# METRIC GEOMETRY OF CARNOT-CARATHÉODORY SPACES WITH $C^1$-SMOOTH VECTOR FIELDS

**Sergey Vodopyanov**
Sobolev Institute of Mathematics, Novosibirsk.

The Fourth Geometry Meeting
dedicated to the centenary of A.D.Alexandrov
August, 20 – 24, 2012

# REFERENCES


1. S. K. Vodopyanov and V. M. Chernikov, "Sobolev Spaces and Hypoelliptic Equations" I. *Siberian Advances in Mathematics.* 1996. T. 6, № 3. P. 27–67; II. 1996. T. 6, № 4. P. 64–96.

2. S. K. Vodopyanov and M. B. Karmanova, "Local approximation theorem on Carnot manifolds under minimal smoothness", *Dokl. AN*, 427, No. 6, 731–736 (2009).

3. M. Karmanova and S. Vodopyanov, "Geometry of Carnot–Carathéodory spaces, differentiability and coarea formula", *Analysis and Mathematical Physics*, Birkhäuser, 284–387 (2009).

4. S. G. Basalaev and S. K. Vodopyanov, "Approximate differentiability of mappings of Carnot–Carathéodory spaces", *Eurasian Math. J.*, 3 (2012): arXiv:1206.5197.

5. A. V. Greshnov, "A proof of Gromov Theorem on homogeneous nilpotent approximation for $C^1$-smooth vector fields", *Mathematicheskie Trudy*, 15, No. 2 (2012).

# Classical definition of Carnot-Carathéodory space.

$\mathbb{M}$ is connected smooth manifold, $\dim \mathbb{M} = N$

$T\mathbb{M}$ is a tangent bundle;

"horizontal" subbundle is

$H\mathbb{M} = \text{span}\{X_1, \dots, X_n\} \subseteq T\mathbb{M}$ ($n < N$, $X_i \in C^\infty$)

There is a filtration $H\mathbb{M} = H_1 \subsetneq H_2 \subsetneq \dots \subsetneq H_M = T\mathbb{M}$ such that

$$\text{span}\{H_1, [H_1, H_i]\} = H_{i+1}, \quad \dim H_i = \text{const}.$$

$\Longrightarrow (\mathbb{M}, H\mathbb{M}, \langle \cdot, \cdot \rangle_{H\mathbb{M}})$ defines a subriemannian geometry

$M$ is a depth of the subriemannian space $\mathbb{M}$

## Classical definition of Carnot-Carathéodory space.

$\mathbb{M}$ is connected smooth manifold, $\dim \mathbb{M} = N$

$T\mathbb{M}$ is a tangent bundle;

"horizontal" subbundle is

$H\mathbb{M} = \text{span}\{X_1, \dots, X_n\} \subseteq T\mathbb{M}$ ($n < N$, $X_i \in C^\infty$)

There is a filtration $H\mathbb{M} = H_1 \subsetneq H_2 \subsetneq \dots \subsetneq H_M = T\mathbb{M}$ such that

$$\text{span}\{H_1, [H_1, H_i]\} = H_{i+1}, \quad \dim H_i = \text{const}.$$

$\Longrightarrow (\mathbb{M}, H\mathbb{M}, \langle\cdot,\cdot\rangle_{H\mathbb{M}})$ defines a subriemannian geometry

$M$ is a depth of the subriemannian space $\mathbb{M}$

• Субриманова геометрия описывает физические процессы, в которых движение возможно лишь вдоль нескольких выделенных ("допустимых"="горизонтальных") направлений

# Examples

**1.** Heisenberg group $\mathbb{H}^n$

$$\mathbb{M} = \mathbb{R}^{2n+1} : X_i = \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2}\frac{\partial}{\partial t}, \ X_{n+i} = \frac{\partial}{\partial x_i} - \frac{x_i}{2}\frac{\partial}{\partial t}, \ X_{2n+1} = \frac{\partial}{\partial t}$$

$$H_1 = \mathrm{span}\{X_1, X_2, \dots, X_{2n}\}, \ H_2 = [H_1, H_1] = \mathrm{span}\{X_{2n+1}\}$$

# Examples

**1.** Heisenberg group $\mathbb{H}^n$

$\mathbb{M} = \mathbb{R}^{2n+1}: \; X_i = \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2}\frac{\partial}{\partial t}, \;\; X_{n+i} = \frac{\partial}{\partial x_i} - \frac{x_i}{2}\frac{\partial}{\partial t}, \;\; X_{2n+1} = \frac{\partial}{\partial t}$

$H_1 = \text{span}\{X_1, X_2, \ldots, X_{2n}\}, \;\; H_2 = [H_1, H_1] = \text{span}\{X_{2n+1}\}$

**2.** Carnot group is a connected simply connected group Lie $G$ with stratified Lie algebra $V$:

$$V = V_1 \bigoplus V_2 \bigoplus \ldots \bigoplus V_M; \;\; [V_1, V_i] = V_{i+1}, \;\; [V_1, V_m] = \{0\}$$

**!** A Carnot group is a tangent cone to a subriemannian space in a regular point (Mitchell 1985; Gromov, Bellaiche 1996)

## • Mathematical foundation of thermodynamics

• 1909, Carathéodory in order to prove the existence of entropy derived the following statement:

*Let* $\mathbb{M}$ *be a connected manifold endowed with a corank one distribution. If there exist two points that can not be connected by a horizontal path then the distribution is integrable.*

## • **Mathematical foundation of thermodynamics**

• 1909, Carathéodory in order to prove the existence of entropy derived the following statement:

*Let $\mathbb{M}$ be a connected manifold endowed with a corank one distribution. If there exist two points that can not be connected by a horizontal path then the distribution is integrable.*

$D \subset T\mathbb{M}$ is a *corank one distribution* if $\exists$ a smooth 1-form $\theta$ s. t. $D_x = \{v \in T_x\mathbb{M} : \theta(x)\langle v\rangle = 0\}$. An absolutely continuous path $\gamma$ is called *horizontal* if $\dot{\gamma}(x) \in D_x$.

