symmetric Riemannian manifold of negative sectional curvature is locally audible.

5. Solenoidal tensor fields

Investigating the local audibility of a metric g, we have first of all to eliminate metrics that are isometric and close to g but do not coincide with g.

Given a Riemannian manifold (M, g), let $C^\infty(S^2\tau'_M)$ be the space of smooth symmetric rank two covariant tensor fields on M. The divergence $\delta_g : C^\infty(S^2\tau'_M) \to C^\infty(\tau'_M)$ is defined in coordinates by the equality $(\delta_g f)_i = g^{jk}\nabla_j f_{ik}$, where ∇ is the covariant derivative of the metric g. A tensor field f is said to be solenoidal if $\delta_g f = 0$. The above-mentioned elimination of "unnecessary" metrics is implemented with the help of the following (Croke–Dairbekov–Sh [2000])

Lemma

Let (M, g) be of negative curvature. For every $k \geq 2$ and $0 < \alpha < 1$, there exists a neighborhood $V \subset C^{k,\alpha}(S^2\tau'_M)$ of the metric g such that, for every metric $g' \in V$, there exists a diffeomorphism φ of the manifold M onto itself such that the tensor field $\varphi^* g'$ is solenoidal in the metric g, i.e., $\delta_g(\varphi^* g') = 0$.

5. Solenoidal tensor fields

Investigating the local audibility of a metric g, we have first of all to eliminate metrics that are isometric and close to g but do not coincide with g.
Given a Riemannian manifold (M, g), let $C^\infty(S^2\tau'_M)$ be the space of smooth symmetric rank two covariant tensor fields on M. The divergence $\delta_g : C^\infty(S^2\tau'_M) \to C^\infty(\tau'_M)$ is defined in coordinates by the equality $(\delta_g f)_i = g^{jk}\nabla_j f_{ik}$, where ∇ is the covariant derivative of the metric g. A tensor field f is said to be solenoidal if $\delta_g f = 0$. The above-mentioned elimination of “unnecessary” metrics is implemented with the help of the following (Croke–Dairbekov–Sh [2000])

Lemma

Let (M, g) be of negative curvature. For every $k \geq 2$ and $0 < \alpha < 1$, there exists a neighborhood $V \subset C^{k,\alpha}(S^2\tau'_M)$ of the metric g such that, for every metric $g' \in V$, there exists a diffeomorphism φ of the manifold M onto itself such that the tensor field $\varphi^* g'$ is solenoidal in the metric g, i.e., $\delta_g(\varphi^* g') = 0$.

5. Solenoidal tensor fields

Investigating the local audibility of a metric g, we have first of all to eliminate metrics that are isometric and close to g but do not coincide with g.

Given a Riemannian manifold (M, g), let $C^\infty(S^2\tau'_M)$ be the space of smooth symmetric rank two covariant tensor fields on M. The divergence $\delta_g : C^\infty(S^2\tau'_M) \to C^\infty(\tau'_M)$ is defined in coordinates by the equality $(\delta_g f)_i = g^{jk}\nabla_j f_{ik}$, where ∇ is the covariant derivative of the metric g. A tensor field f is said to be solenoidal if $\delta_g f = 0$. The above-mentioned elimination of "unnecessary" metrics is implemented with the help of the following (Croke–Dairbekov–Sh [2000])

Lemma

Let (M, g) be of negative curvature. For every $k \geq 2$ and $0 < \alpha < 1$, there exists a neighborhood $V \subset C^{k,\alpha}(S^2\tau'_M)$ of the metric g such that, for every metric $g' \in V$, there exists a diffeomorphism φ of the manifold M onto itself such that the tensor field $\varphi^* g'$ is solenoidal in the metric g, i.e., $\delta_g(\varphi^* g') = 0$.

In virtue of the lemma, the definition of local audibility takes the following equivalent form.

Proposition

A negatively curved manifold (M, g) *is locally audible if and only if the following statement is true. If* g_k $(k = 1, 2, \dots)$ *is a sequence of metrics on* M *converging to* g *in the* C^∞*-topology and such that every* g_k *is isospectral to* g *and satisfies* $\delta_g g_k = 0$ *then* $g_k = g$ *starting with some* k_0*.*

The latter statement is proved for negatively curved metrics if the sequence $g_k \to g$ is replaced with with a smooth one-parameter family g_t $(-\varepsilon < t < \varepsilon,\ g_0 = g)$.

