Right-angled hyperbolic polyhedra

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August 20 – 24, 2012

A.D.Alexandrov: Back to Euclid!

Back to right-angled building blocks!

How to construct closed orientable connected hyperbolic 3-manifolds?

Which properties can be obtained from a construction?

Construction and classification of hyperbolic 3-manifolds

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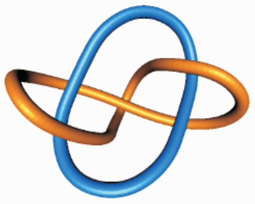

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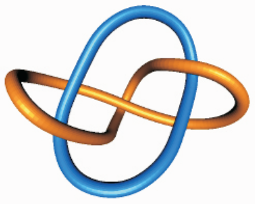

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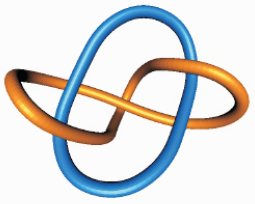

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$2\pi/5$-dodecahedron

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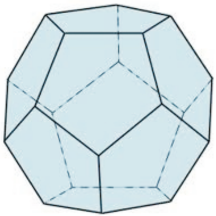

To apply Poincare polyhedral theorem one needs to find such a pairing of faces that edges split in classes with the sum of dihedral angles 2π in each class.

30 edges with $2\pi/5$ will split (if so) in 6 classes.

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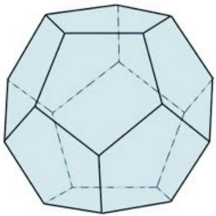

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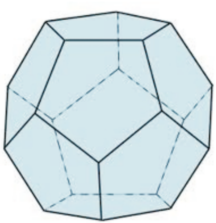

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3-Manifolds form Platonic solids

[Richardson – Rubinstein, 1984], the final list in [Everitt, 2006]

Spherical: M_1^E from $2\pi/3$-tetrahedron;
M_2^E, M_3^E from $2\pi/3$-cube;
M_4^E, M_5^E, M_6^E from $2\pi/3$-octahedron;
M_7^E, M_8^E from $2\pi/3$-dodecahedron;

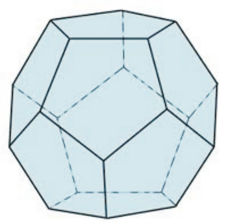
Euclidean: $M_9^E, \dots, M_{14}^E$ from $\pi/2$-cube;

Hyperbolic: $M_{15}^E, \dots, M_{22}^E$ from $2\pi/5$-dodecahedron;
$M_{23}^E, \dots, M_{28}^E$ from $2\pi/3$-icosahedron.

[Cavicchioli - Spaggiari – Telloni, 2009, 2010],
[Kozlovskaya - V., 2011], [Cristofori - Kozlovskaya - V., 2012]:
Covering properties of these manifolds and of their generalizations.

Right-angled dodecahedron

Consider $\pi/2$-dodecahedron in a hyperbolic space $\mathbb{H}^3$.



There are 30 edges with dihedral angles $\pi/2$, so they can not be split in cycles with sum 2π.

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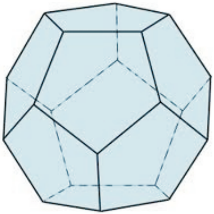

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Right-angled building blocks

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Compact right-angled polyhedra in $\mathbb{H}^n$.

For a polyhedron P let $a_k(P)$ be the number of its k-dimensional faces and

$$a_k^\ell = \frac{1}{a_k} \sum_{\dim F = k} a_\ell(F)$$

be the average number of ℓ-dimensional faces in a k-dimensional polyhedron.

[Nikulin, 1981]

$$a_k^\ell < C_{n-\ell}^{n-k} \frac{C_{[\frac{n}{2}]}^{\ell} + C_{[\frac{n+1}{2}]}^{\ell}}{C_{[\frac{n}{2}]}^{k} + C_{[\frac{n+1}{2}]}^{k}}$$

for $\ell < k \leqslant \left[\frac{n}{2}\right]$.

Existence of right-angled polyhedra in $\mathbb{H}^n$

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for $\ell < k \leqslant \left[\frac{n}{2}\right]$.