## • Development

• Carathéodory 1909, Rashevskiy 1938, Chow 1939: arbitrary two points of any connected C-C space $\mathbb{M}$ can be joined by a "horizontal" curve.

## • Development

• Carathéodory 1909, Rashevskiy 1938, Chow 1939: arbitrary two points of any connected C-C space $\mathbb{M}$ can be joined by a "horizontal" curve.

It follows that $(\mathbb{M}, d_c)$ is a metric space with the subriemannian distance

$$d_c(u,v) = \inf\{L(\gamma) \mid \gamma \text{ is horizontal, } \gamma(0) = u, \gamma(1) = v\}$$

not comparable to Riemannian one.

- **Hörmander, 1967: Hypoelliptic equations**

**A problem:** *when a distribution solution $f$ to the equation*

$$(X_1^2 + \ldots + X_{n-1}^2 - X_n)f = \varphi \in C^\infty$$

*is a smooth function?*

Here $X_i \in C^\infty$.

• **Hörmander, 1967: Hypoelliptic equations**

**A problem:** *when a distribution solution $f$ to the equation*

$$(X_1^2 + \ldots + X_{n-1}^2 - X_n)f = \varphi \in C^\infty$$

*is a smooth function?*

Here $X_i \in C^\infty$.

• **Particular case: Kolmogorov's equations**

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} - \frac{\partial u}{\partial t} = f$$

• physics (diffusion process), economics (arbitrage theory, some stochastic volatility models of European options), etc.

## Hypoelliptic Equations

• **Hörmander (1967)**: sufficient conditions on fields $X_1, \ldots, X_n$:

*There exists* $M < \infty$ *such that*

• Lie$\{X_1, X_2, \ldots, X_n\} = \text{span}\{X_I(v) \mid |I| \leq M\} = T_v\mathbb{M}$ *for all* $v \in \mathbb{M}$ *where*

$$X_I(v) = \text{span}\{[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots](v) : X_{i_j} \in H_1\}$$

*for* $I = (i_1, i_2, \ldots, i_k)$.

## Hypoelliptic Equations

• **Hörmander (1967)**: sufficient conditions on fields $X_1, \ldots, X_n$:

*There exists $M < \infty$ such that*

• Lie$\{X_1, X_2, \ldots, X_n\} = \text{span}\{X_I(v) \mid |I| \leq M\} = T_v\mathbb{M}$ *for all* $v \in \mathbb{M}$ *where*

$$X_I(v) = \text{span}\{[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots](v) : X_{i_j} \in H_1\}$$

*for* $I = (i_1, i_2, \ldots, i_k)$.

• $M$ *is called the depth of the sub-Riemannian space* $\mathbb{M}$.

## Hypoelliptic Equations

• **Hörmander (1967)**: sufficient conditions on fields $X_1, \dots, X_n$:

*There exists* $M < \infty$ *such that*

• Lie$\{X_1, X_2, \dots, X_n\} = \text{span}\{X_I(v) \mid |I| \leq M\} = T_v\mathbb{M}$ *for all* $v \in \mathbb{M}$ *where*

$$X_I(v) = \text{span}\{[X_{i_1}, [X_{i_2}, \dots, [X_{i_{k-1}}, X_{i_k}] \dots](v) : X_{i_j} \in H_1\}$$

*for* $I = (i_1, i_2, \dots, i_k)$.

• $M$ *is called the depth of the sub-Riemannian space* $\mathbb{M}$.

• **Stein (1971)**: The program of studying of geometry of Hörmander vector fields;
*description of singularities of fundamental solutions*

## Quasilinear equations of subelliptic type

Let a function $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^N$ meet the following conditions:

($\mathcal{A}$1) the mapping $\Omega \ni x \mapsto \mathcal{A}(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^n$,
the mapping $\mathbb{R}^n \ni \xi \mapsto \mathcal{A}(x, \xi)$ is continuous for a. a. $x \in \Omega$;

there are some constants $0 < \alpha \leq \beta < \infty$ such that

($\mathcal{A}$2) $\langle \mathcal{A}(x, \xi), \xi \rangle \geq \alpha |\xi|^p$;

($\mathcal{A}$3) $|\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p-1}$;

($\mathcal{A}$4) $\langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta \rangle > 0$;

($\mathcal{A}$5) $\mathcal{A}(x, \lambda\xi) = \lambda |\lambda|^{p-2} \mathcal{A}(x, \xi)$ for all $\lambda \in \mathbb{R} \setminus 0$.

## Quasilinear equations of subelliptic type

- $u : \Omega \to \mathbb{R}$ is called an $\mathcal{A}$-solution to the equation

$$-\,\mathrm{div}_h(\mathcal{A}(x, \nabla_0 u)) = 0 \quad \text{in} \quad \Omega \quad \text{if}$$

$u \in W^1_{p,\mathrm{loc}}$ and

$$\int_\Omega \mathcal{A}(x, \nabla_0 u) \nabla_0 \psi \, dx = 0 \quad \text{for all test functions } \psi \in C^1_0(\Omega).$$

Here $\nabla_0 u = (X_1 u, X_2 u, \ldots, X_n u)$ where $X_1, X_2, \ldots, X_n$ are vector fields meeting Hörmander condition.

