THE EULER INTERNATIONAL MATHEMATICAL INSTITUTE
SAINT PETERSBURG STATE UNIVERSITY
THE NATIONAL RESEARCH UNIVERSITY — HIGHER SCHOOL OF ECONOMICS

INTERNATIONAL CONFERENCE

CONSTRUCTIVE NONSMOOTH ANALYSIS AND RELATED TOPICS

ABSTRACTS

SAINT PETERSBURG
2012

УДК 519.85
ББК 22.18
К65

К65 **Конструктивный негладкий анализ и смежные вопросы**. Тезисы докладов международной конференции — СПб.: Изд-во Санкт-Петербургкого университета, 2012. — 182 с.

ISBN 978-5-9651-0643-1

Настоящее издание содержит тезисы докладов, представленных на Международную конференцию «Конструктивный негладкий анализ и смежные вопросы» (CNSA-2012), г. Санкт-Петербург, 18–23 июня 2012 г.

Тематика конференции включает в себя следующие вопросы: конструктивные средства негладкого анализа, негладкие задачи вариационного исчисления и теории управления, задачи негладкой механики, недифференцируемая оптимизация, приложения негладкого анализа, негладкое математическое моделирование, задачи математической диагностики, негладкие методы в теории игр, нелинейные чебышевские аппроксимации и негладкая оптимизация. Сборник представляет интерес для специалистов по указанным областям науки.

Конференция проведена при поддержке Российского Фонда Фундаментальных Исследований (проект № 12-01-06029-г), Национального исследовательского университета — Высшей школы экономики в Нижнем Новгороде.

УДК 519.85
ББК 22.18

Ответственные за выпуск: В. Ф. Демьянов, Г. Ш. Тамасян

Подготовил оригинал-макет: Г. Ш. Тамасян

ISBN 978-5-9651-0643-1

International Conference

CONSTRUCTIVE NONSMOOTH ANALYSIS AND RELATED TOPICS

Saint Petersburg, June 18–23, 2012

ORGANIZERS

The Euler International Mathematical Institute

Saint Petersburg State University

The National Research University – Higher School Of Economics

SPONSORED BY

Russian Foundation for Basic Research

Laboratory of Algorithms and Technologies for Networks Analysis,
Nizhniy Novgorod Branch, Russia

Organizers of the conference are: Vladimir F. Demyanov (St. Petersburg State University), Alexander B. Kurzhanski (Lomonosov Moscow State University), Nadia Zalesskaya (Euler International Mathematical Institute), Tatiana Vinogradova (Euler International Mathematical Institute), Lyudmila Polyakova (St. Petersburg State University), Grigoriy Tamasyan (St. Petersburg State University)

Co-organizers: Panos Pardalos (The National Research University – Higher School of Economics, University of Florida), Boris Goldengorin (The National Research University – Higher School of Economics, N. Novgorod Branch)

The International Program Committee: J.-P. Aubin (France), M. Balinski (France), F.L. Chernous'ko (Russia), F. Clarke (France), V.F. Demyanov (Russia), A.D. Ioffe (Israel), F. Giannessi (Italy), J.-B. Hiriart-Urruty (France), A.B. Kurzhanski (Russia), S.S. Kutateladze (Russia), V.N. Malozemov (Russia), D. Pallaschke (Germany), P. Pardalos (USA), L.A. Petrosyan (Russia), B.T. Polyak (Russia), R.T. Rockafellar (USA), V.M. Tikhomirov (Russia), V.N. Ushakov (Russia), Ya.I. Zabotin (Russia).

Invited Speakers: A. Bagirov (Australia), J. Borwein (Australia), J.-P. Crouzeix (France), G. Di Pillo (Italy), S.I. Dudov (Russia), H. Frankowska (France), M. Gaudioso (Italy), V.V. Gorokhovik (Belarus), J. Gwinner (Germany), Yu. Ledyaev (USA), A. Lewis (USA), M.M. Mäkelä (Finland), R. Mifflin (USA) E.A. Nurminski (Russia), A. Maugeri (Italy), A.I. Minchenko (Belarus), B. Mordukhovich (USA), T.M. Rassias (Greece), G. Stavroulakis (Greece), A.S. Strekalovski (Russia), M. Thera (France), R. Tichatschke (Germany), R. Wets (USA), Z.-Q. Xia (China).

CONTENTS — СОДЕРЖАНИЕ

Adjoint exhausters in extremum conditions

Abbasov M.
abbasov.majid@gmail.com
St. Petersburg State University, Universitetskii prospekt 35, Petergof, 198504, St. Petersburg, Russia

In the talk, new formulations of extremality conditions in terms of adjoint exhausters are stated. The results obtained allow one to construct descent and ascent directions and provide visual geometric interpretation.

Несобственные экзостеры в условиях экстремума

Пусть $f : X \to \mathbb{R}$, где $X \subset \mathbb{R}^n$ — открытое множество. Будем говорить, что у функции f в точке x существует нижний экзостер $E_*(x)$, если имеет место разложение

$$f(x+g) = f(x) + \max_{C \in E_*(x)} \min_{v \in C} (v, g) + o_x(g), \tag{1}$$

а $E_*(x)$ – семейство выпуклых компактов в $\mathbb{R}^n$.

Будем говорить, что у функции f в точке x существует верхний экзостер $E^*(x)$, если имеет место разложение

$$f(x+g) = f(x) + \min_{C \in E^*(x)} \max_{v \in C} (v, g) + o_x(g), \tag{2}$$

а $E^*(x)$ – семейство выпуклых компактов в $\mathbb{R}^n$.

Бесконечно малые в (1), (2) удовлетворяют соотношению

$$\lim_{\alpha \downarrow 0} \frac{o_x(\alpha g)}{\alpha} = 0 \quad \forall\ g \in \mathbb{R}^n,$$

либо

$$\lim_{||g|| \to 0} \frac{o_x(g)}{||g||} = 0 \quad \forall\ g \in \mathbb{R}^n. \tag{3}$$

В работе [1] были получены изящные условия максимума функции f в точке x, когда имеет место разложение (1) и условия минимума, когда имеет место разложение (2). Поэтому нижний экзостер был назван собственным для задачи на максимум, а верхний – собственным для задачи на минимиум. Соответственно верхний экзостер был назван несобственным для задачи на максимум, а нижний – несобственным для задачи на минимиум.

Проблема поиска условий экстремума в терминах несобственных экзостеров впервые была рассмотрена в [2]. В настоящей работе развиваются описанные там идеи, и предлагается иная формулировка условий [4, 5].

Пусть справедливо разложение (1). Обозначим

$$h(g) = \max_{C \in E_*(x)} \min_{w \in C} (w, g).$$

Тогда

— условие $h(g) \geqslant 0$ для любого $g \in \mathbb{R}^n$, является необходимым условием минимума функции f в точке x [3],
— если имеет место (3), то условие $h(g) > 0$ для любого $g \in \mathbb{R}^n$, является достаточным условием строгого локального минимума функции f в точке x [3].

Теорема 1. Для того, чтобы $h(g) \geqslant 0$ для любого $g \in \mathbb{R}^n$, необходимо и достаточно, чтобы в замкнутом положительном полупространстве, порожденным произвольной гиперплоскостью, проходящей через нуль, всегда лежало целиком хотя бы одно из множеств семейства $E_*(x)$, то есть для любого $g \in \mathbb{R}^n$ должно существовать $\widehat{C} \in E_*(x)$ такое, что для любого $v \in \widehat{C} \quad (v, g) \geqslant 0$.

Следствие 1. Для того, чтобы $h(g) \geqslant 0$ для любого $g \in \mathbb{R}^n$, необходимо и достаточно, чтобы в каждом из двух замкнутых полупространств, порожденных произвольной гиперплоскостью, проходящей через нуль, лежало целиком хотя бы одно из множеств семейства $E_*(x)$ (то есть чтобы такая гиперплоскость разделяла какие-то два множества семейства).

Теорема 2. Для того, чтобы $h(g) > 0$ для любого $g \in \mathbb{R}^n$, необходимо и достаточно, чтобы в открытом положительном полупространстве, порожденным произвольной гиперплоскостью, проходящей через нуль, всегда лежало целиком хотя бы одно из множеств семейства $E_*(x)$, то есть для любого $g \in \mathbb{R}^n$ должно существовать $\widehat{C} \in E_*(x)$ такое, что для любого $v \in \widehat{C} \quad (v, g) > 0$.

Следствие 2. Для того, чтобы $h(g) > 0$ для любого $g \in \mathbb{R}^n$, необходимо и достаточно, чтобы в каждом из двух открытых полупространств, порожденных произвольной гиперплоскостью, проходящей через нуль, лежало целиком хотя бы одно из множеств семейства $E_*(x)$ (то есть чтобы такая гиперплоскость строго разделяла какие-то два множества семейства).

Аналогично формулируются условия нестрогого (строгого) локального максимума в терминах верхнего экзостера.

Полученные результаты дают теоретическую возможность поиска направлений спуска (подъема) и допускают наглядную геометрическую интерпретацию.

Работа выполнена при поддержке РФФИ, проект № 12-01-00752.

ЛИТЕРАТУРА

[1] Demyanov V. F., “Exhausters and convexificators – new tools in nonsmooth analysis”, *Quasidifferentiability and related topics*, Nonconvex Optim. Appl., 43, Kluwer Acad. Publ., Dordrecht, 85–137,(2000).

[2] Рощина В. А., “Ограниченные экзостеры и условия экстремума”, Процессы управления и устойчивость: Труды 36-й международной научной конференции аспирантов и студентов / Под ред. Н. В. Смирнова, В. Н. Старкова. СПб.: Изд-во С.-Петерб. ун-та, С. 521–524, (2005)

[3] Демьянов В.Ф., Рубинов А.М. “Основы негладкого анализа и квазидифференциальное исчисление”, М.: Наука, 432 с., (1990)

[4] Abbasov M. E., Demyanov V. F., “Proper and adjoint exhausters in Nonsmooth analysis: optimality conditions”, Forthcoming.

[5] Abbasov M. E., “Extremality conditions in terms of adjoint exhausters”, *Vestnik of Saint-Petersburg University*, Ser. 10. Applied mathematics, informatics, control processes. N. 2. 3–8, (2011).

B^{-1}-convex Sets and Functions

Adilov G., Yesilce I.

gabiladilov@gmail.com, ilknuryesilce@gmail.com

Faculty of Education, Mersin University Yenisehir Campus, Mersin, 33343, Turkey

A subset U of $\mathbb{R}^n_{++}$ is B^{-1}-convex if for all $x, y \in U$ and all $\lambda \in [0,1]$ one has $\lambda x \wedge y \in U$. These sets were introduced and studied in [2]. In this paper, we examine some properties of B^{-1}-convex sets and also define and analyse B^{-1}-convex functions.

And finally, we compare B^{-1}-convexity with B-convexity (were introduced in [3] and studied in [1], [3], [4], [5]).

References

[1] *G. Adilov and A. Rubinov,* “B-convex Sets and Functions”, *Numerical Functional Analysis and Optimization*, 27 (3-4), 237-257 (2006).

[2] *G. Adilov and I. Yesilce,* “$B^{-1}-$convex Sets and $B^{-1}-$measurable Maps”, *Numerical Functional Analysis and Optimization*, 33 (2), 1-11 (2012).

[3] *W. Briec and C.D. Horvath,* “B-convexity”, *Optimization*, 53, 103-127 (2004).

[4] *W. Briec, C.D. Horvath,* “Halfspaces and Hahn–Banach like properties in B-convexity and Max-Plus convexity”, *Pacific J. Optim.*, 4, (2), 293–317 (2008).

[5] *W. Briec, C.D. Horvath and A.M. Rubinov,* “Separation in B-convexity”, *Pacific J. Optimiz.*, 1, 13-30 (2005).

Quantitative stability of a generalized equation. Application to non-regular electrical circuits

Adly S., Cibulka R., Outrata J.V.
samir.adly@unilim.fr, cibi@kma.zcu.cz, outrata@utia.cas.cz
XLIM UMR-CNRS 6172, Université de Limoges, France, Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic

The paper is devoted to the study of several stability properties (such as Aubin property, calmness and isolated calmness) of a special non-monotone generalized equation. Given matrices $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ with $m \leqslant n$, consider the problem of finding, for a vector $p \in \mathbb{R}^n$, the solution $z \in \mathbb{R}^n$ to the inclusion

$$p \in f(z) + BF(Cz). \tag{1}$$

In [1], the authors considered the special case of the above inclusion with the linear single-valued part f, with $B = C^T$, and F being the Clarke subdifferential of the so-called Moreau-Panagiotopoulos super potential. To be more precise, this potential was supposed to be

$$J(x) := j_1(x_1)+j_2(x_2)+\cdots+j_m(x_m) \quad \text{whenever} \quad x = (x_1, \dots, x_m)^T \in \mathbb{R}^m,$$

with $j_i : \mathbb{R} \to \mathbb{R}$ being a locally Lipchitz continuous function for each $i = 1, \dots, m$.

We will refer to several additional assumptions in the main results. Namely,

- $(A1)$ B is injective;
- $(A2)$ f is continuously differentiable on $\mathbb{R}^n$;
- $(A3)$ F has closed graph;
- $(A4)$ C is surjective; and
- $(A5)$ there are $F_j : \mathbb{R} \rightrightarrows \mathbb{R}$, $j \in \{1, \dots, m\}$ such that $F(x) = \prod_{j=1}^{m} F_j(x_j)$ whenever $x = (x_1, \dots, x_m)^T \in \mathbb{R}^m$.