In virtue of the lemma, the definition of local audibility takes the following equivalent form.

Proposition

A negatively curved manifold (M, g) *is locally audible if and only if the following statement is true. If* g_k $(k = 1, 2, \dots)$ *is a sequence of metrics on* M *converging to* g *in the* C^∞*-topology and such that every* g_k *is isospectral to* g *and satisfies* $\delta_g g_k = 0$ *then* $g_k = g$ *starting with some* k_0.

The latter statement is proved for negatively curved metrics if the sequence $g_k \to g$ is replaced with with a smooth one-parameter family g_t $(-\varepsilon < t < \varepsilon,\ g_0 = g)$.

6. Eigenvalue spectrum and Length spectrum

$$\mathrm{Sp}\,(\Delta_g) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \to +\infty\}$$

$$k(t) = \sum_{k=0}^{\infty} \cos(\sqrt{\lambda_k} t) \in \mathcal{S}'(\mathbb{R}) \text{ is the wave kernel}$$

Chazarain [1974], Duistermaat–Guillemin [1975]:

$$\operatorname{singsupp} k \subseteq (LSp(M,g)) \cup \{0\} \cup (-LSp(M,g)),$$

and the equality holds for a negatively curved manifold.

$LSp(M,g) = \{\text{lengths of closed geodesics}\}$ is the length spectrum.

Guillemin–Kazhdan [1980]: If two negatively curved manifolds have coincident eigenvalue spectra, then their length spectra coincide too.

6. Eigenvalue spectrum and Length spectrum

$$\mathrm{Sp}\,(\Delta_g) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \to +\infty\}$$

$$k(t) = \sum_{k=0}^{\infty} \cos(\sqrt{\lambda_k} t) \in \mathcal{S}'(\mathbb{R}) \text{ is the wave kernel}$$

Chazarain [1974], Duistermaat–Guillemin [1975]:

$$\mathrm{singsupp}\, k \subseteq (LSp(M,g)) \cup \{0\} \cup (-LSp(M,g)),$$

and the equality holds for a negatively curved manifold.

$LSp(M,g) = \{\text{lengths of closed geodesics}\}$ is the length spectrum.

Guillemin–Kazhdan [1980]: If two negatively curved manifolds have coincident eigenvalue spectra, then their length spectra coincide too.

6. Eigenvalue spectrum and Length spectrum

$$\mathrm{Sp}\,(\Delta_g) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \to +\infty\}$$

$$k(t) = \sum_{k=0}^{\infty} \cos(\sqrt{\lambda_k}\, t) \in \mathcal{S}'(\mathbb{R}) \text{ is the wave kernel}$$

Chazarain [1974], Duistermaat–Guillemin [1975]:

$$\mathrm{singsupp}\, k \subseteq (LSp(M,g)) \cup \{0\} \cup (-LSp(M,g)),$$

and the equality holds for a negatively curved manifold.

$LSp(M,g) = \{\text{lengths of closed geodesics}\}$ is the length spectrum.

Guillemin–Kazhdan [1980]: If two negatively curved manifolds have coincident eigenvalue spectra, then their length spectra coincide too.

6. Eigenvalue spectrum and Length spectrum

$$\mathrm{Sp}\,(\Delta_g) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \to +\infty\}$$

$$k(t) = \sum_{k=0}^{\infty} \cos(\sqrt{\lambda_k}\, t) \in \mathcal{S}'(\mathbb{R}) \text{ is the wave kernel}$$

Chazarain [1974], Duistermaat–Guillemin [1975]:

$$\mathrm{singsupp}\, k \subseteq (LSp(M,g)) \cup \{0\} \cup (-LSp(M,g)),$$

and the equality holds for a negatively curved manifold.

$LSp(M,g) = \{\text{lengths of closed geodesics}\}$ is the length spectrum.

Guillemin–Kazhdan [1980]: If two negatively curved manifolds have coincident eigenvalue spectra, then their length spectra coincide too.