## Quasilinear equations of subelliptic type

• A function $u : \Omega \to \mathbb{R}$ is called an $\mathcal{A}$-solution in $\Omega$ to the equation

$$-\,\mathrm{div}_h(\mathcal{A}(x, \nabla_0 u)) = 0 \quad \text{if}$$

$u \in W^1_{p,\mathrm{loc}}(\Omega)$ and

$$\int\limits_{\Omega} \mathcal{A}(x, \nabla_0 u)\nabla_0 \psi \, dx = 0 \quad \text{for all test functions } \psi \in C_0^1(\Omega).$$

Here $\nabla_0 u = (X_1 u, X_2 u, \dots, X_n u)$ where $X_1, X_2, \dots, X_n$ are vector fields meeting Hörmander condition.

PROBLEM is to prove regularity properties of the $\mathcal{A}$-solution to this equation; $|u(x) - u(y)| \le M d_{cc}^{\lambda}(x, y)$, $\lambda \in (0, 1)$.
It is known for $C^\infty$-vector fields [1996; Chernikov, V.].

## Geometric control theory

◇ The linear system of ODE ($x \in \mathbb{M}^N$, $n < N$)

$$\dot{x} = \sum_{i=1}^{n} a_i(t) X_i(x), \qquad X_i \in C^\infty.$$

# Geometric control theory

◇ The linear system of ODE ($x \in \mathbb{M}^N$, $n < N$)

$$\dot{x} = \sum_{i=1}^{n} u_i(t) X_i(x), \qquad X_i \in C^\infty. \tag{1}$$

• Problem: To find bounded measurable functions $u_i(t)$ such that system (1) has a solution with the initial data $x(0) = p$, $x(1) = q$.

## Geometric control theory

◇ The linear system of ODE ($x \in \mathbb{M}^N$, $m < N$)

$$\dot{x} = \sum_{i=1}^{n} u_i(t) X_i(x), \qquad X_i \in C^\infty.$$

• Problem: To find measurable functions $u_i(t)$ such that system (1) has a solution with the initial data $x(0) = p$, $x(1) = q$.

If system (1) has a solution for every $q \in U(p)$ then it is called *locally controllable.*

## Geometric control theory

◇ The linear system of ODE ($x \in \mathbb{M}^N$, $n < N$)

$$\dot{x} = \sum_{i=1}^{n} u_i(t) X_i(x), \qquad X_i \in C^\infty.$$

• Problem: To find bounded measurable functions $u_i(t)$ such that system (1) has a solution with the initial data $x(0) = p$, $x(1) = q$.

If system (1) has a solution for every $q \in U(p)$ then it is called *locally controllable.*

• It is locally controllable if Lie$\{X_1, X_2, \dots, X_n\} = T\mathbb{M}$, i.e. the "horizontal" distribution $H\mathbb{M} = \{X_1, X_2, \dots, X_n\}$ is bracket-generating.

## Main classical results (proved for smooh enough vector fields)

- 1938–1939 Rashevskiy–Chow theorem;

## Main classical results (proved for smooh enough vector fields)

• 1938–1939 Rashevskiy–Chow theorem;

• 1982–1986 Mitchell-Gershkovich-Nagel-Stein-Wainger:
Ball–Box theorem (a ball in the Carnot-Carathéodory metric looks like a box);

## Main classical results (proved for smooh enough vector fields)

• 1909–1938–1939 Carathéodory–Rashevskiy–Chow theorem;

• 1982–1986 Mitchell-Gershkovich-Nagel-Stein-Wainger: Ball–Box theorem (a ball in the Carnot-Carathéodory metric looks like a box);

• 1986–1996 Gromov–Mitchell theorem on convergence of rescaled $CC$-spaces with respect to a fixed point to a nilpotent tangent cone;

## Main classical results (proved for smooh enough vector fields)

- 1909–1938–1939 Carathéodory–Rashevskiy–Chow theorem;

- 1982–1986 Mitchell-Gershkovich-Nagel-Stein-Wainger: Ball–Box theorem (a ball in the Carnot-Carathéodory metric looks like a box);

- 1986–1996 Gromov–Mitchell theorem on convergence of rescaled $CC$-spaces with respect to a fixed point to a nilpotent tangent cone;

- 1996 Gromov theorem on convergence of rescaled vector fields to *nilpotentized* vector fields constituting a basis of graded nilpotent group;

## Main classical results (proved for smooh enough vector fields)

- 1909–1938–1939 Carathéodory–Rashevskiy–Chow theorem;

- 1982–1986 Mitchell-Gershkovich-Nagel-Stein-Wainger: Ball–Box theorem (a ball in the Carnot-Carathéodory metric looks like a box);

- 1986–1996 Gromov–Mitchell theorem on convergence of rescaled $CC$-spaces with respect to a fixed point to a nilpotent tangent cone;

- 1996 Gromov theorem on convergence of rescaled vector fields to *nilpotentized* vector fields constituting a basis of graded nilpotent group;

- 1996 M. Gromov, A. Bellaïche approximation theorem on local behavior of metrics in the given space and in a local tangent cone.