Denote by Φ the set-valued mapping from $\mathbb{R}^n$ into itself defined by $\Phi(z) = f(z) + BF(Cz)$ whenever $z \in \mathbb{R}^n$. Our aim is to investigate stability properties such as Aubin continuity, calmness and isolated calmness of the solution mapping $S := \Phi^{-1}$. Especially, the verifiable conditions ensuring these properties are given in terms of the input data f, F, B and C.

We show that this theory is of great interest for the design of electrical circuits involving nonsmooth and non-monotone electronic devices like diacs (DIode Alternating Current) and SCR (Silicon Controlled Rectifiers). Circuits with other devices like Zener diodes, thyristors, varactors and transistors can be analyzed in the same way.

References

[1] Adly, Samir; Outrata Jiří V., Qualitative stability of a class of non-monotone variational inclusions. Application in electronics, to appear in Journal of Convex Analysis.

[2] K. Addi, S. Adly, B. Brogliato, D. Goeleven, *A method using the approach of Moreau and Panagiotopoulos for the mathematical formulation of non-regular circuits in electronics*. Nonlinear Anal. Hybrid Syst. 1 (2007), no. 1, 30P43,

[3] B. S. Mordukhovich. *Stability theory for parametric generalized equations and variational inequalities via nonsmooth analysis*, Trans. Amer. Math. Soc. 343 (1994), no. 2, 609P657.

[4] B. S. Mordukhovich and J. Outrata. On second-order subdifferentials and their applications. *SIAM J. Optim.*, 12(2001), pp. 139–169.

Nonsmooth Analysis and Identification of Structures

Aliawdin P.

Piotr-Aliawdin@yandex.ru

University of Zielona Gyra, Poland Prof. Z. Szafrana 1, 65-516, Zielona Gyra, Poland

Introduction. Protection of buildings and structures from seismic actions is very important. One way of protection is equipment of their load-bearing structures with systems of different elements- absorbing the energy of external actions [1]. This paper presents a nonsmooth mathematical model of the limit analysis of systems with destructive brittle and plastic elements providing such protection. Next, the identification of computational models is considered on examples of construction vibration and heat transfer in road pavement structure.

1. A mathematical model of structures with seismic-protected systems. Some elements such systems can be abruptly shut down (elastic-brittle ones), and some can be damaged as a result of plastic flow (elastic-plastic ones).

Let us assume the problem of load-bearing capacity of such structures as a generalized dynamic shakedown problem [2, 3] taking into account small system displacements under low cyclic external actions. First we write the FEM equation of motion for a damped discrete elastic system under loading F, expressed in matrix notation, as follows:

$$[M]\ddot{u} + [C]\dot{u} + [K]u = F, \tag{1}$$

where: $[M]$, $[C]$, $[K]$ are accordingly structural mass, damping and stiffness matrices; $\ddot{u}$, $\dot{u}$, u are accordingly nodal acceleration, velocity and displacement vectors; F is a vector of load as a function of time t.

The vector F belongs to the set Ω_F, described by the vectors of single loadings F_j, $j \in J$; J is a set of single loadings. The set Ω_F has to include a natural structures stress state $F = 0$ (i.e. $0 \in \Omega_F$) and generally is nonconvex. For the purpose of simplification it may be approximated by the convex domain.

The "elastic" solution of equation (1), for the known initial conditions, is used then as basis for analysis of real inelastic system. Namely the problem of load-bearing capacity of structures made of perfectly elastic-plastic and elastic-brittle elements, under variable loads is formulated as follows. Find the vectors of single loadings F_j, $j \in J$, a vector of load F, as well as the vector of residual forces $S^r = (S^r_{pl}, S^r_{br})$ such, that

$$\sum_{j \in J} T^T_{F_j} F_j \longrightarrow \max, \tag{2}$$

$$\varphi_{pl/\Omega F d}(S^e_{pl} + S^r_{pl}, K_{pl}) \leqslant 0, \tag{3}$$

$$\varphi_{br}(S^e_{br} + S^r_{br}, K_{br}) \leqslant 0, \tag{4}$$

$$S^e = \omega_F F, \tag{5}$$

$$A_{pl} S^r_{pl} + A_{br} S^r_{br} = 0, \tag{6}$$

$$S^r_{br} \geqslant 0_{br}. \tag{7}$$

Here T_{F_j} are the vectors of weight coefficients corresponding to the vectors of single j-loading F_j, $j \in J$; $\varphi_{pl/\Omega F}$ are the yield functions, depending on the set Ω_F external actions (loadings F_j) for elastic-plastic elements; φ_{br} are the strength functions for brittle elements; ω_F is the matrix of loads influence on the elastic forces $S^e = (S^e_{pl}, S^e_{br})$; $A = (A_{pl}, A_{br})$ is a matrix of equilibrium Eqns (6). The subscripts *pl* and *br* relate to the elastic-plastic and elastic-brittle elements, superscripts e and r — to the elastic and residual forces.

After finding the failure mechanism (active constraints (3), (4) in problem (1)–(7) one must take into account the dynamic effects of this destruction in the some iterative procedure [2].

Note that in addition to loads F_j in this problem it is possible to consider the dislocations d_j, $j \in J$, as similar external actions. By changing the dislocation d_j we can also optimize the state of structures prestressing. In the particular case of one-pass loading the problem (1)–(7) is simplified, while $|J| = 1$, this problem is also applicable for the analysis of the progressive destruction of structures.

2. The procedure of parameter identification for dynamic system. The identification was performed by two criteria, namely the criterion of least squares and the criterion of minimax [4].

Criterion of a minimax tuning of FE model for the natural frequencies dynamic system,

$$\rho(x) = \max_{i \in I} \left| \frac{f_{mi} - f_{ci}(x)}{(f_i^+ + f_i^-)/2} \right|, \tag{8}$$

has to minimize under eqns (1), and known initial conditions, where $f_m i$ are measured values i-th natural frequencies, $f_i = \omega_i/(2\pi)$, $i \in I$; I is a set of analyzed frequencies;

$f_{ci} \equiv f_{ci}(x)$ are values i-th natural frequencies, calculated from eqns (1) and known initial conditions, depending on the vector x;

x is a vector of parameters of system $x \in R^n$; n is a number parameters;

f_i^+, f_i^- are accordingly high and low values of i-th natural frequencies of vibration given by the designer.

As far as the identification of system is ill-posed problem, criterion of optimality (8) is modified,

$$\rho_\alpha(x) = \rho(x) + \alpha \|x\|^2, \tag{9}$$

where α is a Tikhonov regulation parameter, $\alpha > 0$;

$$\|x\| = \sqrt{\sum_{i \in 1:n} x_i^2}.$$

Choice of regulation parameter α on k-step of calculations process, $k = 0, 1, \ldots, K$, was made by the formula

$$\alpha_k = \alpha_0 q^k, \quad q > 0. \tag{10}$$

The next parameter α_{k+1} is find from the following $(K+1)$-th problem:

$$\|x_{\alpha_{k+1}} - x_{\alpha_k}\| \longrightarrow \min_{k \in 0:K}, \tag{11}$$

up to comprehensible accuracy for solving an initial problem (8).

3. Numerical analysis of dynamic and heat transfer identification problems for the constructions. First we analyze the problem of a construction vibration of 9-storey residential building located near the subway shallow in Minsk [2]. Two inverse problems are formulated here, to identify material parameters of structure by smooth (least squares) and nonsmooth (minimax) optimization.

Secondly the same approach was used for heat transient regime in real road pavement structure [5] respectively the boundary conditions and heat material parameters. It was shown that the criterion of least squares does not give trust results and only nonsmooth criterion of minimax lead to a trust result of the identification.

References

[1] A.V. Minasian, "Bearing capacity reserves of seismic-protected systems in terms of energy viewpoint", Proceedings of the International Conference "Actual problems of research on the theory of structures", Moscow, V.A. Kucherenko TSNIISK, 1, 270-276 (2009). (in Russian).

[2] P. Aliawdin, Y. Muzychkin, "Limit analysis of structures with destructible elements under impact loadings", Engineering Transactions, 59, No. 3, 139-159 (2011).

[3] P. Aliawdin, Limit analysis of structures under variable loads, Minsk, Technoprint, (2005). (in Russian).

[4] V.F. Dem'anov et al., Quasidifferentiability and Nonsmooth Modelling in Mechanics, Engineering and Economics, Nonconvex Optimization and Its Applications. Vol. 10, Kluwer Academic Publishers, (1996).

[5] P. Aliawdin, J. Poiczycski, "Heat transfer in road pavement structure and identification of its layers thermal parameters", Civil and environmental engineering reports, University of Zielona Gyra Press, Poland, No. 5, 63- 78 (2010).

Solvability of Systems of Linear Interval Equations via the Codifferential Descent Method

Angelov T.

angelov.t@gmail.com

St. Petersburg State University, Universitetskii prospekt 35, Petergof, 198504,
St. Petersburg, Russia

A system of linear interval equations

$$\mathbf{A}x = \mathbf{b}, \tag{1}$$

is considered in the works [1, 2, 3]. Here $\mathbf{A} = (\mathbf{a}_{ij})$ is an interval $(m \times n)$ matrix and $\mathbf{b} = (\mathbf{b}_i)$ is an interval m-vector. System (1) is understood as a family of all systems of linear equations $Ax = b$ of the same structure with the matrices $A \in \mathbf{A}$ and vectors $b \in \mathbf{b}$.

An interval $\mathbf{a}$ is defined by its lower $\underline{\mathbf{a}}$ and upper $\overline{\mathbf{a}}$ bounds, so that $\mathbf{a} = [\underline{\mathbf{a}}, \overline{\mathbf{a}}] = \{x \in \mathbb{R} \,|\, \underline{\mathbf{a}} \leqslant x \leqslant \overline{\mathbf{a}}\}$. We will need the following definitions:

$\operatorname{mid} \mathbf{a} = \frac{1}{2}(\overline{\mathbf{a}} + \underline{\mathbf{a}})$ — midpoint of an interval $\mathbf{a}$,

$\operatorname{rad} \mathbf{a} = \frac{1}{2}(\overline{\mathbf{a}} - \underline{\mathbf{a}})$ — radius of an interval $\mathbf{a}$,

$$\langle \mathbf{a} \rangle = \begin{cases} \min\{|\overline{\mathbf{a}}|, |\underline{\mathbf{a}}|\}, & \text{if } 0 \notin \mathbf{a}, \\ 0, & \text{otherwise}, \end{cases}$$ — mignitude of an interval $\mathbf{a}$, the closest point to zero inside an interval $\mathbf{a}$.

By the (weak) *solution set* of a system of linear interval equations (1) is meant the set

$$\Xi(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \,|\, Ax = b \text{ for some } A \in \mathbf{A},\ b \in \mathbf{b}\},$$

constructed of all possible solutions of the systems $Ax = b$ with $A \in \mathbf{A}$ and $b \in \mathbf{b}$ [2, 3].

The following statement holds [1].

Statement. The expression

$$\operatorname{Uni}(x, \mathbf{A}, \mathbf{b}) = \min_{1 \leqslant i \leqslant m} \left\{ \operatorname{rad} \mathbf{b}_i - \left\langle \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right\rangle \right\} \tag{2}$$

defines the functional $\operatorname{Uni} : \mathbb{R}^n \to \mathbb{R}$, such that the belonging of a vector $x \in \mathbb{R}^n$ to the solution set $\Xi(\mathbf{A}, \mathbf{b})$ of the system of linear interval equations $\mathbf{A}x = \mathbf{b}$ is equivalent to the nonnegativity of the functional Uni in x,

$$x \in \Xi(\mathbf{A}, \mathbf{b}) \quad \Longleftrightarrow \quad \operatorname{Uni}(x, \mathbf{A}, \mathbf{b}) \geqslant 0.$$

Taking into account that the functional (2) is nonsmooth and multi-extremal, it is more natural to study it by means of tools of codifferential calculus [4], than by means of the ones discussed in the paper [1].

Optimization of the functional (2) by the codifferential descent method is suggested. Special software package dealing with such problems is created.

References

[1] *S. P. Shary.* Solvability of Interval Linear Equations and Data Analysis under Uncertainty // Automation and Remote Control. Springer. Vol. 73(2) (2012). P.310-322.

[2] *S. P. Shary.* Finite-dimensional Interval Analysis. Novosibirsk, (2011). Electronic book, accessible at http://www.nsc.ru/interval/Library/InteBooks (In Russian)

[3] *A. Neumaier* Interval Methods for Systems of Equations. Cambridge: Cambridge University Press, (1990).

[4] *V. F. Demyanov, A. M. Rubinov.* Foundations of Nonsmooth analysis and Quasidifferential Calculus, Moscow, Nauka (1990)

A Nonmonotone Proximal Bundle Method with (Potentially) Continuous Decisions on Stepsize

Astorino A., Frangioni A., Fuduli A., Gorgone E.
astorino@icar.cnr.it, frangio@di.unipi.it,
antonio.fuduli@unical.it, egorgone@deis.unical.it
Istituto di Calcolo e Reti ad Alte Prestazioni C.N.R., Via P. Bucci Cubo 41C, Rende (CS), 87036, Italia

We discuss a numerical algorithm for minimization of a convex nondifferentiable function belonging to the family of bundle methods. Unlike all of its brethren, the approach does not rely on measuring descent of the objective function at the so-called "serious steps", while "null steps" only serve at improving the descent direction in case of unsuccessful steps. Rather, a merit function is defined which is optimized at each iteration, leading to a (potentially) continuous choice of the stepsize between zero (the null step) and one (the serious step). By avoiding the discrete choice the convergence analysis is simplified, and we can more easily obtain sharp efficiency estimates for the method. Simple choices for the step selection actually reproduce the dichotomic 0/1 behavior of standard proximal bundle methods, but shedding new light on the rationale behind the process, and ultimately with different rules. Yet, using nonlinear upper models of the function in the step selection process can lead to actual fractional steps.