In particular, for a_2^1, the average number of sides in a 2-dimensional face, we get:

$$a_2^1 < \begin{cases} \dfrac{4(n-1)}{n-2} & \text{if} \quad n \quad \text{even} \\ \dfrac{4n}{n-1} & \text{if} \quad n \quad \text{odd} \end{cases}$$

But $a_2^1 \geqslant 5$.

Corollary from the Nikulin inequality: There exist no compact right-angled polyhedra in $\mathbb{H}^n$ for $n > 4$.

[Vinberg, 1985] There exist no compact Coxeter polyhedra in $\mathbb{H}^n$ for $n > 29$.

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Existence of finite-volume right-angled polyhedra

Finite-volume right-angled polyhedra in $\mathbb{H}^n$.

[Dufour, 2010] There exist no finite volume right-angled polyhedra in $\mathbb{H}^n$ for $n > 12$.
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Right-angled polyhedra in $\mathbb{H}^3$

[Andreev, 1970] Bounded acute-angled polyhedron in $\mathbb{H}^3$ is uniquely determined by its combinatorial type and dihedral angles.

[Pogorelov, 1967] A polyhedron can be realized in $\mathbb{H}^3$ as a bounded right-angled polyhedron if and only if
(1) any vertex is incident to 3 edges;
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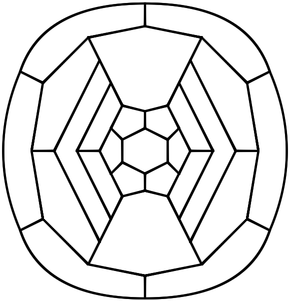
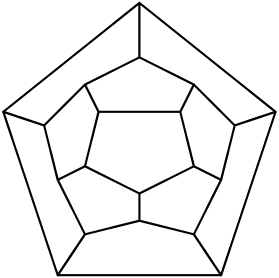
A polyhedron which satisfies (1) and (2), but not (3):

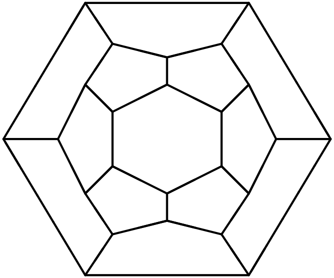


there is a closed circuit which separates two 6-gonal faces, but intersects 4 edges only.

An infinite family of right-angled hyperbolic polyhedra

For integer $n \geqslant 5$ consider right-angled $(2n+2)$-hedra Rn.
$R5$ and $R6$ look as:





Rn are called Löbelll polyhedra.
[Frank Richard Löbell, 1931] The first example of closed orientable hyperbolic 3-manifold – constructed from 8 copies of $R6$.

Two types of moves

Let $\mathcal{R}$ be the set of all compact right-angled polyhedra.

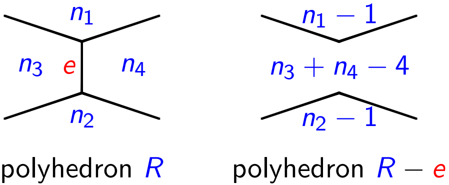
[Inoue, 2008] Two types of moves on $\mathcal{R}$.

Definition of Composition / Decomposition:

Let $R_1, R_2 \in \mathcal{R}$; $F_1 \subset R_1$ and $F_2 \subset R_2$ be a pair of k-gonal faces. Then a composition is $R = R_1 \cup_{F_1 = F_2} R_2$.

Two types of moves

Edge surgery: combinatorial transformation from R to $R - e$



If $R \in \mathcal{R}$ and e is such that faces F_1 and F_2 have at least 6 sides each and e is not a part of prismatic 5-circuit, then $R - e \in \mathcal{R}$.

Two types of moves

Theorem [Inoue, 2008].

