Concentration, Laplacian, and Ricci curvature

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Concentration topology

An mm-space is a complete separable metric space (X, d_X) with a Borel probability measure μ_X.

Gromov's concentration topology

For two mm-spaces X and Y,
$d_{conc}(X, Y) :=$ "difference between
1-Lip functions on X and those on Y"

$\{X_n\}_{n=1}^{\infty}$ concentrates to X, $X_n \overset{conc}{\to} X$
$\overset{def}{\Longleftrightarrow} d_{conc}(X_n, X) \to 0$ as $n \to \infty$

- $X_n \overset{conc}{\to} (\{p\}, \delta_p) \Longleftrightarrow$ For $\forall$ 1-Lip $f_n : X_n \to \mathbf{R}$, $\exists\, c_n \in \mathbf{R}$ s.t.

$$\lim_{n\to\infty} \mu_{X_n}(|f_n - c_n| \geq \varepsilon) = 0, \quad \forall \varepsilon > 0.$$

"Any 1-Lip function on X_n is close to a constant for large n."

Examples

(1) $S^n \stackrel{conc}{\to} (\{p\}, \delta_p)$ (P. Lévy) (2) $S^n \times M \stackrel{conc}{\to} M$

- The concentration topology is weaker than the measured Gromov-Hausdorff topology.
- The concentration topology is useful to study manifolds M_n with $\dim M_n \to \infty$.

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- The concentration topology is useful to study manifolds M_n with $\dim M_n \to \infty$.

Main Thm A

Let X and X_n, $n = 1, 2, \dots$, be mm-spaces s.t. X is proper. If X_n satisfies $\mathrm{CD}(K, \infty)$ and if $X_n \overset{conc}{\to} X$, then X satisfies $\mathrm{CD}(K, \infty)$.

$\mathrm{CD}(K, \infty)$ is a generalization of $\mathrm{Ric} \geq K$ and is defined by Lott-Villani-Sturm using optimal mass-transport.

Idea of Proof of Main Thm A

A key point is to establish the correspondence between the L^2 Wasserstein space on X_n and that on X.
There are almost 1-Lip maps $f_n : X_n \to X$, $n = 1, 2, \dots$.
A difficult point is that the fibers may be large.
Take two points $x_0, x_1 \in X$ and small $\varepsilon > 0$.

$$\nu_0 := \frac{\mu_{X_n}|_{f_n^{-1}(B_\varepsilon(x_0))}}{\mu_{X_n}(f_n^{-1}(B_\varepsilon(x_0)))} \quad \text{and} \quad \nu_1 := \frac{\mu_{X_n}|_{f_n^{-1}(B_\varepsilon(x_1))}}{\mu_{X_n}(f_n^{-1}(B_\varepsilon(x_1)))},$$

To get the correspondence, it suffices to prove

$$W_2(\nu_0, \nu_1) \doteqdot d_X(x_0, x_1).$$

$W_2(\nu_0, \nu_1) \ge d_X(x_0, x_1) - \delta$ is easy. To get the opposite estimate, we use the Kantorovich-Rubinstein duality:

$$W_1(\nu_0, \nu_1) = \sup \left\{ \int_{X_n} \varphi \, d\nu_0 - \int_{X_n} \varphi \, d\nu_1 \mid \varphi : X_n \to \mathbf{R} \ \ 1-\text{Lip} \right\}.$$

Eigenvalues of Laplacian and concentration

Let M and $M_n, n = 1, 2, \ldots$ be connected and closed Riemannian manifolds.

Known Results.

- If $\lambda_1(M_n) \to +\infty$, then $M_n \overset{conc}{\to} \{p\}$ (Gromov-V. Milman).
- If $M_n \overset{conc}{\to} \{p\}$ and if $\mathbf{Ric}_{M_n} \geq 0$, then $\lambda_1(M_n) \to +\infty$ (E. Milman).

Under $\mathbf{Ric}_{M_n} \geq 0$, we have

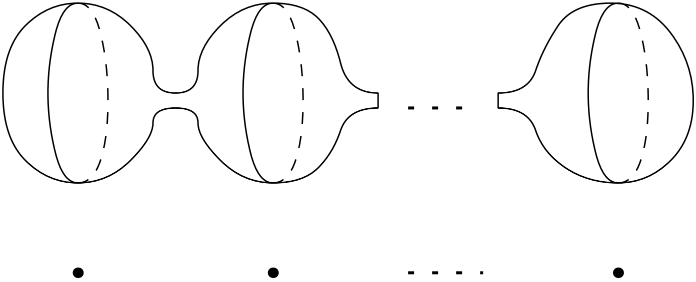
$$M_n \overset{conc}{\to} \{p\} \iff \lambda_1(M_n) \to +\infty.$$

What happens if $\lambda_k(M_n) \to +\infty$ for a number k.