References

[1] *A. Astorino, A. Frangioni, M. Gaudioso, and E. Gorgone,* "Piecewise Quadratic Approximations in Convex Numerical Optimization", *SIAM Journal on Optimization*, 21, No. 4, 1418-–1438 (2011).

[2] *A. Frangioni,* "Generalized Bundle Methods", *SIAM Journal on Optimization*, 13, No. 1, 117-–156 (2002).

[3] *F. Facchinei, S. Lucidi,* "Nonmonotone Bundle-Type Scheme for Convex Nonsmooth Minimization", *Journal of Optimization Theory and Applications*, 76, No. 2, 241-–257 (1993).

Smoothing techniques in nonsmooth optimization and applications

Bagirov A.
a.bagirov@ballarat.edu.au
School of Science, Information Technology and Engineering, University of Ballarat, University Drive, Mount Helen, Ballarat, Victoria, 3350, Australia

In this talk we discuss various smoothing techniques for solving nonsmooth optimization problems with known structures. Such techniques involve the exponential and hyperbolic smoothing functions. The possible extension of these techniques to general nonsmooth optimization problems is also discussed. The smoothing techniques are applied for solving quasidifferentiable optimization problems. The efficiency of algorithms based on smoothing techniques will be demonstrated by applying them to the well known nonsmooth optimization test problems.

Metric Projection, Weak and Strong Convexity

Balashov M.
balashov73@mail.ru
Moscow Institute of Physics and Technology, Institutskii pereulok 9, Dolgoprudny, Moscow Region, 141700, Russia

Current investigations were inspired by works [1, 2, 3, 4, 5, 6, 7].

We discuss the relationship between the uniqueness and continuity of the metric projection on the closed sets in some class of Banach spaces (all over reals) and the properties of considered sets. Firstly we consider when the next problem

$$\inf_{a\in A}\|x-a\| \tag{1}$$

has unique solution for any point x from some neighborhood of the set A, namely for each $x\in U(R,A)=\left\{y\in E \mid 0<\varrho(y,A)=\inf\limits_{a\in A}\|y-a\|<R\right\}$.

We shall define $\pi(x,A)=\{a\in A \mid \|x-a\|=\varrho(x,A)\}$. For closed bounded set $A\subset E$ we also define $T_A(R)=\left\{x\in E \mid \sup\limits_{a\in A}\|x-a\|>R\right\}$.

Secondly, we consider the similar problem

$$\sup_{a\in A}\|x-a\|. \tag{2}$$

The studied question will be the same: when for each point x, x is sufficiently far from the set A (namely, $x\in T_A(R)$), problem (2) has unique solution?

Definition 1. ([7]). A subset $A \subset E$ is called *proximally smooth* with the constant $R > 0$, if the distance function $x \to \varrho(x, A)$ is continuously differentiable on the set $U(R, A)$.

Definition 2. ([7, 9]). We say that a subset $A \subset E$ satisfies *P-supporting condition* with the constant $R > 0$, if for any such point $x \in U(R, A)$, that $a \in \pi(x, A)$, we have inequality

$$\varrho\left(a + \frac{R}{\|x-a\|}(x-a), A\right) \geqslant R.$$

Theorem 1. ([9, Theorem 2.4]). *Let E be uniformly convex and uniformly smooth Banach space; $A \subset E$ be closed subset, $R > 0$. The next conditions are equivalent:*

(i). *The set A satisfies to the P-supporting condition with the constant R.*

(ii). *The set A is proximally smooth with the constant R.*

(iii). *The projection mapping $x \to \pi(x, A)$ is singleton and continuous on the set $U(R, A)$.*

Theorem 1 was proved in [11] for the moduli of convexity and smoothness of the space E of the power type.

Theorem 1 determines some class of nonconvex sets with the same approximation properties (in $U(R, A)$) as convex sets. We shall call such class *weakly convex sets.*

A function $f : E \supset U \to \mathbb{R}$ is called *weakly convex with the modulus* $\mu : [0, \operatorname{diam} U) \to [0, +\infty)$, ($\mu(t)$ increasing, $\mu(0) = 0$) on the set U, if it is continuous on the set U and for any pair of points $x_0, x_1 \in U$, such that $[x_0, x_1] \subset U$, and for any number $\lambda \in [0, 1]$, we have $f((1-\lambda)x_0 + \lambda x_1) \leqslant (1-\lambda)f(x_0) + \lambda f(x_1) + \lambda(1-\lambda)\mu(\|x_0 - x_1\|)$.

Theorem 2. *For any uniformly convex and uniformly smooth Banach space any nonconvex proximally smooth set has weakly convex distance function in some neighborhood with modulus μ of the type $t^2 = O(\mu(t))$, $t \to +0$ and $\mu(t) = o(t)$, $t \to +0$. The converse statement also holds true.*

In the case of the Hilbert space we obtain precise relationships between the size of neighborhood of the set and modulus of weak convexity in the previous theorem.

We shall say that a closed convex set A from a Banach space E is *summand of the ball of radius $R > 0$,* if there exists another closed convex set B such that $A + B = B_R(0)$. As we proved in [12, Theorem 4.2.7] in the Hilbert space any summand of the ball of radius $R > 0$ A should be an intersection of shifts of the ball $B_R(0)$, i.e. $A = \bigcap\limits_{x \in X} B_R(x)$. We shall call a set of the type $A = \bigcap\limits_{x \in X} B_R(x) \neq \varnothing$ *strongly convex of radius $R > 0$.*

Theorem 3. ([8, Theorems 1, 7]) *Let a Banach space E be strictly convex and reflexive, the subset $A \subset E$ be closed and convex. Then*

1) If A is summand of the ball of radius R, then for any $x \in T_A(R)$ problem (2) has unique solution;

2) If $\dim E < \infty$ and for any $x \in T_A(R)$ problem (2) has unique solution, then the set A is summand of the ball of radius R.

We also shall discuss generalization of Theorem 3 for the case of the Hilbert and Banach spaces.

Theorem 4. ([10]) *Let $\mathbb{H}$ be the Hilbert space and $A \subset \mathbb{H}$ be a closed convex set. Then the properties*

1) $A = \bigcap\limits_{x \in X} B_R(x) \neq \varnothing$,

2) for any $\varrho > 0$ and any points $x_0, x_1 \in \mathbb{H} \setminus (A \cup U(\varrho, A))$, $a_i = \pi(x_i, A)$, $i = 0, 1$, we have

$$\|a_0 - a_1\| \leqslant \frac{R}{R + \varrho} \|x_0 - x_1\|,$$

are equivalent.

References

[1] *L. P. Vlasov, Approximative properties of sets in normed linear spaces, Russ. Math. Surv.,* (1973), 28 (6), 1-66.

[2] *V. S. Balaganskii, L. P. Vlasov,* The problem of convexity of Chebyshev sets, *Russ. Math. Surv.,* (1996), 51 (6), 1127-1190.

[3] *S. B. Stechkin,* Approximative properties of sets in linear normed spaces, *Rev. Roum. Math. Pur. et Appl.,* (1963), 8(1), 5-18.

[4] *M. Edelstein,* Farthest points of sets in uniformly convex Banach spaces, *Israel J. Math.,* (1966), 4(3), 171-176.

[5] *J.-P. Vial,* Strong and weak convexity of sets and functions, *Math. Oper. Res.,* (1983), 8:2, 231–259.

[6] *E. S. Polovinkin,* Strongly convex analysis, *Sbornik: Math.,* 187:2 (1996), 259–286.

[7] *F. H. Clarke, R. J. Stern, P. R. Wolenski,* Proximal smoothness and lower–C^2 property, *J. Convex Anal.,* (1995), 2:1-2, 117-144.

[8] *M. V. Balashov, G. E. Ivanov,* On farthest points of sets, *Mathematical Notes,* 2006, 80:2, 159-166.

[9] *M. V. Balashov, G. E. Ivanov,* Weakly convex and proximally smooth sets in Banach spaces, *Izvestiya: Mathematics,* (2009), 73(3), 455-499.

[10] *M. V. Balashov, M. O. Golubev,* About Lipschitz condition of the metric projection in the Hilbert space, *Proceedings of 54th Scientific Conference of MIPT "Problems of Fundamental and Applied Natural and Technical Sciences in the Modern Information Society". Control and Applied Mathematics,* (2011), Vol. 1, 34. (in Russian)

[11] *F. Bernard, L. Thibault, N. Zlateva,* Characterization of proximal regular sets in super reflexive Banach spaces, *J. Convex Anal.,* (2006), 13:3-4, 525-559.

[12] *E. S. Polovinkin, M. V. Balashov, Elements of Convex and Strongly Convex Analysis,* Fizmatlit, Moscow, 2007. 436 p. (in Russian)

Dual fuzzy cones

Baskov O.

Saint Petersburg State University, Universitetskiy pr., 35, Saint Petersburg, 198504, Russia

Since the second half of the 20th century the fuzzy set theory has been extensively developed. It turned out that various practical problems can be handily formulated and explored with the use of the fuzzy logic. Known theorems started to get generalized to the fuzzy case. This report is devoted to a generalization of the concept of dual cones.

In crisp case a dual cone of a given set C consists of all vectors x whose scalar product with any vector of C is nonnegative:

$$\operatorname{Dual} C = \{y : \forall x \in C \Rightarrow xy \geqslant 0\}.$$

If there exists such vector $y \in C$ that $xy < 0$, then the vector x does not belong to the dual cone of C. Based on these considerations, the concept of the dual cone can be generalized as follows. For vectors x, y such that $xy < 0$, where y belongs to C with some grade α (we write $\lambda_C(y) = \alpha$), x does not belong to $\operatorname{Dual} C$ with the degree of certainty equal to α. Thus, we define the membership fuction of the dual cone of C to be

$$\lambda_{\operatorname{Dual} C}(x) = \inf_{y : xy < 0} (1 - \lambda_C(y)).$$

It has been shown that thus defined fuzzy set $\operatorname{Dual} C$ is a fuzzy cone. Moreover, it is always convex and closed. Some other properties of the crisp dual cones may be generalized as well. For instance, if two fuzzy sets A and B satisfy $A \subset B$, then $\operatorname{Dual} A \supset \operatorname{Dual} B$. It also true that the dual cone of the dual cone of a given fuzzy set C coincides with the minimal closed convex cone containing C:

$$\operatorname{Dual} \operatorname{Dual} C = \operatorname{cl} \operatorname{cone} C.$$

Particularly, the dual cone of the dual cone of a finitely generated fuzzy cone is the cone itself.

The concept of dual fuzzy cone can be used to solve a fuzzy multicriteria choice problem in a framework of the so-called axiomatic approach of reducing the Pareto set which is based on a certain kind of numerical data on a fuzzy preference relation of the Decision Maker [1]. Under 'reasonable' axioms the data is represented as a set of vectors $u^1, \ldots, u^k$ with associated degrees of certainty $\alpha^1, \ldots, \alpha^k$. The cone of the preference relation then must contain a fuzzy convex conical hull $M = \text{cone}\left\{u^1/\alpha^1, \ldots, u^k/\alpha^k, e^1/1, \ldots, e^m/1\right\}$, where $e^1, \ldots, e^m$ form the basis of the criteria space R^m, and x/γ denotes the vector x with associated degree of certainty γ. The generators of the dual cone of M allow to construct new criteria, the Pareto set with respect to which is narrower than the initial Pareto set. For finding the generators a generalization of a special case of the Motzkin – Burger algoithm [2] has been made.

The author thanks the Russian Fund for Basic Research for financial support (project No. 11-07-00449).

References

[1] V.D. Noghin, "The Edgeworth – Pareto Principle and the Relative Importance of Criteria in the Case of a Fuzzy Preference Relation", *Computational Mathematics and Mathematical Physics*, 43, No. 11, 1604–1612 (2003).

[2] S.N. Chernikov, *Linear inequalities*, M., Nauka Publ. (1968).

Tolerance-based Algorithm for the Assignment Problem

Batsyn M., Bychkov I., Goldengorin B., Pardalos P.

mbatsyn@hse.ru, il.bychkov@gmail.com, b.goldengorin@rug.nl, pardalos@ufl.edu

Laboratory of Algorithms and Technologies for Networks Analysis and Department of Applied Mathematics and Informatics, National Research University Higher School of Economics, 136 Rodionova, Nizhny Novgorod, 603093, Russia

The Assignment Problem (AP) is the problem of assigning n workers to n jobs so that the total cost of this assignment is minimal, each worker is assigned to only one job and for each job only one worker is assigned to it. The cost paid to worker i for job j is denoted as $c(i, j)$. Though it is well known that the AP is polynomially solvable, efficient algorithms for solving it are important. The AP is used as a relaxation for many NP-hard problems and so it should be solved as fast as possible.

The approaches proposed for the solution of the AP can be classified into three classes: primal-dual (shortest path) algorithms, pure primal algorithms, and pure dual algorithms. An experimental evaluation of a best representative algorithm from each of these classes shows that the best algorithm is the Hungarian algorithm based on the primal-dual (shortest path) approach and König-Egervary's theorem [1]. The time complexity of the Hungarian algorithm is $O(n^3)$. In paper [1] eight codes are selected and compared on a wide set of dense instances containing both randomly generated and benchmark instances. These codes represent the most popular and efficient methods for solving the AP. Since the Jonker and Volgenant's code (JV-algorithm described in [3]) has a good and stable average performance for all tested instances and it is based on the Hungarian algorithm, we compare the algorithm suggested in this paper against the JV-algorithm.