## APPLICATIONS of SUBRIEMANNIAN GEOMETRY

- Thermodynamics
- Non-holonomic mechanics
- Geometric Control Theory
- Subelliptic equation
- Geometric measure theory
- Quasiconformal analysis
- Analysis on metric spaces
- Contact geometry
- Complex variable
- Economics
- Transport problem
- Quantum control
- Neurobiology
- Tomography
- Robotecnics

## Carnot–Carathéodory space ($C^1$-smooth vector fields)

- $\mathbb{M}$ is a connected $C^\infty$-smooth manifold with $\dim_{\mathrm{top}}(\mathbb{M}) = N$;

## Carnot–Carathéodory space ($C^1$-smooth vector fields)

- $\mathbb{M}$ is a connected $C^\infty$-smooth manifold with $\dim_{\mathrm{top}}(\mathbb{M}) = N$;
- in $T\mathbb{M}$ there exists a filtration by subbundles

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M};$$

## Carnot–Carathéodory space ($C^1$-smooth vector fields)

- $\mathbb{M}$ is a connected $C^\infty$-smooth manifold with $\dim_{\text{top}}(\mathbb{M}) = N$;
- in $T\mathbb{M}$ there exists a filtration by subbundles

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M};$$

- $\forall v \in \mathbb{M}\ \exists\, U(v)$ and vector fields $X_1, X_2, \ldots, X_N \in C^1$ such that

$H_i\mathbb{M}(v) = \text{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\}$, $\dim H_i\mathbb{M}(v) = \dim H_i$;

## Carnot–Carathéodory space ($C^1$-smooth vector fields)

- $\mathbb{M}$ is a connected $C^\infty$-smooth manifold with $\dim_{\text{top}}(\mathbb{M}) = N$;
- in $T\mathbb{M}$ there exists a filtration by subbundles

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M};$$

- $\forall v \in \mathbb{M}\ \exists U(v)$ with vector fields $X_1, X_2, \ldots, X_N \in C^1$ such that

$H_i\mathbb{M}(v) = \text{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\}$, $\dim H_i\mathbb{M}(v) = \dim H_i$;

- $[H_i, H_j] \subset H_{i+j}$, $i, j = 1, \ldots, M-1$.

## Carnot–Carathéodory space ($C^1$-smooth vector fields)

- $\mathbb{M}$ is a connected $C^\infty$-smooth manifold with $\dim_{\text{top}}(\mathbb{M}) = N$;
- in $T\mathbb{M}$ there exists a filtration by subbundles

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M};$$

- $\forall v \in \mathbb{M}\ \exists U(v)$ with vector fields $X_1, X_2, \ldots, X_N \in C^1$ such that

$H_i\mathbb{M}(v) = \text{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\}$, $\dim H_i\mathbb{M}(v) = \dim H_i$;

- $[H_i, H_j] \subset H_{i+j}$, $i, j = 1, \ldots, M-1$.

It is equivalent to $\quad [X_i, X_j](v) = \sum\limits_{k\colon \deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v)$

where $\deg X_k = \min\{m : X_k \in H_m\}$;

## Carnot–Carathéodory space ($C^1$-smooth vector fields)

- $\mathbb{M}$ is a connected $C^\infty$-smooth manifold with $\dim_{\text{top}}(\mathbb{M}) = N$
- in $T\mathbb{M}$ there exists a filtration by subbundles

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

- $\forall v \in \mathbb{M}\ \exists U(v)$ with vector fields $X_1, X_2, \ldots, X_N \in C^1$ such that

$H_i\mathbb{M}(v) = \text{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\}$, $\dim H_i\mathbb{M}(v) = \dim H_i$;

- $[H_i, H_j] \subset H_{i+j}$, $i, j = 1, \ldots, M-1$;

$\diamond$ *If* $H_{j+1} = \text{span}\{H_j, [H_1, H_j], [H_2, H_{j-1}], \ldots, [H_k, H_{j+1-k}]\}$ where $k = \lfloor \frac{j+1}{2} \rfloor$, $j = 1, \ldots, M-1$, then $\mathbb{M}$ is called the Carnot manifold.

# Basic Concepts

**Exponential mapping**: $u \in \mathbb{M}$, $(v_1, \dots, v_N) \in \mathbb{R}^N$,

$$\begin{cases} \dot{\gamma}(t) = \sum_{i=1}^{N} v_i X_i(\gamma(t)), \quad t \in [0,1], \\ \gamma(0) = u. \end{cases}$$

Then $\exp\Big(\sum_{i=1}^{N} v_i X_i\Big)(u) = \gamma(1)$. For each point $u$, define

$\theta_u : U(0) \to \mathbb{M}$ as $\theta_u(v_1, \dots, v_N) = \exp\Big(\sum_{i=1}^{N} v_i X_i\Big)(u)$.

## Basic Concepts

**Exponential mapping**: $u \in \mathbb{M}$, $(v_1, \ldots, v_N) \in \mathbb{R}^N$,

$$\begin{cases} \dot\gamma(t) = \sum\limits_{i=1}^{N} v_i X_i(\gamma(t)), \quad t \in [0,1], \\ \gamma(0) = u. \end{cases}$$

Then $\exp\Big(\sum\limits_{i=1}^{N} v_i X_i\Big)(u) = \gamma(1)$. For each point $u$, define

$\theta_u : U(0) \to \mathbb{M}$ as $\theta_u(v_1, \ldots, v_N) = \exp\Big(\sum\limits_{i=1}^{N} v_i X_i\Big)(u)$.