Thm (not precise)

If $\lambda_k(M_n) \to +\infty$ as $n \to \infty$ for a number k,
then $\underset{\exists subseq.}{M_{n_i}} \overset{conc}{\to} {}^{\exists}X$ with $\#X \le k$.

Example: connected sum of k copies of an n-sphere



Thm together with Main Thm A implies

Cor

If $\mathbf{Ric}_{M_n} \geq \mathbf{0}$ and if $\lambda_k(M_n) \to +\infty$ for ${}^{\exists}k$, then $M_n \overset{conc}{\to} \{p\}$.

- Even the connectivity of X is highly nontrivial!

Using the corollary we prove

Main Thm B

For ${}^{\forall}k$, ${}^{\exists}C_k > \mathbf{0}$ s.t. if M is a closed Riem mfd with $\mathbf{Ric}_M \geq \mathbf{0}$, then

$$\lambda_k(M) \leq C_k \lambda_1(M).$$

- C_k is independent of the dimension of M.

Proof of Main Thm B

Main Thm B: $\mathbf{Ric}_M \geq 0 \implies \lambda_k(M) \leq C_k \lambda_1(M)$
Suppose Main Thm B is false.
Then, $^\exists k$, $^\exists\{M_n\}$ s.t. $\mathbf{Ric}_{M_n} \geq 0$ & $\dfrac{\lambda_k(M_n)}{\lambda_1(M_n)} \to +\infty$ as $n \to \infty$.
Let $M_n{}'$ be the scale-change of M_n as $\lambda_1(M_n{}') = 1$. Since

$$\lambda_k(M_n{}') = \frac{\lambda_k(M_n{}')}{\lambda_1(M_n{}')} = \frac{\lambda_k(M_n)}{\lambda_1(M_n)} \to +\infty,$$

Cor implies that $M_n' \overset{conc}{\to} \{p\}$. By E. Milman's thm, this is a contradiction to $\lambda_1(M_n{}') = 1$. □

- Cor: $\mathbf{Ric}_{M_n} \geq 0$, $\lambda_k(M_n) \to +\infty \implies M_n \overset{conc}{\to} \{p\}$.
- E. Milman: $M_n \overset{conc}{\to} \{p\}$, $\mathbf{Ric}_{M_n} \geq 0 \implies \lambda_1(M_n) \to +\infty$.

Concentration of Alexandrov spaces

Thm

Let X_n be finite-dimensional Alexandrov spaces of nonnegative curvature with normalized Hausdorff measure.
If $X_n \xrightarrow{conc} X$ for an mm-space X,
then X is an Alexandrov space of nonnegative curvature.

- X maybe infinite-dimensional.
- If curv of $X_n \geq -1$, then X is not necessarily a length space.

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Thank you very much!

L^2 Wasserstein distance

X : a complete separable metric space.
$P(X)$: the set of Borel probability measures on X.
$P_0(X) := \{ \nu \in P(X) \mid \nu$ has compact support $\}$
$\nu_0, \nu_1 \in P(X)$
π : a transport plan between ν_0 and ν_1
$\overset{def}{\Leftrightarrow}$ π is a Borel measure on $X \times X$ s.t.

$$\pi(A \times X) = \nu_0(A) \quad \text{and} \quad \pi(X \times A) = \nu_1(A)$$

for any Borel subset $A \subset X$.

- $\pi(A \times B)$ means the quantity of the transport from A to B.

The L^2-Wasserstein distance between ν_0 and ν_1

$$W_2(\nu_0, \nu_1) := \left(\inf_{\pi} \int_{X \times X} d_X(x, x')^2 \, d\pi(x, x') \right)^{\frac{1}{2}}.$$

(X, d_X, μ_X) : a metric measure space, $\nu \in P(X)$.

The relative entropy of ν w.r.t. μ_X

$$\mathbf{Ent}(\nu) := \mathbf{Ent}(\nu|\mu_X) := \int_{\{\rho>0\}} \rho \log \rho \, d\mu_X$$

if $\nu = \rho \mu_X$.
$\mathbf{Ent}(\nu) := +\infty$ if ν is not absolutely continuous w.r.t. μ_X.

Condition $\mathbf{CD}(K, \infty)$ for (X, d_X, μ_X)

For ${}^\forall \nu_0, \nu_1 \in P_0(X)$, ${}^\forall \varepsilon > 0$, ${}^\forall t \in (0, 1)$, ${}^\exists \nu_t \in P(X)$ s.t.

$$W_2(\nu_t, \nu_i) \le t^{1-i}(1-t)^i W_2(\nu_0, \nu_1) + \varepsilon \quad \text{for } i = 0, 1,$$

$$\mathbf{Ent}(\nu_t) \le (1-t)\,\mathbf{Ent}(\nu_0) + t\,\mathbf{Ent}(\nu_1) - \frac{K}{2} t(1-t) W_2(\nu_0, \nu_1)^2 + \varepsilon.$$

For a complete Riem mfd, $\mathbf{CD}(K, \infty) \iff \mathbf{Ric} \ge K$.