In this paper we develop a tolerance-based branch-and-bound algorithm for solving the AP. Our main contributions are represented by the tolerance based lower bound and branching rule. In spite of the fact that the AP is polynomially solvable by the Hungarian algorithm, we show that the suggested branch-and-bound algorithm (which is exponential) can be more efficient on a number of instances.

A notion of a tolerance comes from sensitivity analysis. In a minimization problem the upper tolerance of an element is the maximal increase of its value, such that the current optimal solution of this problem remains optimal. The lower tolerance of an element is the maximal decrease of its value, such that the current optimal solution of this problem remains optimal. As it is shown in [2], the upper tolerance of an element (contained in the optimal solution) can be calculated as the change in the value of the optimal solution when this element is forbidden. For elements which are not in the optimal solution the upper tolerance is equal to infinity. In the same way the lower tolerance of an element (not contained in the optimal solution) is equal to the change in the value of the optimal solution when this element is inserted to the optimal solution.

For solving the AP with branch-and-bound algorithm we consider the Relaxed Assignment Problem (RAP) for which the constraint requiring that each job is assigned to exactly one worker is removed. The optimal solution of the RAP can contain unassigned jobs (without any worker) and overassigned jobs (with two or more workers). To solve the RAP it is necessary to find the minimal element in each row of the cost matrix $c(i, j)$. Consider the cost matrix in the following example (the RAP solution is shown by square brackets):

$$\begin{array}{cccc} 5 & [1] & 2 & 5 \\ 8 & 3 & [1] & 5 \\ 3 & [1] & 8 & 4 \\ [3] & 4 & 5 & 8 \end{array}$$

This solution is not feasible for the AP because job 2 is overassigned and job 4 is unassigned. The objective function is $f = 6$ and it is a lower bound for the AP solution. We describe the suggested tolerance-based branch-and-bound algorithm on this example. Elements $(1, 2)$ and $(3, 2)$ can't be present in the optimal solution of AP simultaneously, so one of them should be forbidden. If, for example, we forbid element $(1, 2)$ then the increase in the objective function of the RAP (or upper tolerance of $(1, 2)$) will be $2-1 = 1$, because the next minimal value in the first row is 2. Besides, one of the elements in the 4-th column should be present in the optimal solution of AP. So we should insert one of them to the optimal solution. If, for example, we insert element $(2, 4)$ then the increase in the objective function of the RAP (or lower tolerance of $(2, 4)$) will be $5 - 1 = 4$. The tolerance-based approach includes the following steps. First the upper tolerances $u(i, j)$ of all the unfeasible elements from the optimal solution ($(1, 2)$ and $(3, 2)$ in this case) are computed: $u(1, 2) = 2-1 = 1$, $u(3, 2) = 3-1 = 2$. The minimal upper tolerance is $u = 1$. Then we compute the lower tolerances $l(i, j)$ of all the unfeasible elements which are not in the optimal solution ($(1, 4)$,$(2, 4)$,$(3, 4)$ and $(4, 4)$ in this case): $l(1, 4) = 5-1 = 4$, $l(2, 4) = 5 - 1 = 4$, $l(3, 4) = 4 - 1 = 3$, $l(4, 4) = 8 - 3 = 5$. The minimal lower tolerance is $l = 3$. We compute the maximum of these minimal upper and lower tolerances: $\max\{u,\ l\} = 3$, which is called the bottleneck tolerance, because the solution of the RAP should be increased at least for this value in order to obtain a feasible solution of the source AP. A new lower bound for the AP solution is $6 + 3 = 9$. The "bottleneck" is provided by element $(3, 4)$ and it is efficient to branch by this element. We consider two branches: in the first branch element $(3, 4)$ is inserted to the optimal solution of the RAP, in the second branch it is forbidden. So for the first branch we have the following solution:

$$\begin{array}{cccc} 5 & [1] & 2 & 5 \\ 8 & 3 & [1] & 5 \\ 3 & 1 & 8 & [4] \\ [3] & 4 & 5 & 8 \end{array}$$

It is a feasible solution for the AP with $f = 9$. This gives us an upper bound for the optimal solution of the AP. In the second branch we have forbidden element $(3, 4)$ with the minimal lower tolerance. So the new minimal lower tolerance is $l = 4$ and the tolerance-based lower bound is equal to $6 + 4 = 10$. It is greater then the current upper bound of 9, which means that the second branch is stopped here and $f = 9$ is the optimal solution of the AP.

The suggested branch-and-bound algorithm is additionally improved by a heuristic construction of a feasible AP solution on every branch. It works faster than the Hungarian algorithm for the instances on which the Hungarian algorithm makes a lot of "covering" steps.

References

[1] *M. Dell'Amico, P. Toth* Algorithms and Codes for Dense Assignment Problems: the State of the Art. *Discrete Applied Mathematics*, 100 (1-2). 2000. P. 17-48.

[2] *Sierksma G., Turkensteen M., B.Goldengorin* Tolerance based algorithms for the ATSP. *Lecture Notes in Computer Science*, 3353. 2004. P. 222-234.

[3] *R. Jonker, A. Volgenant* A Shortest Augmenting Path Algorithm for Dense and Sparse Linear Assignment Problems. *Computing*, 38. 1987. P. 325-340.

Network game of emission reduction

Belitskaia A., Petrosyan L.
aanytka@yandex.ru, spbuoasis7@peterlink.ru
St. Petersburg University, Universitetskii prospekt 35, Petergof, Saint-Petersburg, 198504, Russia

In this paper a network game theoretical model of emission reduction is considered. Let $I = \{1, ..., n\}$ be the set of players involved in the game.

The dynamics of the game is governed by the following system of differential equations:

$$\dot{S}_i(t) = \sum_{j \in K_i} (e_j \frac{1}{2k_j}) + \frac{e_i}{2} - \delta S_i(t),$$

$$S_i(t_0) = S_i^0, \qquad i = 1, ..., n,$$

where δ is the natural rate of pollution absorption,

$e_i(t)$ is the emission of player i at time t,

$S_i(t)$ is the stock of accumulated pollution of player i by time t,

K_i is the set of players, which influence the evolution of the stock of accumulated pollution of player i,

k_j is the number of players which evolution of the stock of accumulated pollution depends on emission of player j, e_j.

The game starts at the time instant t_0 from the initial state $S_0 = (S_1^0, S_2^0, ..., S_n^0)$.

$C_i(e_i)$ will be the emission reduction cost incurred by player i while limiting its emission to level e_i:

$$C_i(e_i(t)) = \frac{\gamma}{2}(e_i(t) - \bar{e}_i)^2, \qquad 0 \leqslant e_i(t) \leqslant \bar{e}_i, \qquad \gamma > 0.$$

$D_i(S_i(t))$ denotes its damage cost:

$$D_i(S_i) = \pi S_i(t), \qquad \pi > 0.$$

The payoff function of the player i is defined as

$$K_i(S_i^0, t_0) = \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \left(C_i(e_i(t)) + D_i(S_i(t))\right) dt,$$

where ρ is the common social discount rate.

Suppose that each player seeks to minimize a stream of discounted sum of emission reduction cost and damage cost.

In this paper the feedback Nash equilibrium is computed for the network emission reduction game. The corresponding optimal trajectories for all players $i \in I$ where founded.

References

[1] L.A. Petrosjan, G. Zaccour, "Time-consistent Shapley Value Allocation of Pollution cost Reduction", *Journal of Economic Dynamics and Control*, Vol. 27, 381-–398 (2003).

[2] David W. K. Yeung, Leon A. Petrosyan, *Cooperative Stochastic Differential Games, Springer Science+Business Media*, Springer Science+Business Media, (2006).

[3] L.A. Petrosjan, "Cooperative differential games on networks", *Trudy Inst. Mat. i Mekh. UrO RAN*, Tom 16 No5, 143-–150 (2010).

Recent Progress in Maximal Monotone Operator Theory

Borwein J.M.

jon.borwein@gmail.com

CARMA, University of Newcastle, Callaghan NSW 2308, Australia

Maximal monotone operator theory has recently turned fifty years old. In the first part of the talk, I shall try to explain why maximal(ly) monotone operators are both interesting and important objects while briefly survey the history of the subject — culminating with a description of the remarkable progress made during the past decade. In the second part of the talk, I shall describe in more detail some of the recent striking results on the structure of monotone operators in non-reflexive space. Part I follows the paper [1] and the book chapter [2, Ch. 8]. Part II is based upon [3, 4, 5].

References

[1] Jonathan M. Borwein, "Fifty years of maximal monotonicity", *Optimization Letters*, 4 (2010) 473-490. Available at `http://carma.newcastle.edu.au/jon/fifty.pdf`

[2] J.M. Borwein and J. Vanderwerff, *Convex Functions: Constructions, Characterizations and Counterexamples*, Encyclopedia of Maths and Applications, 109, Cambridge University Press (2010). See `http://carma.newcastle.edu.au/ConvexFunctions`

[3] Heinz H. Bauschke, Jonathan M., Xianfu Wang and Liangjin Yao, "Fitzpatrick-Phelps type coincides with dense type and negative-infimum type", *Optimization Letters*. E-published, August 2011. DOI 10.1007/s11590-011-0383-2. Available at `http://arxiv.org/abs/1104.0750`

[4] Heinz H. Bauschke, Jonathan M. Borwein, Sean Wang and Liangjin Yao, “The Brezis-Browder theorem in an arbitrary Banach space”. Submitted to *J. Functional Analysis*, Oct 2011. Available at `http://arxiv.org/abs/1110.5706`

[5] Heinz H. Bauschke, Jonathan M. Borwein, Sean Wang and Liangjin Yao, “Construction of pathological maximally monotone operators on non-reflexive Banach spaces”. Submitted to *Set Valued and Variational Analysis*, August 2011. Available at `http://arxiv.org/abs/1108.1463`

Approaching the Maximal Monotonicity of Bifunctions via Representative Functions

Boţ R.I., Grad S.-M.

bot@mathematik.tu-chemnitz.de, grad@mathematik.tu-chemnitz.de

Chemnitz University of Technology, Faculty of Mathematics, 09107 Chemnitz, Germany

In this talk, which is based on [1], we provide an approach to maximal monotone bifunctions based on the theory of representative functions. To begin we extend to general Banach spaces the statements due to A.N. Iusem (see [3]) and, respectively, N. Hadjisavvas and H. Khatibzadeh (see [2]), where sufficient conditions that guarantee the maximal monotonicity of a bifunction were proposed. When the space we work with is taken reflexive, the mentioned results from the literature are rediscovered via easier proofs that involve representative functions and convex analysis techniques and do not require renorming arguments as done in the original papers.

References

[1] R.I. Boţ, S.-M. Grad, “Approaching the maximal monotonicity of bifunctions via representative functions”, *Journal of Convex Analysis*, 19, No. 3 (2012).

[2] N. Hadjisavvas, H. Khatibzadeh, “Maximal monotonicity of bifunctions”, *Optimization*, 59, No. 2, 147–160 (2010).

[3] A.N. Iusem, “On the maximal monotonicity of diagonal subdifferential operators”, *Journal of Convex Analysis*, 18, No. 2, 489–503 (2011).

Clasical linear vector optimization duality revisited

Boţ R.I., Grad S.-M., Wanka G.
bot@mathematik.tu-chemnitz.de, grad@mathematik.tu-chemnitz.de,
wanka@mathematik.tu-chemnitz.de
Chemnitz University of Technology, Department of Mathematics, D-09107
Chemnitz, Germany

This talk will bring into attention a vector dual problem that successfully *cures* the *trouble* encountered by some classical vector duals to the classical linear vector optimization problem (see [4, 5]) in finite-dimensional spaces.

This "new-old" vector dual is based on a vector dual introduced by Boţ and Wanka for the case when the image space of the objective function of the primal problem is partially ordered by the corresponding nonnegative orthant, extending it for the framework where an arbitrary nontrivial pointed convex cone partially orders the mentioned space. The vector dual problem we propose (cf. [1]) has, different to other recent contributions to the field (see [3]) which are of set-valued nature, a vector objective function.

Weak, strong and converse duality for this "new-old" vector dual problem are delivered and it is compared with other vector duals considered in the same framework in the literature (see [1, 2]).

We also show that the efficient solutions of the classical linear vector optimization problem coincide with its properly efficient solutions (in any sense) when the image space is partially ordered by a nontrivial pointed closed convex cone, too, extending the classical result due to Focke and, respectively, Isermann.

References

[1] R.I. Boţ, S.-M. Grad, G. Wanka, "Classical linear vector optimization duality revisited", *Optimization Letters*, 6, No. 1, 199–210 (2012).

[2] R.I. Boţ, S.-M. Grad, G. Wanka: *Duality in Vector Optimization*, Springer-Verlag, (2009).

[3] A.H. Hamel, F. Heyde, A. Löhne, C. Tammer, K. Winkler, "Closing the duality gap in linear vector optimization", *Journal of Convex Analysis*, 11, 163–178 (2004).

[4] H. Isermann, "On some relations between a dual pair of multiple objective linear programs", *Zeitschrift für Operations Research*, 22, A33–A41 (1978).

[5] J. Jahn, *Vector Optimization - Theory, Applications, and Extensions*, Springer-Verlag, (2004).