Croke–Sh [1998]: If a solenoidal symmetric tensor field $f = (f_{ij})$ on a negatively curved manifold integrates to zero over every closed geodesic, then $f \equiv 0$.

$$\oint_\gamma f = \oint_\gamma f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)\,dt$$

7. A compactness estimate implies the local audibility

Let (M, g) be a Riemannian manifold of negative sectional curvature and g_m $(m = 1, 2, \dots)$ be a sequence of Riemannian metrics on M converging to g in the C^∞-topology. Assume every g_m to be isospectral to the metric g and solenoidal, i.e., $\delta_g g_m = 0$. In virtue of the proposition, we have to prove that g_m coincides with g starting with some m_0. We assume this false and try to get a contradiction. Passing to a subsequence, we can assume the tensor field $f_m = g_m - g$ to be not identically equal to zero for every n. Let γ be a closed geodesic of the metric g and γ_m be the closed geodesic of the metric g_m in the same free homotopy class as γ. Then γ_m converges uniformly to γ as $n \to \infty$ since there is a unique closed geodesic in a free homotopy class on a negatively curved manifold.

Since γ_m minimizes the energy functional E_{g_m} in its homotopy class, we can write

$$\oint_\gamma f_m = \oint_\gamma (g_m - g) = E_{g_m}(\gamma) - E_g(\gamma) \geq E_{g_m}(\gamma_m) - E_g(\gamma) = 0.$$

The last equality of the chain holds for a sufficiently large m since the metrics g_m and g have coincident length spectra. Thus, for every closed geodesic γ of the metric g,

$$\oint_\gamma f_m \geq 0 \quad \text{for} \quad m > m_0(\gamma).$$

Swopping the roles of g and g_m, we infer also that

$$\oint_{\gamma_m} f_m \leq 0. \quad \text{for} \quad m > m_0(\gamma)$$

We normalize the tensor field f_m by setting $F_m = f_m / \|f_m\|_{H^k}$ with an appropriately chosen k. The inequalities hold for F_m as well.

Since γ_m minimizes the energy functional E_{g_m} in its homotopy class, we can write

$$\oint_\gamma f_m = \oint_\gamma (g_m - g) = E_{g_m}(\gamma) - E_g(\gamma) \geq E_{g_m}(\gamma_m) \; - \; E_g(\gamma) = 0.$$

The last equality of the chain holds for a sufficiently large m since the metrics g_m and g have coincident length spectra. Thus, for every closed geodesic γ of the metric g,

$$\oint_\gamma f_m \geq 0 \quad \text{for} \quad m > m_0(\gamma).$$

Swopping the roles of g and g_m, we infer also that

$$\oint_{\gamma_m} f_m \leq 0. \quad \text{for} \quad m > m_0(\gamma)$$

We normalize the tensor field f_m by setting $F_m = f_m / \|f_m\|_{H^k}$ with an appropriately chosen k. The inequalities hold for F_m as well.

Since γ_m minimizes the energy functional E_{g_m} in its homotopy class, we can write

$$\oint_\gamma f_m = \oint_\gamma (g_m - g) = E_{g_m}(\gamma) - E_g(\gamma) \geq E_{g_m}(\gamma_m) \; - \; E_g(\gamma) = 0.$$

The last equality of the chain holds for a sufficiently large m since the metrics g_m and g have coincident length spectra. Thus, for every closed geodesic γ of the metric g,

$$\oint_\gamma f_m \geq 0 \quad \text{for} \quad m > m_0(\gamma).$$

Swopping the roles of g and g_m, we infer also that

$$\oint_{\gamma_m} f_m \leq 0. \quad \text{for} \quad m > m_0(\gamma)$$

We normalize the tensor field f_m by setting $F_m = f_m / \|f_m\|_{H^k}$ with an appropriately chosen k. The inequalities hold for F_m as well

$$\oint\limits_{\gamma} F_m \geq 0, \quad \oint\limits_{\gamma_m} F_m \leq 0 \quad \text{for } m \geq m_0(\gamma).$$

Assume for a moment the sequence F_m to converge in the H^k-norm: $\|F_m - F\|_{H^k} \to 0$ as $n \to \infty$. Passing to the limit, we have

$$\oint\limits_{\gamma} F = 0$$

for every closed geodesic γ of the metric g. Of course, F is a solenoidal tensor field. By the previous theorem, $F \equiv 0$. This contradicts to the equality $\|F\|_{H^k} = 1$.

$$\oint\limits_{\gamma} F_m \geq 0, \quad \oint\limits_{\gamma_m} F_m \leq 0 \quad \text{for } m \geq m_0(\gamma).$$

Assume for a moment the sequence F_m to converge in the H^k-norm: $\|F_m - F\|_{H^k} \to 0$ as $n \to \infty$. Passing to the limit, we have

$$\oint\limits_{\gamma} F = 0$$

for every closed geodesic γ of the metric g. Of course, F is a solenoidal tensor field. By the previous theorem, $F \equiv 0$. This contradicts to the equality $\|F\|_{H^k} = 1$.