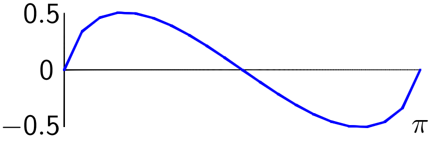
For any $P_0 \in \mathcal{R}$ there exists a sequence of unions of right-angled hyperbolic polyhedra $P_1, \ldots, P_k$ such that each set P_i is obtained from P_{i-1} by decomposition or edge surgery, and P_k consists of Löbell polyhedra. Moreover,

$$\mathrm{vol}(P_0) \geqslant \mathrm{vol}(P_1) \geqslant \mathrm{vol}(P_2) \geqslant \ldots \geqslant \mathrm{vol}(P_k).$$

Lobachevsky function

Volumes of hyperbolic 3-polyhedra can by calculated in terms of the Lobachevsky function

$$\Lambda(x) = -\int_0^x \log|2\sin(t)|\,\mathrm{d}t.$$

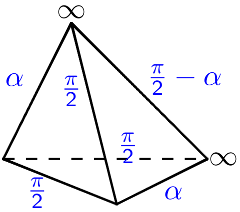


– periodic: $\Lambda(x+\pi) = \Lambda(x)$;
– odd: $\Lambda(-x) = -\Lambda(x)$;
– maximum $\Lambda^{max} = \Lambda(\pi/6) = 0.507\ldots$.

[Defining identity] For any positive $m \in \mathbb{Z}$ Lobachevsky function satisfies the following relation:

$$\Lambda(m\theta) = m \sum_{k=0}^{m-1} \Lambda\left(\theta + \frac{k\pi}{m}\right).$$

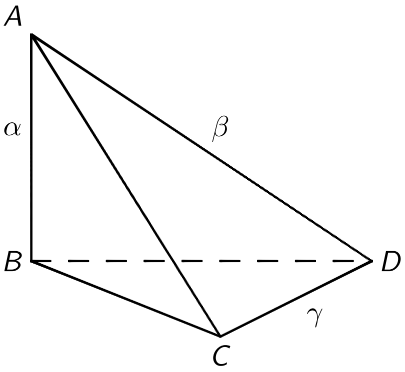
Geometric meaning of the Lobachevsky function:



Volume of this tetrahedra is equal to $\frac{1}{2}\Lambda(\alpha)$.

Doubly-rectengular tetrahedra

A tetrahedron $ABCD$ is said to be doubly-rectangular if AB is orthogonal to BCD and CD is orthogonal to ABC.



Denote it by $R(\alpha, \beta, \gamma)$.

Schläfli variation formula.

Let P_t be a smooth family of compact polyhedra in a complete connected n-dimensional space of constant curvature k. Then

$$(n-1) \cdot k \cdot d\mathrm{vol}(P_t) = \sum_F \mathrm{vol}_{n-2}(F)\, d\theta(F),$$

where the sum is taken over all faces of co-dimension two.

The volume formula for doubly-rectangular tetrahedra

Theorem [Kellerhals, Vinberg]
Let $R = R(\alpha, \beta, \gamma)$ be doubly-rectangular tetrahedron in $\mathbb{H}^3$. Then

$$\mathrm{vol}(R) = \frac{1}{4}\left(\Lambda(\alpha+\delta) - \Lambda(\alpha-\delta) + \Lambda\left(\frac{\pi}{2}+\beta-\delta\right)\right.$$
$$\left. + \Lambda\left(\frac{\pi}{2}-\beta-\delta\right) + \Lambda(\gamma+\delta) - \Lambda(\gamma-\delta) + 2\Lambda\left(\frac{\pi}{2}-\delta\right)\right),$$

where

$$0 \leq \delta = \arctan \frac{\sqrt{\cos^2\beta - \sin^2\alpha \sin^2\gamma}}{\cos\alpha \cos\gamma} < \frac{\pi}{2}.$$

Theorem [V., 1998]

For any $n \geqslant 5$ the following formula holds for volumes of Löbell polyhedra

$$\mathrm{vol}(Rn) = \frac{n}{2}\left(2\Lambda(\theta_n) + \Lambda\left(\theta_n + \frac{\pi}{n}\right) + \Lambda\left(\theta_n - \frac{\pi}{n}\right) + \Lambda\left(\frac{\pi}{2} - 2\theta_n\right)\right),$$

where

$$\theta_n = \frac{\pi}{2} - \arccos\left(\frac{1}{2\cos(\pi/n)}\right).$$

The initial list of right-angled polyhedra

Theorem [Inoue, 2008]

The dodecahedron $R5$ and the Löbell polyhedron $R6$ are first and second smallest volume compact right-angled hyperbolic polyhedra.