On strict h-polyhedral separability of two sets

Cherneutsanu E.
katerinache@yandex.ru
St. Petersburg State University, Universitetskii prospekt 35, Petergof, 198504,
St. Petersburg, Russia

In the report, we analyze a gradient-type method for solving the problem of strict separation of the convex hull of a set A from a set B by means of h - hyperplanes. Corresponding examples are provided.

О строгом h-отделении двух множеств

Пусть в пространстве $\mathbb{R}^n$ заданы два конечных множества

$$A = \{a_i\}_{i=1}^m \quad \text{и} \quad B = \{b_j\}_{j=1}^k.$$

Задачу строгого отделения выпуклой оболочки множества A от множества B с помощью h гиперплоскостей, определяемых уравнениями

$$\langle w^s, x\rangle = \gamma_s, \quad s \in 1:h,$$

где $w^s \neq \mathbb{O}$ при всех s, можно формализовать следующим образом [1, 2]:

$$F(G) := \frac{1}{m}\sum_{i=1}^{m} \max_{s\in 1:h}\big[\langle w^s, a_i\rangle - \gamma_s + c\big]_+ + \\ + \frac{1}{k}\sum_{j=1}^{k} \min_{s\in 1:h}\big[-\langle w^s, b_j\rangle + \gamma_s + c\big]_+ \to \min_{G}. \quad (1)$$

Здесь G — матрица размера $h \times (n+1)$ со строками $g^s = (w^s, \gamma_s)$, $s \in 1:h$; $c > 0$ — параметр. Матрицу G указанных размеров будем называть *подходящей*, если у неё все w^s отличны от нулевого вектора.

Очевидно, что $F(G) \geqslant 0$. Справедливо следующее утверждение: *выпуклая оболочка множества A и множество B строго h-отделимы тогда и только тогда, когда существует подходящая матрица G_*, такая, что $F(G_*) = 0$.*

Задача (1) сводится к конечному числу задач линейного программирования (ЛП) (см. [3]), однако количество таких задач ЛП и их размеры могут быть очень велики. Поэтому представляют интерес локальные методы решения задачи (1) (см. [1, 4]). Следует иметь в виду, что в этом случае приходится минимизировать функции, зависящие от матрицы G.

В докладе анализируется метод градиентного типа для решения задачи (1) (см. [5]). Приводятся примеры 3-отделения. Выясняется роль параметра c.

Работа выполнена при поддержке РФФИ, проект № 12-01-00752.

ЛИТЕРАТУРА

[1] Astorino A., Gaudioso M. "Polyhedral separability througt successive LP" JOTA, Vol. 112. No. 2, 265–293 (2002)

[2] Малозёмов В. Н., Чернэуцану Е. К. "Строгая h-отделимость двух множеств" Семинар «DHA & CAGD». Избранные доклады. 18 декабря 2010 г. http://dha.spb.ru/PDF/hSeparability.pdf

[3] Чернэуцану Е. К. "Строгая h-отделимость и линейное программирование" Семинар «DHA & CAGD». Избранные доклады. 29 января 2011 г. http://dha.spb.ru/PDF/hSeparabilityLP.pdf

[4] Малозёмов В. Н., Чернэуцану Е. К. "Численный метод строгого h-отделения" Семинар «DHA & CAGD». Избранные доклады. 15 октября 2011 г. http://dha.spb.ru/PDF/hSepNumerical.pdf

[5] Чернэуцану Е. К. "Численные эксперименты по строгой h-отделимости" Семинар «DHA & CAGD». Избранные доклады. 17 декабря 2011 г. http://dha.spb.ru/PDF/hSepEx.pdf

Dynamics and Optimization of Multibody Systems in the Presence of Dry Friction

Chernousko F.

chern@ipmnet.ru

Institute for Problems in Mechanics, Russian Academy of Sciences,
pr. Vernadskogo 101-1, Moscow, 119526, Russia

Motions of various multibody mechanical systems over a horizontal plane in the presence of dry friction forces are considered.

Snake-like systems consist of several rigid rods connected consecutively by cylindrical joints where actuators are placed. Such systems move as a result of twisting at the joints caused by torques generated by the actuators.

Another kind of multibody systems considered in the paper are systems containing movable internal masses. The controlled relative motion of these masses result in the displacement of the system as a whole.

Of course, such motions are possible only in the presence of external forces. In this paper, we take into account dry friction forces obeying Coulomb's law. The dry friction force $\bar{F}$ acting upon mass m moving along a plane is defined as

$$\bar{F} = -fmg\bar{v}/|\bar{v}| \quad \text{if} \quad \bar{v} \neq 0, \qquad |\bar{F}| \leqslant fmg \quad \text{if} \quad \bar{v} = 0.$$

Here, $\bar{v}$ is the velocity of mass, g is the gravity acceleration, and f is the coefficient of friction. Note that, at the state of rest ($\bar{v} = 0$), the friction force is not determined uniquely. This feature leads to specific peculiarities in investigating systems with friction.

Principles of motion considered in the paper are of interest for biomechanics [1]. On the other hand, these ideas are widely used in robotics, especially for mobile mini-robots [2-7].

For the snake-like systems, both quasistatic and dynamic modes of motion are considered. In quasistatic motions, the dynamic terms in the equations of motion are omitted, and these slow motions are indeed a sequence of equilibrium positions. In dynamic motions, full equations of motion are taken into account.

It is shown that progressive quasistatic motions are impossible for two-link and three-link systems but can be performed by multilink systems. For two-link and three-link systems, progressive periodic motions are designed that consist of alternating slow and fast phases [8,9]. Wave-like quasistatic progressive motions of multilink systems are describe d in [10].

Progressive periodic motions of systems with internal moving masses are also proposed. Here, different possible controls of internal masses are considered: either relative velocities or accelerations of the internal masses can play the role of controls.

The most important characteristic of a progressive motion is its average velocity. Hence, it is natural to maximize this velocity with respect to parameters of the system and controls applied.

A number of such optimization problems are solved. The average speed of the progressive motions for two-link and three-link systems is maximized with respect to their geometrical and mechanical parameters (lengths of links, masses, etc.) as well as to certain control parameters [11,12]. Similar problems are solved also for systems with internal moving masses [13]. As a result, optimal parameters and controls are obtained that correspond to the maximum possible speed of the system under consideration.

Computer simulation and experimental data are presented that illustrate and confirm the obtained results. These results are useful for mobile robots moving in various environment and inside tubes.

The work was supported by the Russian Foundation for Basic Research (Projects 11-01-00513 and 11-01-12110).

References

[1] *J.Gray, Animal locomotion,* Norton, New-York (1968).

[2] *S.Hirose, Biologically Inspired Robots: Snake-like Locomotors and Manipulators,* Oxford University Press (1993).

[3] *J.-M.Breguet, R.Clavel,* “Stick and slip actuators: design, control, performance and applications”, *Proc. Intern. Symp.on Micromechatronics and Human Science (MHS),* IEEE, New-York 89-95 (1998).

[4] *F. Schmoeckel, H.Worn,* “Remotedly controllable mobile microrobots acting as nano positioners and intelligent tweezers in scanning electron microscopes (SEMs), *Proc. Intern. Conf. on Robotics and Automation,* IEEE, New-York, 3903-3913 (2001).

[5] *A. Fidlin, J.J. Thomsen,* “Predicting vibration-induced displacement for a resonant friction slider”, *European Journal of Mechanics. A/Solids*, 20, 155-166 (2001).

[6] *P. Vartholomeos, E. Papadopulos,* “Dynamics, design and similation of a novel microrobotic platform employing vibration microactuators.”, *Trans ASME. Journal of Dynamic Systems, Measurement and Control*, 128, No. 1, 122-133 (2006).

[7] *K.Zimmermann, I. Zeidis, C.Behn, Mechanics of Terrestrial Locomotion,* Springer, Heidelberg (2009).

[8] *F.L. Chernousko,* “The motion of a multilink system along a horizontal plane”, *Journal of Applied Mathematics and Mechanics*, 64, No. 1, 5–15 (2000).

[9] *F.L. Chernousko,* “Controllable motions of a two-link mechanism along a horizontal plane”,*Journal of Applied Mathematics and Mechanics*, 65, No. 4, 565–577 (2001).

[10] *F.L. Chernousko,* “The wave-like motion of a multilink system on a horizontal plane”, *Journal of Applied Mathematics and Mechanics*, 64, No. 4, 497–507 (2000).

[11] *F.L. Chernousko,* “Snake-like locomotions of multilink mechanisms”, *Journal of Vibration and Control*, 9, No. 1–2, 237-256 (2003).

[12] *F.L. Chernousko,* “Modelling of snake-like locomotions”, *Applied Mathematics and Computation*, 164, No. 2, 415–434 (2005).

[13] *F.L. Chernousko,* “Analysis and optimization of the motion of a body controlled by a movable internal mass”, *Journal of Applied Mathematics and Mechanics*, 70, No. 6, 915-941 (2006).

Convex level-sets integration

Crouzeix J.-P.

jp.crouzeix@isima.fr

Limos, Blaise Pascal University, Complexe scientifique des Cézaux, 63173 AUBIERE Cedex, France

Given a multivalued map $F : \Omega \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^n$ with Ω convex subset of $\mathbb{R}^n$, the common convex integration problem consists in finding a convex function $f : \Omega \to \mathbb{R}$ such that F coincides with the Fenchel-subgradient of f. It is known that if F is single-valued and continuously differentiable a necessary and sufficient condition is the symmetry and positive semi-definiteness of the matrix $F'(x)$ at any $x \in \Omega$. In the multivalued case, a necessary and sufficient condition is the maximal monotonicity of F. In both cases, f is defined up to a constant and $F(x)$ belongs to the normal cone at the point x to the level set $\{y \in \Omega \,:\, f(y) \leqslant f(x)\}$.

More complex is the convex level sets integration problem which consists to find a function f with convex level sets such that, at any $x \in \Omega$, $F(x)$ belongs to the normal cone at the point x to the level set $\{y \in \Omega \,:\, f(y) \leqslant f(x)\}$. If f is a solution, then $k \circ f$, with $k : \mathbb{R} \to \mathbb{R}$ increasing, is also a solution. Solutions are defined up to a scalarization.

The single-valued case has been solved by Crouzeix–Rapcsak. If F is continuously differentiable and does not vanish on Ω, a necessary and sufficient condition is the symmetry and positive semi-definiteness of the matrix $F'(x)$ on the orthogonal subspace to $F(x)$ at any $x \in \Omega$.

This talk intend to give a state of art on the very recent advances in the multi-valued case. One considers theoretical aspects and numerical constructions.

An important application is the revealed preference problem in consumer theory.

References

[1] *J.-P. Crouzeix and T. Rapcs«ak,* "Integrability of pseudomonotone differentiable maps and the revealed preference problem", *Journal of Convex Analysis*, 2, 431–446 (2005).

[2] *J.-P. Crouzeix, A. Eberhard and D. Ralph,* "A geometrical insight on pseudoconvexity and pseudomonotonicity", *Mathematical Programming serie B*, 123, 61–83 (2010).

[3] *J.-P. Crouzeix, A. Keraghel and N. Rahmani,* "Integration of pseudomonotone maps and the revealed preference problem", *Optimization*, 60, 783–800 (2011).

[4] *A. Eberhard, D. Ralph and J.-P Crouzeix,* "A constructive proof of the existence of a utility in revealed preference theory", Techn. paper, RMIT Melbourne, Australia (2012).

[5] *A. Fostel, H. Scarf and M. Todd,* "Two new proofs of Afriat's theorem", *Economic Theory*, 24, 211–219 (2004).

Bilevel programming, Lyapunov functions and regions of attraction

Demenkov M.
demenkov@ipu.ru, max.demenkov@gmail.com
Institute of Control Sciences, Russian Academy of Sciences, Profsoyuznaya 65, Moscow 117997, Russia

Bilevel optimization involves a hierarchy of two optimization problems. Some variables in the upper level problem are constrained to be optimal in the lower level problem [1]. The origin of the bilevel programming problems can be traced back to the Stackelberg games introduced in the 1930s, although its modern formulation arised in the 1970s [2].

We investigate a connection between bilevel optimization and stability of nonlinear dynamical systems of the form

$$\dot{x}(t) = f(x(t)),$$

where $x \in R^n$ and f is locally Lipschitz. We assume that $x_0 = 0$ is an exponentially stable equilibrium. In this case, an important system characteristic is the size of its *region of attraction*, which is a set of initial points $x(0)$ with the property $x(t) \to 0$ as $t \to \infty$.

We can estimate the region of attraction through the sublevel sets of a Lyapunov function [3]. A Lyapunov function for the equilibrium x_0 is a positive definite continuous Lipschitz function $V(x,p)$ of the state space vector x (parameterized by vector p), which is decreasing along the solution trajectories of the system:

$$\dot{V}(x,p) < 0, x \neq 0,$$

where $\dot{V}(x,p) =< \nabla_x V(x,p), f(x) >$ if f and V are smooth functions, or $\dot{V}(x,p) = D^+V(x,p)$ (the Dini derivative [12]) otherwise. An estimation of the region of attraction is given by a compact set $\Omega(p,\gamma)$ defined as follows:

$$\Omega(p,\gamma) = \{x : V(x,p) \leqslant \gamma\},\ \Omega(p,\gamma) \subseteq \Psi(p),\ \Psi(p) = \{x : \dot{V}(x,p) \leqslant 0\}.$$

In case f and V are polynomial and p is fixed, our estimation can be computed in polynomial time by means of semidefinite programming (SDP), if one can represent V as sum of squares (SOS) of polynomials [4]. This approach relies on a convex relaxation of the problem with fixed p. To obtain a better estimate, we need to vary p, and then our problem ceased to be convex. Some heuristic algorithms have been proposed for this case, using SDP with SOS polynomials as their major part [6].