The problem is thus reduced to the question: does the sequence F_m contain a subsequence converging in H^k? Since the embedding $H^{k+1} \subset H^k$ is compact, it suffices to prove the boundedness of the sequence F_m in the H^{k+1}-norm,

$$\|F_m\|_{H^{k+1}} \leq C.$$

This means in terms of the sequence g_m that

$$\|g_m - g\|_{H^{k+1}} \leq C\|g_m - g\|_{H^k}.$$

Similar compactness estimates appeared already in spectral geometry. Such an estimate (for $k = 0$) serves as a base for main results of Sh–Uhlmann [2000] and Dairbekov–Sh [2003] that are devoted to the spectral rigidity of Riemannian manifolds with the geodesic flow of Anosov type.

The problem is thus reduced to the question: does the sequence F_m contain a subsequence converging in H^k? Since the embedding $H^{k+1} \subset H^k$ is compact, it suffices to prove the boundedness of the sequence F_m in the H^{k+1}-norm,

$$\|F_m\|_{H^{k+1}} \leq C.$$

This means in terms of the sequence g_m that

$$\|g_m - g\|_{H^{k+1}} \leq C\|g_m - g\|_{H^k}.$$

Similar compactness estimates appeared already in spectral geometry. Such an estimate (for $k = 0$) serves as a base for main results of Sh–Uhlmann [2000] and Dairbekov–Sh [2003] that are devoted to the spectral rigidity of Riemannian manifolds with the geodesic flow of Anosov type.

8. Heat invariants

The fundamental solution $K(x, y, t)$ to the heat equation on an n-dimensional Riemannian manifold (M, g) admits the asymptotic representation

$$K(x, x, t) \sim (4\pi t)^{-n/2}(a_0(x) + a_1(x)t + a_2(x)t^2 + \dots)$$

as $t \to +0$, where the coefficients are determined by the local geometry of the manifold in a neighborhood of the point x. More precisely, every function $a_k(x)$ is an invariant polynomial in the curvature tensor and its covariant derivatives. The integrals of the coefficients $a_k(M, g) = \int_M a_k(x)\, dV(x)$ are called heat invariants. They are determined by the spectrum of the Laplacian:

$$\operatorname{tr}_{L^2} e^{-t\Delta_g} = \sum_{k=0}^{\infty} e^{-\lambda_k t}$$
$$\sim (4\pi t)^{-n/2}\Big(a_0(M, g) + a_1(M, g)t + a_2(M, g)t^2 + \dots\Big).$$

The first three invariants are as follows:

$$a_0(M,g) = \mathrm{Vol}(M,g), \quad a_1(M,g) = \frac{1}{6}\int_M S\,dV,$$

$$a_2(M,g) = \frac{1}{360}\int_M (5S^2 - 2|Ric|^2 + 2|R|^2)\,dV,$$

where R, Ric, and S are the curvature tensor, Ricci tensor, and scalar curvature respectively.

Gilkey [1989]: for $k \geq 2$

$$a_{k+1}(M,g) = \int\limits_M \Big(c_k|\nabla^{(k-1)}S|^2 + c_k'|\nabla^{(k-1)}Ric|^2 + P_k(\nabla, R)\Big)dV.$$

with positive c_k and c_k', where $P_k(\nabla, R)$ is some invariant polynomial in the variables $\nabla^{(l)}R$ ($l \leq k-2$). It is a homogeneous polynomial of degree $2k+2$ if the degree of homogeneity of $\nabla^{(l)}R$ is assumed to be equal to $l+2$. This is a universal polynomial, i.e., independent of the manifold.