Theorem [Shmel'kov – V., 2011]

The initial list of small compact right-angled hyperbolic polyhedra:

1	4.3062 . . .	$R5$	7	8.6124 . . .	$R5 \cup R5$
2	6.2030 . . .	$R6$	8	8.6765 . . .	$R6_3^3$
3	6.9670 . . .	$R6^1$	9	8.8608 . . .	$R6_1^3$
4	7.5632 . . .	$R7$	10	8.9456 . . .	$R6_2^3$
5	7.8699 . . .	$R6_1^2$	11	9.0190 . . .	$R8$
6	8.0002 . . .	$R6_2^2$			

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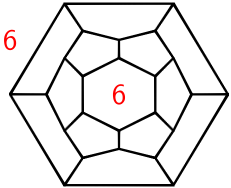
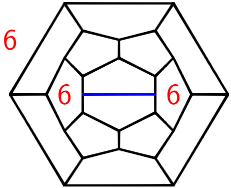
Applying edge surgeries.

The polyhedron $R6$ and possible faces to apply surgeries:

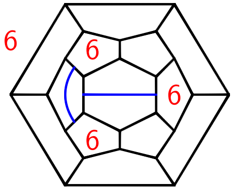


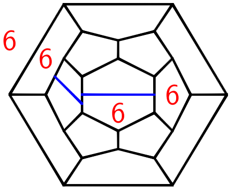
The polyhedron $R6^1$ (obtained from $R6$ by a surgery) and possible faces to apply surgeries:



Applying edge surgeries.

Polyhedra $R6_1^2$ and $R6_2^2$ obtained from $R6^1$ by edge surgeries.

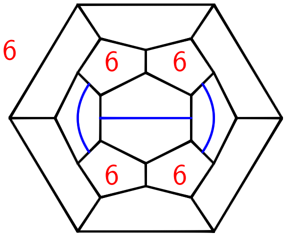


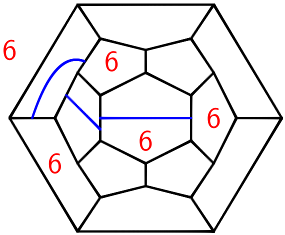


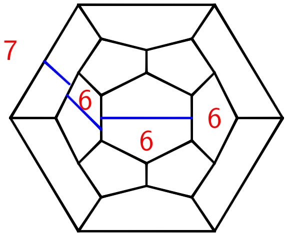
There are few possibilities to apply edge surgeries to them.

Applying edge surgeries.

Polyhedra $R6^3_1$, $R6^3_2$ and $R6^3_3$ $(= R7^1)$







vol Rn are given by the explicit formula.

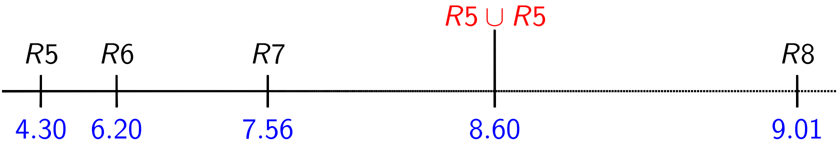
In other cases numerical calculations were done by the computer program developed by K. Shmel'kov.

In symmetric cases results coincide with calculations by the computer program **Orb** developed by C. Hodgson.

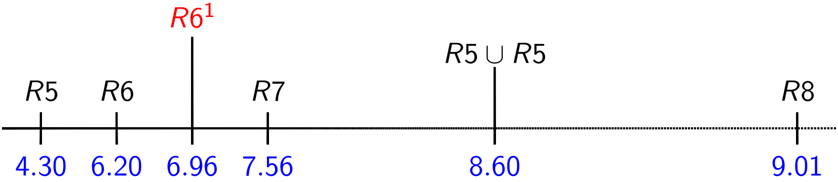
The set of volumes of right-angled polyhedra.