**Dilatations** $\Delta_\tau^u$: if $u \in \mathbb{M}$ и $v = \exp\Big(\sum\limits_{i=1}^{N} v_i X_i\Big)(u)$ then

$$\Delta_\tau^u(v) = \exp\Big(\sum_{i=1}^{N} v_i \tau^{\deg X_i} X_i\Big)(u)$$

## The New Approach to regular CC-spaces: a Local Lie Group at $u \in \mathbb{M}$ for $C^1$-Smooth Case

$$[X_i, X_j](v) = \sum_{k:\, \deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v).$$

**Theorem 1 (2009; Karmanova, V.).** Coefficients $\{c_{ijk}(u)\}_{\deg X_k = \deg X_i + \deg X_j} = \{\bar{c}_{ijk}\}$ satisfy Jacobi identity:

$$\sum_k \bar{c}_{ijk}(u)\bar{c}_{kml}(u) + \sum_k \bar{c}_{mik}(u)\bar{c}_{kjl}(u) + \sum_k \bar{c}_{jmk}(u)\bar{c}_{kil}(u) = 0$$

for all $i, j, m, l = 1, \dots, N$, and

$\bar{c}_{ijk} = -\bar{c}_{jik}$ for all $i, j, k = 1, \dots, N$.

Then the collection $\{\bar{c}_{ijk}\}$ defines nilpotent graded Lie algebra.

## The New Approach to regular CC-spaces: a Local Lie Group at $u \in \mathbb{M}$ for $C^1$-Smooth Case

According to the second Lie theorem we take *basis vector fields* $\{(\widehat{X}_i^u)'\}_{i=1}^N$ *in* $\mathbb{R}^N$ *constituting a Lie algebra in such a way that*

$$[(\widehat{X}_i^u)', (\widehat{X}_j^u)'](v) = \sum_{k:\, \deg X_k = \deg X_i + \deg X_j} \bar{c}_{ijk} (\widehat{X}_k^u)'(v),$$

$(\widehat{X}_i^u)' = e_i$, $i = 1, \ldots, N$,

*and* $\exp = \mathrm{Id}$.

The corresponding Lie group is nilpotent graded Lie group $\mathbb{G}_u\mathbb{M}$

## A Local Lie Group $\mathcal{G}^u\mathbb{M}$

In a neighborhod $\mathcal{G}_u \subset \mathbb{M}$ of $u$ push-forwarded vector fields

$\widehat{X}_i^u = D\theta_u(\widehat{X}_i^u)'$ define a structure of local Lie group

in such a way that

$$\theta_u : \mathbb{G}_u\mathbb{M} \to \mathcal{G}_u\mathbb{M}$$

is a local isomorphism of Lie groups.

## A Local Lie Group $\mathcal{G}^u\mathbb{M}$

In a neighborhod $\mathcal{G} \subset \mathbb{M}$ of $u$ push-forwarded vector fields

$\widehat{X}_i^u = D\theta_u(\widehat{X}_i^u)'$ they a structure of local Lie group

in such a way that

$$\theta_u : \mathbb{G}_u\mathbb{M} \to \mathcal{G}^u\mathbb{M}$$

is a local isomorphism of Lie groups.

- vector fields $\widehat{X}_i^u$ are left-invariant

Then $(\mathcal{G}, \widehat{X}_1^u, \ldots, \widehat{X}_N^u, \cdot) = \mathcal{G}^u\mathbb{M}$ **is a local Lie group**

- In the case of Carnot manifolds it is called **the local Carnot group**

## Quasimetric

Let $v = \exp\Big(\sum_{i=1}^{N} v_i \widehat{X}_i^u\Big)(w)$. Then

$$d_\infty^u(v,w) = \max_{i=1,\dots,N}\{|v_i|^{\frac{1}{\deg X_i}}\}$$

- $d_\infty^u(v,w) \geq 0$; $d_\infty^u(v,w) = 0 \Leftrightarrow v = w$

- $d_\infty^u(v,w) = d_\infty^u(w,v)$

- generalized triangle inequality: for a neighborhood $U \Subset \mathbb{M}$, there exists a constant $c = c(U)$ such that for any $v, s, w \in U$ we have

$$d_\infty^u(v,w) \leq c(d_\infty^u(v,s) + d_\infty^u(s,w))$$

## Quasimetric

- $d_\infty$ is defined similarly (with $X_i$ instead of $\widehat{X}_i^u$, $i = 1, \ldots, N$): if $v = \exp\Big(\sum_{i=1}^{N} v_i X_i\Big)(w)$ then

$$d_\infty(v, w) = \max_{i=1,\ldots,N} \{|v_i|^{\frac{1}{\deg X_i}}\}.$$

- $d_\infty(v, w) \geq 0$; $d_\infty(v, w) = 0 \Leftrightarrow v = w$.

- $d_\infty(v, w) = d_\infty(w, v)$.

- generalized triangle inequality: Do we have locally

$$d_\infty(v, w) \leq c(d_\infty(v, s) + d_\infty(s, w)) \quad \text{for some constant } c?$$

## Gromov type nilpotentization theorem

**Theorem 2 [2012; Greshnov].** For $x \in \mathrm{Box}(g, r_g)$ consider

$$X_i^\varepsilon(x) = (\Delta^g_{\varepsilon^{-1}})_* \varepsilon^{\deg X_i} X_i(\Delta^g_\varepsilon x), \quad i = 1, \dots, N.$$

Then the following expansion holds:

$$X_i^\varepsilon(x) = \widehat{X}_i^g(x) + \sum_{j=1}^N a_{ij}(x) \widehat{X}_j^g(x)$$

where $a_{ij}(x) = o(\varepsilon^{\max\{0, \deg X_j - \deg X_i\}})$ for $x \in \mathrm{Box}(g, \varepsilon r_g)$ and $o(\cdot)$ is uniform in $g$ belonging to some compact set of $\mathbb{M}$ as $\varepsilon \to 0$.