The first three invariants are as follows:

$$a_0(M,g) = \mathrm{Vol}(M,g), \quad a_1(M,g) = \frac{1}{6}\int_M S\,dV,$$

$$a_2(M,g) = \frac{1}{360}\int_M (5S^2 - 2|Ric|^2 + 2|R|^2)\,dV,$$

where R, Ric, and S are the curvature tensor, Ricci tensor, and scalar curvature respectively.
Gilkey [1989]: for $k \geq 2$

$$a_{k+1}(M,g) = \int\limits_M \Big(c_k |\nabla^{(k-1)} S|^2 + c_k' |\nabla^{(k-1)} Ric|^2 + P_k(\nabla, R) \Big) dV.$$

with positive c_k and c_k', where $P_k(\nabla, R)$ is some invariant polynomial in the variables $\nabla^{(l)} R$ ($l \leq k-2$). It is a homogeneous polynomial of degree $2k+2$ if the degree of homogeneity of $\nabla^{(l)} R$ is assumed to be equal to $l+2$. This is a universal polynomial, i.e., independent of the manifold.

9. Derivation of the compactness estimate

Let (M, g) be a Riemannian manifold and let $f \in C^\infty(S^2\tau'_M)$ be a sufficiently small solenoidal tensor field. Assume the metrics g and $g + f$ to be isospectral.

First of all, $\mathrm{Vol}(M, g + f) = \mathrm{Vol}(M, g)$. This gives

$$\left| \int_M \mathrm{tr}\, f\, dV \right| \leq C \|f\|_{L^2}^2.$$

Next, we equate the values of the heat invariant

$$a_{k+1}(M, g + f) - a_{k+1}(M, g) = 0.$$

We expand the left-hand side of the equation into Tailor series in f and obtain

$$\|f\|_{H^{k+1}}^2 = \int_M Q_k(\nabla, R, f)\, dV,$$

where $Q_k(\nabla, R, f)$ is a power series in $\nabla^{(l)} f$ $(l \leq k)$ with coefficients depending on $\nabla^{(l)} R$.

9. Derivation of the compactness estimate

Let (M, g) be a Riemannian manifold and let $f \in C^\infty(S^2\tau'_M)$ be a sufficiently small solenoidal tensor field. Assume the metrics g and $g + f$ to be isospectral.

First of all, $\mathrm{Vol}(M, g + f) = \mathrm{Vol}(M, g)$. This gives

$$\left| \int_M \mathrm{tr}\, f \, dV \right| \leq C \|f\|^2_{L^2}.$$

Next, we equate the values of the heat invariant

$$a_{k+1}(M, g + f) - a_{k+1}(M, g) = 0.$$

We expand the left-hand side of the equation into Tailor series in f and obtain

$$\|f\|^2_{H^{k+1}} = \int_M Q_k(\nabla, R, f)\, dV,$$

where $Q_k(\nabla, R, f)$ is a power series in $\nabla^{(l)} f$ $(l \leq k)$ with coefficients depending on $\nabla^{(l)} R$.

Let us distinguish linear terms of $Q_k(\nabla, R, f)$

$$Q_k(\nabla, R, f) = Q'_k(\nabla, R, f) + L_k(\nabla, R, f).$$

Since $Q'_k(\nabla, R, f)$ does not contain linear terms it admits the estimate

$$\left| \int_M Q'_k(\nabla, R, f)\, dV \right| \leq C \|f\|^2_{H^k}.$$

The main difficulty is about estimating the integral

$$\int_M L_k(\nabla, R, f)\, dV$$

of linear terms.

Let us distinguish linear terms of $Q_k(\nabla, R, f)$

$$Q_k(\nabla, R, f) = Q'_k(\nabla, R, f) + L_k(\nabla, R, f).$$

Since $Q'_k(\nabla, R, f)$ does not contain linear terms it admits the estimate

$$\left| \int_M Q'_k(\nabla, R, f)\, dV \right| \leq C\|f\|^2_{H^k}.$$

The main difficulty is about estimating the integral

$$\int_M L_k(\nabla, R, f)\, dV$$

of linear terms.

Assume g to be a metric of constant sectional curvature K_0. Since $R_{ijkl} = K_0(g_{ik}g_{jl} - g_{il}g_{jk})$, the linear form

$$L_k(\nabla, R, f) = L_k(\nabla, f)$$

consists of summands that are obtained from derivatives

$$\nabla_{l_1 \dots l_{2m}} f_{ij} \quad (2m \le k)$$

by raising a half of indices with the help of the tensor g^{ij} followed by the contraction in all indices grouped in pairs. For $m > 0$, every such summand has obviously a divergent form and gives the zero contribution into the integral. For $m = 0$, we have the unique summand $\operatorname{tr} f$ that has been already estimated by $\|f\|^2_{L^2}$. We finally obtain