Löbell polyhedra:

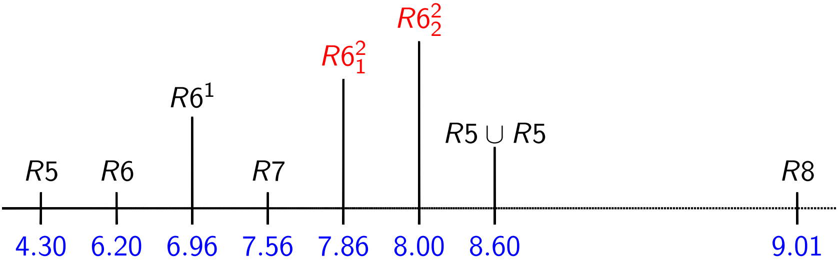
Composition of $R5$ with $R5$:



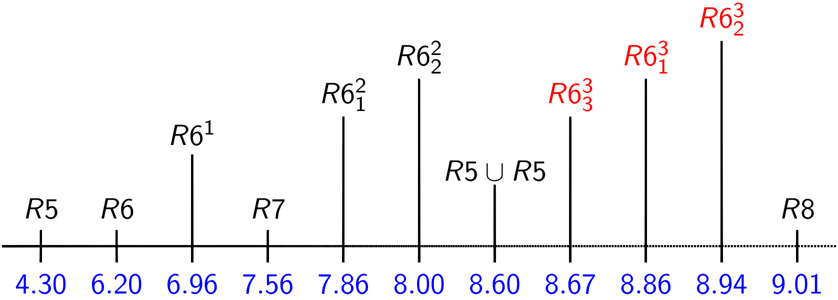
Edge surgery on $R6$:



Edge surgeries on $R6^1$:



Edge surgery on $R6_1^2$ and $R6_2^2$:



Volume bounds from combinatorics

Theorem [Atkinson, 2009]

Let P be a compact right-angled hyperbolic polyhedron with N vertices. Then

$$(N-2)\cdot\frac{v_8}{32}\leqslant \mathrm{vol}(P) < (N-10)\cdot\frac{5v_3}{8},$$

where v_8 is the maximal octahedron volume, and v_3 is the maximal tetrahedron volume.

There exists a sequence of compact right-angled polyhedra P_i with N_i vertices such that $\mathrm{vol}(P_i)/N_i$ tends to $5v_3/8$ as $i\to\infty$.

The following result demonstrates that $5v_3/8$ is a double-limit point for the normalized volume function $\omega(R)=\mathrm{vol}(R)/\mathrm{vert}(R)$.

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Volume bounds from combinatorics

[Repovš – V., 2011] For each integer $k \geqslant 1$ there is a sequence of compact right-angled hyperbolic polyhedra $R_k n$ such that

$$\lim_{n \to \infty} \frac{\mathrm{vol}(R_k n)}{\mathrm{vert}(R_k n)} = \frac{k}{k+1} \cdot \frac{5 v_3}{8}.$$

Polyhedra $R_k n$ are constructed from Löbell polyhedra Rn.

Volume bounds from combinatorics

[Repovš – V., 2011] Let P be a compact right-angled hyperbolic polyhedron, with V vertices and F faces. If P is not a dodecahedron, then

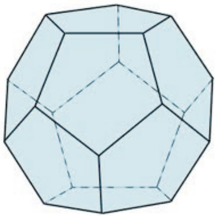
$$\mathrm{vol}(P) \geqslant \max\{(V-2)\cdot\frac{v_8}{32}, 6.203\ldots\}$$

and

$$\mathrm{vol}(P) \geqslant \max\{(F-3)\cdot\frac{v_8}{16}, 6.203\ldots\}.$$

This improves Atkinson's bound for $V \leqslant 56$ and $F \leqslant 30$.

Let R be a bounded right-angled polyhedron in $\mathbb{H}^3$.
(The simplest example is the right-angled dodecahedron.)



Let G be the group generated by reflections in faces of R.

In the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ we fix three generators $\alpha = (1, 0, 0)$, $\beta = (0, 1, 0)$, $\gamma = (0, 0, 1)$ and the sum $\delta = \alpha + \beta + \gamma = (1, 1, 1)$.

Elements $\alpha, \beta, \gamma, \delta$ will be referred as four colors.