## Gromov type nilpotentizaton theorem

**Theorem 2 [2012; Greshnov].** For $x \in \mathrm{Box}(g, r_g)$ consider

$$X_i^{\varepsilon}(x) = (\Delta^g_{\varepsilon^{-1}})_* \varepsilon^{\deg X_i} X_i(\Delta^g_{\varepsilon} x), \quad i = 1, \dots, N.$$

Then the following expansion holds:

$$X_i^{\varepsilon}(x) = \widehat{X}_i^g(x) + \sum_{j=1}^{N} a_{ij}(x) \widehat{X}_j^g(x)$$

where $a_{ij}(x) = o(\varepsilon^{\max\{0, \deg X_j - \deg X_i\}})$ for $x \in \mathrm{Box}(g, \varepsilon r_g)$ and $o(\cdot)$ is uniform in $g$ belonging to some compact set of $\mathbb{M}$ as $\varepsilon \to 0$.

**Corollary 1 (Gromov Type Theorem):** *We have $X_i^{\varepsilon} \to \widehat{X}_i^g$ as $\varepsilon \to 0$, $i = 1, \dots, N$, at the points of* $\mathrm{Box}(g, r_g)$ *and this convergence is uniform in $g$ belonging to some compact neighborhood.*

## Gromov type nilpotentizaton theorem

**Theorem 2 [2012; Greshnov].** For $x \in \mathrm{Box}(g, r_g)$ consider

$$X_i^\varepsilon(x) = (\Delta^g_{\varepsilon^{-1}})_* \varepsilon^{\deg X_i} X_i(\Delta^g_\varepsilon x), \quad i = 1, \dots, N.$$

Then the following expansion holds:

$$X_i^\varepsilon(x) = \widehat{X}_i^g(x) + \sum_{j=1}^N a_{ij}(x)\widehat{X}_j^g(x)$$

where $a_{ij}(x) = o(\varepsilon^{\max\{0, \deg X_j - \deg X_i\}})$ for $x \in \mathrm{Box}(g, \varepsilon r_g)$ and $o(\cdot)$ is uniform in $g$ belonging to some compact set of $\mathbb{M}$ as $\varepsilon \to 0$.

**Corollary 1** (**Gromov Type Theorem**): *We have* $X_i^\varepsilon \to \widehat{X}_i^g$ *as* $\varepsilon \to 0$, $i = 1, \dots, N$, *at the points of* $\mathrm{Box}(g, r_g)$ *and this convergence is uniform in* $g$ *belonging to some compact neighborhood.*

**Corollary 2 [2009; Karmanova, V.].** Generalized triangle inequality holds locally for some constant $c$: $d_\infty(v, w) \leq c(d_\infty(v, s) + d_\infty(s, w))$.

## MAIN RESULT: Comparison of Local Geometries

Let $\mathcal{U} \subset \mathbb{M}$ where $\mathbb{M} \in C^1$:

- $\theta_v(B(0, r_v)) \supset \mathcal{U}$ for all $v \in \mathcal{U}$,
- $\mathcal{G}^u\mathbb{M} \supset \mathcal{U}$ for all $u \in \mathcal{U}$,
- $\theta_v^u(B(0, r_{u,v})) \supset \mathcal{U}$ for all $u, v \in \mathcal{U}$.

**Theorem 3 (2009; Karmanova, V.).** *Let $u, u', v \in \mathcal{U} \Subset \mathbb{M}$. Assume that $d_\infty(u, u') = O(\varepsilon)$ and $d_\infty(u, v) = O(\varepsilon)$, and consider points*

$$w_\varepsilon = \exp\Big(\sum_{i=1}^{N} w_i \varepsilon^{\deg X_i} \widehat{X}_i^u\Big)(v) \text{ and } w'_\varepsilon = \exp\Big(\sum_{i=1}^{N} w_i \varepsilon^{\deg X_i} \widehat{X}_i^{u'}\Big)(v).$$

*Then*

$$\max\{d_\infty^u(w_\varepsilon, w'_\varepsilon), d_\infty^{u'}(w_\varepsilon, w'_\varepsilon)\} = o(\varepsilon)$$

*where $o(\varepsilon)$ is uniform in $u, u', v \in \mathcal{U}$.*

## Corollaries

**4) Local Approximation Theorem for $d_\infty$-quasimetric (2009; Karmanova, V.):**

Let $v, w \in \mathrm{Box}(g, \varepsilon) \subset \mathbb{M}$. Then

$$|d_\infty(v, w) - d_\infty^u(v, w)| = o(\varepsilon).$$

## Corollaries

Assumption: Suppose that $\mathbb{M}$ is a Carnot manifold.

**5) Rashevsky–Chow type Theorem (2012; Basalaev, V.):** *Any two points $u, v \in \mathbb{M}$ can be connected by a horizontal curve $\gamma$ (i. e., $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$ for almost all $t \in [0, 1]$).*

The intrinsic metric on Carnot–Carathéodory space

$$d_c(u, v) = \inf_{\substack{\gamma \text{ is horizontal} \\ \gamma(0)=u, \gamma(1)=v}} \{L(\gamma)\}$$

## Corollaries

Assumption: Suppose that $\mathbb{M}$ is a Carnot manifold.