Note that SOS polynomials represent only a fraction of all positive definite polynomials that tends to zero as we increase the state space dimension [5], although a positive polynomial can always be expressed as a ratio of two SOS polynomials [4]. Therefore, it is reasonable to consider the problem in its general form, even for polynomial f, V. Let us denote by P a subset of all p for which $V(x,p)$ is a positive definite function. Suppose that we have a size estimation of $\Omega(p,\gamma)$ denoted by $S(p,\gamma)$, which is a positive definite continuous Lipschitz function of its parameters. Observe that for a fixed p, $\Omega(p,\gamma)$ of maximum size in terms of inclusion is defined by the following choice of γ [7]:

$$\gamma = \min_{x \in H(x,p)} V(x,p), \quad H(x,p) = \{x : \dot{V}(x,p) = 0, x \neq 0\}.$$

Note that in general case additional constraints should be added, to guarantee the existence of the minimum [8]. Now, our optimization problem can be formulated as bilevel one:

$$\begin{aligned} &\max_{p \in P} S(p,\gamma) \\ &s.t.\ \gamma = \min_{x \in H(x)} V(x,p). \end{aligned}$$

Note that one can consider minimization of $1/S(p,\gamma)$ as the upper level problem here. To the author's knowledge, problem of this kind first appeared in [9]. In [7] one can find a review of earlier attempts to solve it by various nonlinear optimization techniques. In [10, 11] this problem has been recognised as a nonsmooth one even for smooth f, V and local improvement algorithms have been proposed using generalized Clarke gradient [12]. The required solution of the lower level problem for smooth f, V is achieved in [13] by a method of global interval optimization.

Connecting the problem with bilevel programming, we arrive at two conclusions. First, our problem in its general form is NP-hard, since bilevel programming has been proved to be NP-hard even in its simplest form [1, 2]. Second, one can use algorithms developed for bilevel programming (see e.g. [2, 14, 15]) to estimate the region of attraction, if they can handle nonconvexity of the lower level problem.

References

[1] S. Dempe, *Foundations of bilevel programming. Nonconvex optimization and its applications, (Vol. 61)*, Kluwer Academic, Dordrecht (2002).

[2] B. Colson, P. Marcotte, G. Savard, “An overview of bilevel optimization”, *Ann. Oper. Res.*, 153, 235-–256 (2007).

[3] V. Zubov, *Methods of A. M. Lyapunov and their application*, P. Noordhoff, Groningen, Netherlands (1964).

[4] G. Chesi, *Domain of attraction. Analysis and control via SOS programming*, Springer-Verlag, London (2011).

[5] G. Blekherman. “There are significantly more nonnegative polynomials than sums of squares *Israel Journal of Mathematics*, 153, No. 1, 355-–380 (2006).

[6] A. Packard, U. Topcu, P. Seiler, G. Balas, “Help on SOS”, *IEEE Control Systems magazine*, 30, No. 4, 18-–23 (2010).

[7] D. Shields, C. Storey, “The behaviour of optimal Lyapunov functions”, *Inernational Journal of Control*, 21, No. 4, 561-–573 (1975).

[8] A.V. Gorbunov, “On critical level evaluation for basin of attraction estimation by Lyapunov functions technique In *Proceedings of the XI International conference “Stability and oscillations of nonlinear control systems"*, Moscow, Russia (2010).

[9] J.J. Rodden, *Applications of Liapunov stability theory*, PhD thesis, Stanford University, Stanford, CA, USA (1964).

[10] J. Hauser, M.C. Lai, “Estimating quadratic stability domains by nonsmooth optimization In *Proceedings of the 11th American Control Conference*, Chicago, USA (1992).

[11] V.A. Kamenetskiy, “A method for construction of stability regions by Lyapunov functions”, *Systems & Control Letters*, 26, No. 2, 147-–151 (1995).

[12] F.H. Clarke, *Optimization and nonsmooth analysis*, John Wiley & Sons, New York (1983).

[13] S. Mayer, B. Tibken, S. Warthenpfuhl, “Parallel computation of domains of attraction for nonlinear dynamic systems In *Proceedings of the 6th International Symposium on Parallel Computing in Electrical Engineering (PARELEC)*, Luton, UK (2011).

[14] D. Zhua, Q. Xua, Z. Lina, “A homotopy method for solving bilevel programming problem”, *Nonlinear Analysis*, 57, 917-– 928 (2004).

[15] V. Demyanov, F. Facchinei, “Two-level optimization problems and penalty functions”, *Russian Math. (Iz. VUZ)*, 47, No. 12, 46-– 58 (2003) .

Necessary optimality conditions in pessimistic bilevel programming

Dempe S., Mordukhovich B.S., Zemkoho A.B.

dempe@tu-freiberg.de, boris@math.wayne.edu, zemkoho@mailserver.tu-freiberg.de

TU Bergakademie Freiberg, Department of Mathematics and Computer Science, Freiberg, Germany, Wayne State University, Department of Mathematics, Detroit, USA, TU Bergakademie Freiberg, Department of Mathematics and Computer Science, Freiberg, Germany

We consider the so-called pessimistic version of bilevel programming programs. Minimization problems of this type are challenging to handle partly because the corresponding value functions are often merely upper (while not lower) semicontinuous. Employing advanced tools of variational analysis and generalized differentiation, we provide rather general frameworks ensuring the Lipschitz continuity of the corresponding value functions. Several types of *lower subdifferential* necessary optimality conditions are then derived by using the lower-level value function (LLVF) approach and the Karush-Kuhn-Tucker (KKT) representation of lower-level optimal solution maps. We also derive *upper subdifferential* necessary optimality conditions of a new type, which can be essentially stronger than the lower ones in some particular settings. Finally, certain links are established between the obtained necessary optimality conditions for the pessimistic and optimistic versions in bilevel programming.

References

[1] Dempe, S., Mordukhovich B. S. and Zemkoho A. B., "Sensitivity analysis for two-level value functions with applications to bilevel programming", Preprint 2011-07, TU Bergakademie Freiberg, 2011.

[2] Dempe, S., Mordukhovich B. S. and Zemkoho A. B., "Necessary optimality conditions in pessimistic bilevel programming", Preprint 2012-01, TU Bergakademie Freiberg, 2011.

Optimality conditions for bilevel programming problems

Dempe S., Zemkoho A.B.

dempe@tu-freiberg.de, zemkoho@student.tu-freiberg.de

TU Bergakademie Freiberg, Department of Mathematics and Computer Science, Freiberg, D-09596, Germany

Bilevel programming problems are hierarchical optimization problems where the feasible region is (in part) restricted to the graph of the solution set mapping of a second parametric optimization problem [1]:

$$\min\{F(x,y) : x \in X,\ y \in \Psi(x)\},$$

where $X \subseteq \mathbb{R}^n$, $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\Psi(\cdot)$ is the solution set mapping of a second parametric optimization problem:

$$\min_y\{f(x,y) : g(x,y) \leqslant 0\}. \tag{1}$$

Here, $f, g_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ for all i. To solve them and to derive optimality conditions for these problems this parametric optimization problem (1) needs to be replaced with its (necessary) optimality conditions. This results in a (one-level) nonconvex and nondifferentiable optimization problem:

$$\min\{F(x,y) : x \in X,\ f(x,y) \leqslant \varphi(x),\ g(x,y) \leqslant 0\},$$

where $\varphi(\cdot)$ is the optimal value function of the problem (1) or

$$\min\{F(x,y) : x \in X,\ 0 \in \partial f(x,y) + N_K(x,y)\},$$

where $N_K(x,y)$ is the normal cone to the feasible set mapping

$$K(x,y) := \begin{cases} K(x) := \{y : g(x,y) \leqslant 0\} & \text{if } y \in K(x) \\ \varnothing & \text{else.} \end{cases}$$

Usual regularity conditions (as nonsmooth MFCQ) are violated at every feasible point of the resulting problem. Using new results in Variational Analysis, necessary optimality conditions for the bilevel programming problem [2, 3] will be formulated in the talk .

References

[1] *Dempe S., Foundations of Bilevel Programming*, Kluwer Academie Publishers, Dordrecht, 2002.

[2] *Dempe S., Zemkoho A. B.,* "On the Karush-Kuhn-Tucker reformulation of the bilevel optimization problem", Nonlinear Analysis: Theory, Methods & Applications, 75, 1202-1218 (2012).

[3] *Dempe S., Mordukhovich B. S., Zemkoho A. B.,* "Sensitivity analysis for two-level value functions with applications to bilevel programming", Preprint 2011-07, TU Bergakademie Freiberg, 2011.

Constructive tools in Nonsmooth Analysis

Demyanov V.F.

vfd@ad9503.spb.edu

Saint Petersburg State University, Universitetskaya nab., 7-9, St.Petersburg, Russia

There exist many tools to analyze nonsmooth functions. For convex and max-type functions, the notion of subdifferential is used, for quasidifferentiable functions – that of quasidifferential. By means of these tools one is able to solve, e.g., the following problems: to get an approximation of the increment of a functional, to formulate conditions for an extremum, to find steepest descent and ascent directions, to construct numerical methods. For arbitrary directionally differentiable functions these problems are solved by employing the notions of upper and lower exhausters and coexhausters, which are generalizations of such notions of nonsmooth analysis as sub- and superdifferentials, quasidifferentials and codifferentials. Exhausters allow one to construct homogeneous approximations of the increment of a functional while coexhausters provide nonhomogeneous approximations. It became possible to formulate conditions for an extremum in terms of exhausters and coexhausters. It turns out that conditions for a minimum are expressed by an upper exhauster, and conditions for a maximum are formulated via a lower one. This is why an upper exhauster is called a proper one for the minimization problem (and adjoint for the maximization problem) while a lower exhauster is called a proper one for the maximization problem (and adjoint for the minimization problem). The conditions obtained provide a simple geometric interpretation and allow one to find steepest descent and ascent directions. In the present paper, optimization problems are treated by means of proper exhausters and coexhausters.

The main concept used in mathematical modeling is that of function. The analysis of a model consists in the study of properties of functions involved in the description of the model. The efficiency of such an analysis essentially depends on the possibility of constructing simpler approximations of these functions which preserve their most important properties. In the classical ("smooth") Mathematical Analysis, a linear approximation of the increment of a function is constructed by means of the gradient. In the case of a directionally differentiable (d.d.) function, one constructs a positively homogeneous (p.h.) approximation of the increment of the function, and the directional derivative (d.d.) is such an approximation. This derivative is described in terms of exhauster, and optimality conditions have an elegant and constructive form, allowing one not only to check whether these conditions are satisfied but also (if the conditions are not valid) to find directions of steepest descent and ascent.

The study of a point on an extremum is reduced to multiple application of tools of Convex Analysis [12]. In [5, 8, 9, 10] the notions of upper and lower exhausters were introduced, they are generalizations of the notions of exhaustive families of upper convex and lower concave approximations (u.c.a.'s and l.c.a.'s). the notions of upper convex and lower concave approximations were introduced by B.N.Pshenichnyi [11], and that of exhaustive families of u.c.a.'s and l.c.a.'s – by A.M.Rubinov [5]. It became possible to describe optimality conditions in terms of exhausters. It turns out that conditions for a minimum are expressed by an upper exhauster, and conditions for a maximum are formulated via a lower one [4, 2, 3, 1]. This is why an upper exhauster is called a proper one for the minimization problem (and adjoint for the maximization problem) while a lower exhauster is called a proper one for the maximization problem (and adjoint for the minimization problem).

Since the directional derivative is, in general, not continuous as a function of point, one encounters computational problems while constructing numerical methods. To overcome such problems, in [10] the notions of upper and lower coexhausters were suggested. The application of these notions in some cases allows to get continuous (as functions of point) approximations of the increment of the function in a vicinity of the point under study. The approximations thus obtained are not any more positively homogeneous. As in the case of exhausters, conditions for a minimum are expressed by an upper coexhauster, and conditions for a maximum are formulated via a lower one. Naturally, therefore an upper coexhauster is called a proper one for the minimization problem (and adjoint for the maximization problem) while a lower coexhauster is called a proper one for the maximization problem (and adjoint for the minimization problem).

The work is supported by the Russian Foundation for Basic Research (RFFI) under Grant No 12-01-00752.

References

[1] M. E. Abbasov, V. F. Demyanov, "Extremum conditions for a nonsmooth function in terms of exhausters and coexhausters (In Russian) *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, Vol. 15, No. 4, (2009) English translation: *Proceedings of the Steklov Institute of Mathematics*, Suppl. 2, pp. S1-S10, (2010) Pleiades Publishing, Ltd., 2010.

[2] V. F. Demyanov, V. A. Roshchina, "Constrained optimality conditions in terms of proper and adjoint exhausters *Appl. Comput. Math.* 4 , no. 2, P. 114–124, (2005)

[3] V. F. Demyanov, V. A. Roshchina, "Generalized subdifferentials and exhausters in nonsmooth analysis (In Russian) *Dokl. Akad. Nauk* 416 (2007), no. 1, 18–21; translation in *Dokl. Math.* 76, no. 2, P. 652–655, (2007)

[4] V. F.Demyanov, V. A. Roshchina, "Optimality conditions in terms of upper and lower exhausters *Optimization*, **55**, P. 525–540 (2006).