Right-angled polyhedra and Coxeter groups

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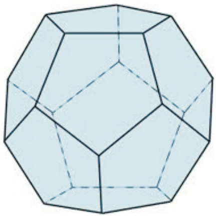

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Construction of manifolds from colorings

Let us color faces of R in colors $\alpha, \beta, \gamma, \delta$ in such a way that any two adjacent faces are getting different colors.
Such a coloring σ defines an epimorphism

$$\varphi_\sigma \colon G \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Denote $G^\sigma = \mathrm{Ker}(\varphi_\sigma)$.

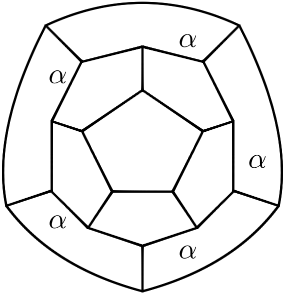
[V., 1987]:

For any bounded right-angled polyhedron R and any 4-coloring σ of its faces the quotient space $\mathbb{H}^3/G^\sigma$ is a closed orientable hyperbolic 3-manifold glued from 8 copies of R.

Remark. This is a constructive way to find a torsion-free subgroup of a right-angled Coxeter group.

Almost right-angled polyhedra

Almost right-angled polyhedra $\mathcal{L}_n(\alpha)$.



Here dihedral angle α is not necessary $\pi/2$.

Volume formula

[Buser – Mednykh – V., 2012]

The volume of the hyperbolic polyhedron $\mathcal{L}_n(\alpha)$, $n \geqslant 5$, where $0 < \alpha < \pi$, is given by the following formula:

$$\mathrm{vol}\,\mathcal{L}_n(\alpha) = \frac{n}{2} \int_{\alpha}^{\pi} \mathrm{arccosh}\ \Big[-\cos\mu\cos(2\pi/n) +$$

$$+ 2\cos(\pi/n)\sqrt{\cos^2\mu\cos^2(\pi/n) + \sin^2\mu}\,\Big]\, d\mu.$$

Why volumes?

Using volumes to study hyperbolic 3-manifolds:

(1) To distinguish manifolds:

By Mostow rigidity theorem volume of a closed hyperbolic 3-manifold is its topological invariant.

Number of manifolds of given volume is finite, but it can be arbitrary large.

(2) To estimate complexity of manifolds:

Let $c(M)$ be complexity (Matveev complexity) of a hyperbolic 3-manifold M, $\mathrm{vol}(M)$ be its volume, and v_3 be volume of the maximal hyperbolic tetrahedra; then $\dfrac{\mathrm{vol(M)}}{v_3} \leqslant c(M)$.

Using volumes to study hyperbolic 3-manifolds:

(3) To describe finite index extensions of groups:

Let G be fundamental group of a hyperbolic 3-manifold, and G^* be its discrete extension such that $[G^* : G] = n$.
Then $\mathrm{vol}(\mathbb{H}^3/G^*) = \frac{1}{n} \cdot \mathrm{vol}(\mathbb{H}^3/G)$.

But volumes of hyperbolic 3-orbifolds are bounded below.

Therefore, discrete extensions can be controlled by volumes.

(4) To describe isometries manifolds:

[Reni – V., 2001] Let $n \geqslant 5$, K be hyperbolic 2-bridge knot, and $M_n(K)$ be (hyperbolic) n-fold cyclic covering of S^3 branched over K. Denote by vol_n volume of the smallest orientable hyperbolic 3-orbifold with torsion of order n. If

$$n \geqslant \sqrt{\frac{\operatorname{vol}(S^3 \setminus K)}{4vol_n}} + 1$$

then $M_n(K)$ doesn't have hidden symmetries.

Problems:

1. [Gromov, 1981:] Does there exists a pair of hyperbolic 3-manifold such that the ratio of their volumes is irrational?

2. Does there exists a pair of compact right-angled hyperbolic polyhedra such that the ratio of their volumes is irrational?

Open Problems

Problems:

1. [Gromov, 1981:] Does there exists a pair of hyperbolic 3-manifold such that the ratio of their volumes is irrational?

2. Does there exists a pair of compact right-angled hyperbolic polyhedra such that the ratio of their volumes is irrational?