**5) Rashevsky–Chow type Theorem (2012; ; Basalaev, V.):** *Any two points $u, v \in \mathbb{M}$ can be connected by a horizontal curve $\gamma$ (i. e., $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$ for almost all $t \in [0, 1]$).*

The intrinsic metric on Carnot–Carathéodory space

$$d_{cc}(u, v) = \inf_{\substack{\gamma \text{ is horizontal} \\ \gamma(0)=u, \gamma(1)=v}} \{L(\gamma)\}$$

**6) Local Approximation Theorem for $d_{cc}$-metric (2009; Karmanova, V.):** *For $v, w \in B_{cc}(u, \varepsilon)$, we have*

$$|d_{cc}(v, w) - d^u_{cc}(v, w)| = o(\varepsilon).$$

## Corollaries (Ball-Box Theorem)

**7) Mitchell-Gershkovich-Nagel-Stein-Wainger theorem type Ball–Box Theorem (2012).** *For* $\mathcal{U} \Subset \mathbb{M}$*, there exist constants* $c(\mathcal{U}) \leq C(\mathcal{U})$ *such that*

$$c(\mathcal{U})d_\infty(x,y) \leq d_{cc}(x,y) \leq C(\mathcal{U})d_\infty(x,y),$$

*where* $x, y \in \mathcal{U}$*, and* $d_{cc}(x,y)$ *is a Carnot–Carathéodory metric.*

## Corollaries

**7) Mitchell-Gershkovich-Nagel-Stein-Wainger theorem type Ball–Box Theorem (2011).** *For $\mathcal{U} \Subset \mathbb{M}$, there exist constants $c(\mathcal{U}) \leq C(\mathcal{U})$ such that*

$$c(\mathcal{U})d_\infty(x,y) \leq d_{cc}(x,y) \leq C(\mathcal{U})d_\infty(x,y),$$

*where $x, y \in \mathcal{U}$, and $d_{cc}(x,y)$ is a Carnot–Carathéodory metric.*

Proof: [2011, V.]

$d_{cc}^u(u,w)(1-o(1)) \leq d_{cc}(u,w) \leq d_{cc}^u(u,w)(1+o(1))$;

$d_\infty^u(u,w)(1-o(1)) \leq d_\infty(u,w) \leq d_\infty^u(u,w)(1+o(1))$;

$d_{cc}^u(u,w) \sim d_\infty^u(u,w)$.

## Application to Quasilinear equations of subelliptic type

**THEOREM [1996 : Chernikov, V.].** *Let* $X_1, X_2, \ldots, X_n$ *are* $C^1$-*vector fields in* $\Omega \subset \mathbb{R}^N$ *extended to a collection of* $C^1$-*vector fields constituting a structure of a Carnot manifold.*

*Then any* $\mathcal{A}$-*solution* $u : \Omega \to \mathbb{R}$ *to the equation*

$$-\operatorname{div}_h(\mathcal{A}(x, \nabla_0 u)) = 0$$

*is Hölder continuous:* $|u(x) - u(y)| \leq M d_{cc}^{\lambda}(x, y)$, $\lambda \in (0, 1)$.

## Application to Geometric control theory

◇ The linear system of ODE ($x \in \mathbb{M}^N$, $n < N$)

$$\dot{x} = \sum_{i=1}^{n} u_i(t) X_i(x), \quad X_i \in C^1.$$

• Problem: To find measurable functions $u_i(t)$ such that system (1) has a solution with the initial data $x(0) = p$, $x(1) = q$.

If system (1) has a solution for every $q \in U(p)$ then it is called *locally controllable.*

• (1) locally controllable if "horizontal" vector fields $\{X_1, \dots, X_n\}$ can be extended to the system of vector fields constituting a structure of a Carnot manifold.

## More Applications

• *sub-Riemannian differentiability theory*: Rademacher-type and Stepanov-type Theorems on sub-Riemannian differentiability of mappings of Carnot manifolds (S. Vodopyanov)

• *geometric measure theory on sub-Riemannian structures*: area formula for intrinsically Lipschitz mappings of Carnot manifolds, coarea formula for $C^{M+1}$-smooth mappings of Carnot manifolds (M. Karmanova; S. Vodopyanov)

• *geometry of non-equiregular Carnot–Carathéodory spaces* (S. Selivanova)

## Sub-Riemannian Differentiability [2007; V.]

**Definition.** A mapping $\varphi : (\mathbb{M}, d_{cc}) \to (\widetilde{\mathbb{M}}, \tilde{d}_{cc})$ *is $hc$-differentiable* at $u \in \mathbb{M}$ if there exists a horizontal homomorphism

$$L_u : (\mathcal{G}^u, d^u_{cc}) \to (\mathcal{G}^{\varphi(u)}, d^{\varphi(u)}_{cc})$$

of local Carnot groups such that

$$\tilde{d}_{cc}(\varphi(w), L_u(w)) = o(d_{cc}(u, w)),\ \ E \cap \mathcal{G}^u \ni w \to u.$$

• For mappings of Carnot groups, this notion coincides with the definition of $\mathcal{P}$-differentiability in the sense of P. Pansu.

• Denote the $hc$-differential of $\varphi$ at $u$ by the symbol $\widehat{D}\varphi(u)$

## Sub-Riemannian Differentiability [2007; V.]

**Rademacher-Type Theorem.** *Suppose that a mapping* $\varphi : (\mathbb{M}, d_{cc}) \to (\widetilde{\mathbb{M}}, \tilde{d}_{cc})$ *is Lipschitz. Then* $\varphi$ *is* $hc$*-differentiable almost everywhere.*

**Stepanov-Type Theorem.** *Suppose that a mapping* $\varphi : (\mathbb{M}, d_{cc}) \to (\widetilde{\mathbb{M}}, \tilde{d}_{cc})$ *is such that*

$$\varlimsup_{y \to x} \frac{\tilde{d}_{cc}(\varphi(y), \varphi(x))}{d_{cc}(y, x)} < \infty$$

*almost everywhere. Then* $\varphi$ *is* $hc$*-differentiable almost everywhere.*

**Theorem.** *Suppose that* $\varphi : (\mathbb{M}, d_{cc}) \to (\widetilde{\mathbb{M}}, \tilde{d}_{cc})$ *is* $C^1_H$*-smooth and contact (i. e.,* $D_H\varphi[H\mathbb{M}] \subset H\widetilde{\mathbb{M}}$*). Then* $\varphi$ *is continuously* $hc$*-differentiable everywhere.*

## Definition of Approximate Sub-Riemannian Differentiability [2000; V.]

Let $E \subset \mathcal{M}$ be a measurable subset of $\mathcal{M}$ and $\varphi : E \to \widetilde{\mathcal{M}}$ be a measurable mapping.