[5] V. F. Demyanov, A. M. Rubinov, "Elements of quasidifferential calculus in *Nonsmooth Problems of Optimization Theory and Control* (ed. V. F. Demyanov), Leningrad University Press, Leningrad, P. 5–127, (1982)

[6] V. F. Demyanov, A. M. Rubinov, *Constructive Nonsmooth Analysis*, Peter Lang Verlag, Frankfurt a/M., (1995)

[7] V. F. Demyanov, J. A. Ryabova, "Exhausters, coexhausters and converters in nonsmooth analysis *Discrete and continuous dynamical systems*, **31**(4), P. 1273–1292, (2011)

[8] V. F. Demyanov, "Exhausters of a positively homogeneous function *Optimization*, **45**, P. 13–29, (1999)

[9] V. F. Demyanov, A. M. Rubinov, "Exhaustive families of approximations revisited in *From Convexity to Nonconvexity* (eds. R. P.Gilbert, P. D.Panagiotopoulos and P. M. Pardalos), Kluwer, Dordrecht, P. 43–50, (2001)

[10] V. F. Demyanov, "Exhausters and convexificators – new tools in nonsmooth analysis in *Quasidifferentiability and Related Topics* (eds. V. F. Demyanov and A. M. Rubinov), Kluwer, Dordrecht, P. 85–137, (2000)

[11] B. N. Pschenichnyi, *Convex Analysis and Extremal Problems*, Nauka, Moscow (1980)

[12] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, N.J.(1970).

A derivative-free approach to constrained global optimization based on non-differentiable exact penalty functions

Di Pillo G., Lucidi S., Rinaldi F.

Department of Computer Control and Management Engineering, Sapienza University of Rome, Italy

In the field of global optimization many efforts have been devoted to globally solving bound constrained optimization problems $\{\min f(x),\ l \leqslant x \leqslant u,\ x, l, u \in \mathbb{R}^{\mathrm{n}},\ f : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}\}$ without using the derivatives of f. In this talk we show how bound constrained derivative-free global optimization methods can be used for globally solving optimization problems where also general constraints $\{g(x) \leqslant 0, g : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{p}}\}$ are present, without using neither the derivatives of f nor the derivatives of g. This is of great practical importance in many real-world problems where only problem functions values are available. To our aim we resort to an exact penalty approach. In particular, we make use of a non-differentiable exact penalty function $P_q(x;\varepsilon)$. We exploit the property that, under weak assumptions, there exists a threshold value $\bar{\varepsilon} > 0$ of the penalty parameter ε such that, for any $\varepsilon \in (0, \bar{\varepsilon}]$, any unconstrained global minimizer of P_q is a global solution of the related constrained problem and conversely. On these bases, we describe an algorithm that, by combining a bound constrained derivative-free global minimization technique for minimizing P_q for given values of the penalty parameter ε and an automatic updating of ε that does not uses derivatives of the problem functions and that occurs only a finite number of times, produces a sequence $\{x_k\}$ such that any limit point of the sequence is a global solution of the general constrained problem. In the algorithm any efficient bound constrained derivative-free global minimization technique can be used. In particular, we adopt an improved version of the DIRECT algorithm. In addition, to improve the performance, the approach is enriched by resorting to local searches, in a multistart framework. Some numerical experimentation confirms the effectiveness of the approach.

Theorems on covering, on coincidence, the Newton's iterations, and the paradox of Zenon

Dmitruk A.
vraimax@mail.ru
Russian Academy of Sciences, CEMI, Moscow, 117418, Russia

The aim of the talk is to discuss interrelations between basic theorems on covering in metric and Banach spaces (in local and nonlocal setting), on fixed point, on coincidence, on approaching the goal, on stepwise descending, and to discuss the Newton like method as the basis of their proof and its prototype in the ancient paradox of Achilles and Tortilla.

Nonsmooth problems of Calculus of Variations with a codifferentiable integrand

Dolgopolik M.V.
maxim.dolgopolik@gmail.com
St. Petersburg State University, Universitetskii prospekt 35, Petergof, 198504, St. Petersburg, Russia

Nonsmooth problems of Calculus of Variations with a codifferentiable integrand are discussed in the report. Necessary optimality conditions are formulated.

Негладкие задачи вариационного исчисления с кодифференцируемой подынтегральной функцией

Пусть $\Omega \subset \mathbb{R}^n$ — открытое ограниченное множество. Обозначим через $C^1(\overline{\Omega})$ линейное пространство, состоящее из всех таких $\varphi \in C^1(\Omega)$, для которых функции φ, $\frac{\partial \varphi}{\partial x_i}$, $i \in \{1, \dots, n\}$, ограничены и равномерно непрерывны на Ω (тогда существует единственное продолжение функции φ и всех её производных на замыкание множества Ω). Пространство $C^1(\overline{\Omega})$ является банаховым пространством относительно нормы

$$\|\varphi\|_1 = \max \left\{ \sup_{x \in \Omega} |\varphi(x)|, \sup_{x \in \Omega} \left| \frac{\partial \varphi}{\partial x_1}(x) \right|, \dots, \sup_{x \in \Omega} \left| \frac{\partial \varphi}{\partial x_n}(x) \right| \right\}.$$

Пусть $C^1_0(\overline{\Omega})$ — это множество всех функций из $C^1(\overline{\Omega})$, обращающихся в 0 на границе множества Ω.

Рассмотрим функционал

$$\mathcal{I}(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx,$$

определённый на пространстве $C^1(\overline{\Omega})$. Здесь функция $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, $f=f(x,u,\xi)$, непрерывна по совокупности аргументов. Зафиксируем произвольное $u_0\in C^1(\overline{\Omega})$ и рассмотрим задачу минимизации функционала $\mathcal{I}$ на замкнутом выпуклом множестве $A=\{v=u+u_0 \mid u\in C_0^1(\overline{\Omega})\}$. При этом мы предполагаем, что функция f — негладкая (необходимые условия экстремума в классическом случае подробно описаны, например, в [1]).

Предположим, что для любых $x\in\Omega$, $u\in\mathbb{R}$ и $\xi\in\mathbb{R}^n$ функция f кодифференцируема по переменным u и ξ при фиксированном x в точке (x,u,ξ), т.е. для любых $(x,u,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$ существуют выпуклые компактные множества $\underline{d}_{u,\xi}f(x,u,\xi), \overline{d}_{u,\xi}f(x,u,\xi)\subset\mathbb{R}^{n+2}$ такие, что для любого приращения аргументов Δu и $\Delta\xi$, соответствующее приращение функции представимо в виде

$$f(x,u+\Delta u,\xi+\Delta\xi)-f(x,u,\xi)=\max_{[a,v_1,v_2]\in\underline{d}_{u,\xi}f(x,u,\xi)}(a+\langle v_1,\Delta u\rangle+\langle v_2,\Delta\xi\rangle)+ \\ +\min_{[b,w_1,w_2]\in\overline{d}_{u,\xi}f(x,u,\xi)}(b+\langle w_1,\Delta u\rangle+\langle w_2,\Delta\xi\rangle)+o(\Delta u,\Delta\xi),$$

где $o(\alpha\Delta u,\alpha\Delta\xi)/\alpha\to 0$ при $\alpha\downarrow 0$. Здесь $\langle\cdot,\cdot\rangle$ обозначает скалярное произведение в $\mathbb{R}^d$. При этом предположим, что многозначное отображение $(x,u,\xi)\to[\underline{d}_{u,\xi}f(x,u,\xi),\overline{d}_{u,\xi}f(x,u,\xi)]$ непрерывно в метрике Хаусдорфа. (Кодифференцируемые функции подробно изучены в [2]).

При небольших дополнительных ограничениях на функцию f и область Ω можно получить следующее необходимое условие экстремума:

Пусть $u^*\in A$ — точка локального минимума функционала $\mathcal{I}$ на множестве A. Тогда для любого измеримого сечения $[b(\cdot),w_1(\cdot),w_2(\cdot)]$ многозначного отображения $x\to\overline{d}_{u,\xi}f(x,u^*(x),\nabla u^*(x))$ такого, что $\int_\Omega b(x)\,dx=0$, существует измеримое сечение $[a(\cdot),v_1(\cdot),v_2(\cdot)]$ многозначного отображения $x\to\underline{d}_{u,\xi}f(x,u^*(x),\nabla u^*(x))$ такое, что $\int_\Omega a(x)\,dx=0$ и

$$\int_\Omega(\langle v_1(x)+w_1(x),h(x)\rangle+\langle v_2(x)+w_2(x),\nabla h(x)\rangle)\,dx=0\quad\forall h\in C_0^1(\overline{\Omega}).$$

В случае $n=1$, $\Omega=(a,b)$, необходимое условие экстремума упрощается и принимает вид:

Пусть $u^*\in A$ — точка локального минимума функционала $\mathcal{I}$ на множестве A. Тогда для любого измеримого сечения $[b(\cdot),w_1(\cdot),w_2(\cdot)]$ многозначного отображения $x\to\overline{d}_{u,\xi}f(x,u^*(x),\nabla u^*(x))$ такого, что $b(x)=0$ для почти всех $x\in(a,b)$, существуют измеримое сечение $[a(\cdot),v_1(\cdot),v_2(\cdot)]$ многозначного отображения $x\to\underline{d}_{u,\xi}f(x,u^*(x),\nabla u^*(x))$

и $c \in \mathbb{R}$ такие, что $a(x) = 0$ для почти всех $x \in (a, b)$ и

$$\int_x^b (v_1(y) + w_1(y))\, dy + v_2(x) + w_2(x) = c \text{ для п.в. } x \in (a, b).$$

Для различных конкретных функций f условия экстремума могут быть модифицированы. Например, пусть $n = 1$, $\Omega = (a, b)$ и

$$f(x, u, \xi) = \max_{i \in I} f_i(x, u, \xi) + \min_{j \in J} g_j(x, u, \xi),$$

где $I = \{1, \dots n\}$, $J = \{1, \dots, m\}$, функции $f_i, g_j \colon (a, b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ непрерывны и непрерывно дифференцируемы по переменным u и ξ на всей своей области определения. В данном случае условия экстремума имеют следующий вид:

Пусть $u^* \in A$ — точка локального минимума функционала $\mathcal{I}$ на множестве A. Тогда для любых измеримых функций $\beta_j \colon (a, b) \to [0, 1]$, $j \in J$, таких, что $\beta_1 + \ldots + \beta_m \equiv 1$ и для любого $j \in J$ справедливо равенство

$$\beta_j(x)[g_j(x, u^*(x), \nabla u^*(x)) - \min_{j \in J} g_j(x, u^*(x), \nabla u^*(x))] = 0 \text{ для п.в. } x \in (a, b),$$

существуют измеримые функции $\alpha_i \colon (a, b) \to [0, 1]$, $i \in I$, и $c \in \mathbb{R}$ такие, что $\alpha_1 + \ldots + \alpha_n \equiv 1$, для любого $i \in I$ выполняется равенства

$$\alpha_i(x)[\max_{i \in I} f_i(x, u^*(x), \nabla u^*(x)) - f_i(x, u^*(x), \nabla u^*(x))] = 0 \text{ для п.в. } x \in (a, b)$$

и

$$\sum_{i \in I} \left(\int_x^b \alpha_i(y) \frac{\partial f_i}{\partial u}(y, u^*(y), \nabla u^*(y))\, dy + \alpha_i(x) \frac{\partial f_i}{\partial \xi}(x, u^*(x), \nabla u^*(x)) \right) +$$
$$+ \sum_{j \in J} \left(\int_x^b \beta_j(y) \frac{\partial g_j}{\partial u}(y, u^*(y), \nabla u^*(y))\, dy + \beta_j(x) \frac{\partial g_j}{\partial \xi}(x, u^*(x), \nabla u^*(x)) \right) = c$$

для п.в. $x \in (a, b)$.

Работа выполнена при финансовой поддержке РФФИ, грант № 12-01-00752.

ЛИТЕРАТУРА

[1] B. Dacorogna, *Direct Methods in the Calculus of Variations*, New–York: Springer Science+Business Media, (2008).

[2] В.Ф. Демьянов, А.М. Рубинов, *Основы негладкого анализа и квазидифференциальное исчисление*, Москва: Наука, (1990).

Variational analysis with smooth substructure

Drusvyatskiy D.
dd379@cornell.edu
Cornell University, 206 Rhodes Hall, Ithaca, NY 14853, USA

Nonsmoothness arises naturally in many applications, but not pathologically so. On the contrary, nonsmooth problems often possess rich underlying structure waiting to be exploited. *Semi-algebraic functions* – those functions arising from polynomial inequalities – nicely examplify such structure.

In the first part of my talk, I will discuss a new theorem showing that the (general/limiting) subdifferential graph of a semi-algebraic function f defined on $\mathbf{R}^n$ has uniform local dimension n at each of its points, a property that may fail even for locally Lipschitz functions. As a direct application, this theorem yields a simple representation formula for the Clarke subdifferential of directionally Lipschitzian semi-algebraic functions, thereby simplifying and unifying some foundational results of Nonsmooth Analysis.

In the second part of the talk, I will discuss a structure that is perhaps more familiar to optimization specialists – an *active manifold.* This notion has recently been axiomatized by the theory of *partly smooth sets.* Existence of such a manifold is exactly what is needed for a local reduction of a nonsmooth problem to a smooth one. It is reassuring to know that such manifolds exist "generically" for semi-algebraic optimization problems.

References

[1] *D. Drusvyatskiy, A.D. Ioffe, A.S. Lewis,* "The dimension of semialgebraic subdifferential graphs", *Nonlinear Analysis: Theory, Methods and Applications*, Volume 75, Issue 3, 1231-1245 (2012).