$$\left| \int_M L_k(\nabla, R, f)\, dV \right| \le C\|f\|^2_{L^2}.$$

Combining our inequalities, we obtain the compactness estimate

$$\|f\|^2_{H^{k+1}} \le C\|f\|^2_{H^k}.$$

Lemma

Let (M, g) be a Riemannian manifold of constant sectional curvature. For every $k \geq 2$, there exists a C^{k+1}-neighborhood V of the metric g such that the compactness estimate

$$\|g' - g\|_{H^{k+1}} \leq C\|g' - g\|_{H^k}$$

holds for every $g' \in V$ satisfying the conditions $\delta_g g' = 0$, $\mathrm{Vol}(M, g) = \mathrm{Vol}(M, g')$ and $a_{k+1}(M, g) = a_{k+1}(M, g')$.

Theorem

A Riemannian manifold of constant negative sectional curvature is locally audible.

Lemma

Let (M, g) be a Riemannian manifold of constant sectional curvature. For every $k \geq 2$, there exists a C^{k+1}-neighborhood V of the metric g such that the compactness estimate

$$\|g' - g\|_{H^{k+1}} \leq C\|g' - g\|_{H^k}$$

holds for every $g' \in V$ satisfying the conditions $\delta_g g' = 0$, $\mathrm{Vol}(M, g) = \mathrm{Vol}(M, g')$ and $a_{k+1}(M, g) = a_{k+1}(M, g')$.

Theorem

A Riemannian manifold of constant negative sectional curvature is locally audible.

10. Why our approach does not work in the case of nonconstant negative curvature?

Given a Riemannian manifold (M, g) and tensor field $f \in C^\infty(S^2\tau'_M)$, the first variation of the spectral invariant a_k in the direction f is defined by

$$\dot{a}_k(M, g; f) = da_k(M, g + tf)/dt|_{t=0}.$$

If the metric g has constant sectional curvature then $\dot{a}_k(M, g; f) = 0$ for every f satisfying $2\dot{a}_0(M, g; f) = \int_M \operatorname{tr} f \, dV = 0$. This is a crucial fact for our approach. Indeed, in this case the difference

$$a_k(M, g + f) - a_k(M, g)$$

can be represented as a power series in f which does not contain linear terms. Leading terms of the series are quadratic in f that allows us to derive the compactness estimate. The presence of linear in f terms stands as a stumbling block for our approach.

We see only one opportunity to fight with linear terms of the series: the use of a linear combination of several invariants. Indeed, if a linear combination

$$a(M,g) = c_0 a_0(M,g) + \cdots + c_k a_k(M,g) \quad (c_k \neq 0)$$

with appropriately chosen constant coefficients turned out to have the zero first variation then we would be able to use the difference $a(M,g+f) - a(M,g)$. Unfortunately, a general metric g seems to have no linear combination satisfying $\dot{a}(M,g;f) = 0$ for every f. But if the curvature tensor of the metric g satisfies some natural differential equation then such linear combinations probably may be found. In such a way, an opportunity arises for proving the local audibility of metrics belonging to some natural classes that are wider than the class of hyperbolic metrics.

We see only one opportunity to fight with linear terms of the series: the use of a linear combination of several invariants. Indeed, if a linear combination

$$a(M,g) = c_0 a_0(M,g) + \cdots + c_k a_k(M,g) \quad (c_k \neq 0)$$

with appropriately chosen constant coefficients turned out to have the zero first variation then we would be able to use the difference $a(M,g+f) - a(M,g)$. Unfortunately, a general metric g seems to have no linear combination satisfying $\dot{a}(M,g;f) = 0$ for every f. But if the curvature tensor of the metric g satisfies some natural differential equation then such linear combinations probably may be found. In such a way, an opportunity arises for proving the local audibility of metrics belonging to some natural classes that are wider than the class of hyperbolic metrics.

The simplest of such equations is $\nabla R = 0$ (locally symmetric metrics). In this case

$$\dot{a}_k(M, g; f) = \int\limits_M L_k(f)\, dV,$$

where L_k $(k = 0, 1, \dots)$ are linear functionals on $C^\infty(S^2\tau'_M)$ with "constant" coefficients. These functionals live in a finite-dimensional space and they cannot be linearly independent.

Theorem

A locally symmetric Riemannian manifold of negative sectional curvature is locally audible.