An approximate differential of a mapping $\varphi$ at a point $g$ is the horizontal homomorphism $L : \mathcal{G}^g \to \mathcal{G}^{\varphi(g)}$ of the local Carnot groups such that the set

$$\{v \in B_{cc}(g,r) \cap \mathcal{G}^g : \tilde{d}_{cc}^{\varphi(g)}(\varphi(v), L(v)) > d_{cc}^g(g,v)\varepsilon\}$$

has $\mathcal{H}^\nu$-density zero at the point $g$ for any $\varepsilon > 0$.

## Whitney Type Theorem [2012; Basalaev, V.]

**Theorem.** *Let $\mathcal{M}$, $\widetilde{\mathcal{M}}$ be Carnot manifolds, $E \subset \mathcal{M}$ be a measurable subset of $\mathcal{M}$ and $f : E \to \widetilde{\mathcal{M}}$ be a measurable mapping. The following conditions are equivalent:*

1) *the mapping $f$ is approximately differentiable almost everywhere in $E$;*

2) *the mapping $f$ has approximate derivatives along the basic horizontal vector fields almost everywhere in $E$;*

3) *there is a sequence of the disjoint sets $Q_1, Q_2, \dots$ such that $\mathcal{H}^\nu\Big(E \setminus \bigcup\limits_{i=1}^{\infty} Q_i\Big) = 0$ and every restriction $f|_{Q_i}$ is a Lipschitz mapping;*

4) $f : E \to \widetilde{\mathcal{M}}$ *meets the condition* $\operatorname{ap}\overline{\lim\limits_{x\to g}} \frac{\widetilde{d}_{cc}(f(g),f(x))}{d_{cc}(g,x)} < \infty$.

## Sub-Riemannian Area Formula [2011; Karmanova]

- the sub-Riemannian Jacobian

$$\mathcal{J}^{SR}(\varphi, y) = \sqrt{\det(\widehat{D}\varphi(y)^* \widehat{D}\varphi(y))}.$$

**Theorem.** *Let $\varphi : \mathbb{M} \to \widetilde{\mathbb{M}}$ be a Lipschitz mapping of Carnot manifolds with respect to $cc$-metrics. Then, the area formula holds:*

$$\int\limits_{\mathbb{M}} f(y)\mathcal{J}^{SR}(\varphi, y)\, d\mathcal{H}^\nu(y) = \int\limits_{\widetilde{\mathbb{M}}} \sum_{y:\, y \in \varphi^{-1}(x)} f(y)\, d\mathcal{H}^\nu(x),$$

*where $f : \mathbb{M} \to \mathbb{E}$ ($\mathbb{E}$ is an arbitrary Banach space) is such that the function $f(y)\sqrt{\det(\widehat{D}\varphi(y)^* \widehat{D}\varphi(y))}$ is integrable. Here Hausdorff measures are constructed with respect to quasimetrics $d_2$ (in the preimage) and $\widetilde{d}_2$ (in the image) with the normalizing factor $\omega_\nu$.*

## Sub-Riemannian Coarea Formula [2009; Karmanova, V.]

- the sub-Riemannian coarea factor

$$\mathcal{J}^{SR}_{\widetilde{N}}(\varphi,x)=\sqrt{\det(\widehat{D}\varphi(x)\widehat{D}\varphi(x)^*)}\cdot\frac{\omega_N}{\omega_\nu}\frac{\omega_{\widetilde{\nu}}}{\omega_{\widetilde{N}}}\frac{\omega_{\nu-\widetilde{\nu}}}{\prod\limits_{k=1}^{M}\omega_{n_k-\tilde{n}_k}}.$$

**Theorem.** *Suppose that $\varphi\in C^{M+1}(\mathbb{M},\widetilde{\mathbb{M}})$ is a contact mapping of two Carnot manifolds,* $\dim H_1\mathbb{M}\geq\dim\widetilde{H}_1\widetilde{\mathbb{M}}$*,* $\dim H_i\mathbb{M}-\dim H_{i-1}\mathbb{M}\geq\dim\widetilde{H}_i\widetilde{\mathbb{M}}-\dim\widetilde{H}_{i-1}\widetilde{\mathbb{M}}$*, $i=2,\dots,M$. Then the following coarea formula*

$$\int\limits_{\mathbb{M}}\mathcal{J}^{SR}_{\widetilde{N}}(\varphi,x)f(x)\,d\mathcal{H}^{\nu}(x)=\int\limits_{\widetilde{\mathbb{M}}}d\mathcal{H}^{\widetilde{\nu}}(z)\int\limits_{\varphi^{-1}(z)}f(u)\,d\mathcal{H}^{\nu-\widetilde{\nu}}(u)$$

*holds, where $f:\mathbb{M}\to\mathbb{E}$ ($\mathbb{E}$ is an arbitrary Banach space) is such that the product $\mathcal{J}^{SR}_{\widetilde{N}}(\varphi,x)f(x):\mathbb{M}\to\mathbb{E}$ is integrable.*

THANK YOU FOR YOUR ATTENTION!