[2] *D. Drusvyatskiy, A.D. Ioffe, A.S. Lewis,* "Characterization of the Clarke subdifferential for directionally Lipschitzian stratifiable functions", submitted to *Mathematics of Operations Research.*

[3] *D. Drusvyatskiy, A.S. Lewis,* "Identifiable sets in optimization", submitted to *SIAM Journal on Optimization.*

On Systematization of the Problems for Estimations of Convex Compact Sets by the Balls

Dudov S.I.
DudovSI@info.sgu.ru
Saratov State University, Astrakhanskaya, 83, Saratov, 410012, Russia

Let D be a fixed convex compact set in $\mathbb{R}^p$ and $n(x)$ be a norm on $\mathbb{R}^p$. We use following notation: $\Omega = \overline{\mathbb{R}^p \setminus D}$,

$$R(x, D) = \max_{y \in D} n(x - y),$$

$$\rho(x, D) = \min_{y \in D} n(x - y),$$

$$P(x, D) = \rho(x, D) - \rho(x, \Omega).$$

Many estimating and approximating problems for convex compact set D by the balls have the form ([1], [2])

$$f(x) \equiv F(R(x, D), P(x, D)) \to \min_{x \in S}. \tag{1}$$

The solutions of some problems (1) can be obtained by solving the following problem:

$$\varphi(x, r) \equiv h(D, Bn(x, r)) \to \min_{x \in \mathbb{R}^p}, \tag{2}$$

where $Bn(x, r) = \{y \in \mathbb{R}^p : n(x - y) \leqslant r\}$,

$$h(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} n(a - b), \ \sup_{b \in B} \inf_{a \in A} n(a - b) \right\}.$$

Notation: $C_f = \operatorname{Arg} \min_{x \in S} f(x)$, $C_\varphi(r) = \operatorname{Arg} \min_{x \in \mathbb{R}^p} \varphi(x, r)$.

Theorem. *Let $S = \mathbb{R}^p$ or $S = D$, the function $F(t_1, t_2)$ be nondecreasing according to each variable and be increasing according to at least one of them. If $x^* \in C_f$ then there exists $r \geqslant 0$ such that $x^* \in C_\varphi(r)$.*

We find the ranges of r for which the solutions of problem (2) give the solutions of the problems of:

– outer and inner estimating of the set D by a ball ($F(t_1, t_2) = t_1$ and $F(t_1, t_2) = t_2$, $S = \mathbb{R}^p$),

– uniform estimating of the set D by balls in the Hausdorff metric ($F(t_1, t_2) = t_1 + t_2$, $S = \mathbb{R}^p$),

– estimating the boundary of the set D by spherical layer of the least width and volume ($F(t_1, t_2) = t_1 + t_2$ and $F(t_1, t_2) = t_1^p + t_2^p$, $S = D$),

– asphericity of the set D ($F(t_1, t_2) = -t_1/t_2$, $S = D$).

This research was supported by programme “Leading Scientific Schools” (No. 4383.2010.1) and RFBR (No. 10-01-00270).

References

[1] *T. Bonnesen and W. Fenchel, Teorie der konvexen Körper*, Springer-Verlag, Berlin (1974).

[2] *S. I. Dudov,* “Relations between several problems of estimating convex compacta by balls”, *Matematicheskii Sbornik*, 198, No. 1, 39-–53 (2007).

Gene selection and cluster validation analysis of a microarray data set from SCID-mice liver infected with a new strain of MHV

Dzalilov Z., Zarei M., Bagirov A., Islam M., Kannourakis G.
z.dzalilov@ballarat.edu.au
School of Science, Information Technology and Engineering, University of Ballarat, University Drive, Mount Helen, Ballarat, Victoria, 3350, Australia,
Ballarat Cancer Research Centre, Ballarat, Australia

Many clustering algorithms have been developed for analysis of gene expression data, but there exist only a few approaches for assessing the performance of these algorithms. The reduction of dimensionality of large scale data sets for tractable analysis is another scientific challenge. Here we analyse a data set of gene expression of Severe Combined Immune Deficient (SCID) mouse liver. This data set contains 28853 gene probes with five features: two of these features are related to normal groups of genes and three to MHV-MI infected diseased groups. To select the most significant genes or features in the dataset, we first apply a gene selection algorithm, and then assess the outcome by two clustering algorithms: the global k-means algorithm and the hierarchical clustering algorithm. Based on predefined information, an external validation method helps us to understand the meaning of stable clusters in our dataset.

Nonsmooth Analysis as a Tool for Matrix Correction of Improper Linear Programming Problems (in Russian)

Erohin V.I.
erohin_v_i@mail.ru
СПбГТИ(ТУ), Московский пр. 26, Санкт-Петербург, 190013, Россия

Негладкий анализ как инструмент матричной коррекции несобственных задач линейного программирования

Задачи линейного программирования (ЛП), не имеющие решения (несобственные), впервые стали предметом систематического исследования в 80-е годы XX века в трудах академика И.И. Еремина, его учеников и коллег [1]–[6], работавших в Институте математики и механики УрО РАН. Позже появились работы профессора В.А. Горелика (ВЦ им. А.А. Дородницына РАН) и его учеников [7]–[15].

Проблемы *матричной* коррекции несобственных задач ЛП допускают разнообразные постановки, главным объединяющим признаком которых является требование изменения (коррекции) коэффициентов как правой, так и левой частей соответствующих уравнений и неравенств, задающих допустимую область исследуемой задачи ЛП. Инструментами решения указанных проблем служат задачи математического программирования специального вида, среди которых встречаются и задачи негладкой оптимизации. Таким образом, на примере несобственных задач ЛП еще раз можно убедиться в актуальности проблем негладкого анализа и проследить их взаимосвязь с теоретическими и практическими проблемами ЛП.

Рассмотрим, например, задачу ЛП в канонической форме:

$$L\left(A,b,c\right): \quad Ax=b,\ x\geqslant 0,\ c^{\top}x\to\max \tag{1}$$

и соответствующую двойственную задачу ЛП в основной форме

$$L^{*}\left(A,b,c\right):\ u^{\top}A\geqslant c^{\top},\ b^{\top}u\to\min, \tag{2}$$

где $A\in\mathbb{R}^{m\times n}$, $b\in\mathbb{R}^{m}$, $c\in\mathbb{R}^{n}$.

Пусть

$$X\left(A,b\right)\triangleq\left\{x\in\mathbb{R}^{n}\left|Ax=b,x\geqslant 0\right.\right\}$$

и $U\left(A,c\right)\ \triangleq\ \left\{u\in\mathbb{R}^{m}\left|u^{\top}A\geqslant c^{\top}\right.\right\}$ — допустимые множества задач $L\left(A,b,c\right)$ и $L^{*}\left(A,b,c\right)$, хотя бы одно из которых является пустым, что делает задачи (1), (2) несобственными [1].

В данной ситуации уместна, в частности, задача *минимальной матричной* коррекции, записанная в виде

$$\left\{\begin{array}{c}\|H\|\to\min,\\ \text{Задачи } L\left(A{+}H,b,c\right), L^{*}(A{+}H,b,c)\ \text{– собственные,}\end{array}\right. \tag{3}$$

где $\|\cdot\|$ – некоторая матричная норма.

Задача (3) допускает многочисленные модификации в виде специальной структуры матриц A и H (например, блочной, тёплицевой, с произвольным множеством некорректируемых элементов), дополнительных ограничений, накладываемых на матрицу $A+H$ и пр. [10]–[13], [15]. При этом использование полиэдральных норм, таких как $\|A\|_{\ell_\infty}=\max\limits_{i,j}|a_{ij}|$ или $\|A\|_{\ell_1}=\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}|a_{ij}|$, позволяет свести задачу (3) к задаче или набору задач ЛП, делая проблему матричной коррекции проблемой негладкой оптимизации [12], [13].

Другим примером является проблема матричной коррекции несобственной задачи ЛП с блочной структурой и совместной подсистемой $A_0x=b_0,\ x\geqslant 0$ [11], [13], которая оказывается дифференцируемой минимаксной задачей (т.е. проблемой негладкой оптимизации) при использовании евклидовой нормы в минимаксной постановке

$$\begin{cases} \max\limits_{i=1,2,\ldots,k} \|H_i\| \to \min, \\ \text{Задачи } L\,(A+H,b,c)\,,\, L^*(A+H,b,c) \,-\, \text{собственные}, \end{cases} \tag{4}$$

где

$$A = \left[\begin{array}{c} A_0 \\ \hline \begin{array}{c|c|c|c} A_1 & 0 & \cdots & 0 \\ \hline 0 & A_2 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & A_k \end{array} \end{array} \right], \quad b = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_k \end{bmatrix},$$

$$H = \left[\begin{array}{c} 0 \\ \hline \begin{array}{c|c|c|c} H_1 & 0 & \cdots & 0 \\ \hline 0 & H_2 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & H_k \end{array} \end{array} \right].$$

ЛИТЕРАТУРА

[1] *Еремин И.И., Мазуров Вл.Д., Астафьев Н.Н. Несобственные задачи линейного и выпуклого программирования*, Наука, (1983).

[2] *Ватолин А.А.* "О задачах линейного программирования с интервальными коэффициентами", *Журн. вычисл. матем. и матем. физ.*, 24, No. 11, 1629–1637 (1984).

[3] *Ватолин А.А.* "Аппроксимация несобственных задач линейного программирования по критерию евклидовой нормы", *Журн. вычисл. матем. и матем. физ.*, 24, No. 12, 1907–1908 (1984).

[4] *Фролов В.Н. Оптимизация плановых программ при слабо согласованных ограничениях*, Наука, (1986).

[5] *Скарин В.Д.* "Об одном подходе к анализу несобственных задач линейного программирования", *Журн. вычисл. матем. и матем. физ.*, 26, No. 3, 439–448 (1986).

[6] *Попов Л.Д.* "Линейная коррекция несобственных выпукло-вогнутых минимаксных задач по максиминному критерию", *Журн. вычисл. матем. и матем. физ.*, 26, No. 9, 1325–1338 (1986).

[7] *Горелик В.А., Кондратьева В.А.* "Параметрическое программирование и несобственные задачи линейной оптимизации" // *Моделирование, декомпозиция и оптимизация сложных динамических процессов*, М:ВЦ РАН, 57-82 (1999).

[8] *Горелик В.А., Муравьева О.В.* “Необходимые и достаточные условия существования минимальной матрицы в задаче коррекции несовместной системы линейных уравнений” // *Моделирование, декомпозиция и оптимизация сложных динамических процессов*, М.:ВЦ РАН, 14–20 (2000).

[9] *Горелик В.А.* “Матричная коррекция задачи линейного программирования с несовместной системой ограничений”, *Журн. вычисл. матем. и матем. физ.*, 41, No. 11, 1697–1705 (2001).

[10] *Ерохин В.И.* “ Оптимальная матричная коррекция и регуляризация несовместных линейных моделей”, *Дискретный анализ и исследование операций. Сер. 2.*, 9, No. 2, 41–77 (2002).

[11] *Горелик В.А., Ерохин В.И., Печенкин Р.В.* “ Оптимальная матричная коррекция несовместных систем линейных алгебраических уравнений с блочными матрицами коэффициентов ”, *Дискрет. анализ и исслед. операций. Сер. 2.*, 12, No. 2, 3–22 (2005).

[12] *Горелик В.А., Ерохин В.И., Печенкин Р.В.* “Оптимальная матричная коррекция несовместных блочных систем линейных алгебраических уравнений с минимаксным критерием”, *Известия РАН, ТиСУ*, 45, No. 5, 575–588 (2006).

[13] *Горелик В.А., Ерохин В.И., Печенкин Р.В. Численные методы коррекции несобственных задач линейного программирования и структурных систем уравнений*, М.:ВЦ РАН, (2006).

[14] *Ерохин В.И.* “Матричная коррекция двойственной пары несобственных задач линейного программирования”, *Журн. вычисл. матем. и матем. физ.*, 47, No. 4, 587–601 (2007).

[15] *Ерохин В.И., Красников А.С.* “Матричная коррекция двойственной пары несобственных задач линейного программирования с блочной структурой”, *Журн. вычисл. матем. и матем. физ.*, 48, No. 1, 80–89 (2008).

Non-smooth variational methods for a Dirichlet problem with one-side growth condition

Faraci F., Iannizzotto A.

ffaraci@dmi.unict.it, iannizzotto@dmi.unict.it

Dipartimento di Matematica e Informatica, Università di Catania, Viale A. Doria 6, 95125 Catania, Italy

In the present talk we deal with a Dirichlet problem of the following type:

$$(P) \qquad \begin{cases} -\Delta u = \lambda\,(f(x,u) + \mu g(x,u)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain ($N > 2$), λ and μ are positive parameters and $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying one-side growth conditions of the type:

(*i*) $\sup_{|t|\leqslant s} |f(\cdot,t)|, \sup_{|t|\leqslant s} |g(\cdot,t)| \in L^1(\Omega)$ for all $s > 0$;

(*ii*) $\max\{sf(x,s), sg(x,s)\} \leqslant a(x) + b|s|^{2^*}$ for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$ $(a \in L^1(\Omega)_+,\ b > 0)$,

plus some technical assumptions on f.

We prove the existence of at least two solutions of problem (P), for λ and μ lying in convenient intervals. Our tools are an abstract multiplicity result for local minimizers of a function defined on a topological space due to B. Ricceri ([3]), and a very general nonsmooth critical point theory for lower semicontinuous functionals on metric spaces developed by Degiovanni, Marzocchi and Zani (see [1] and [2]).

References

[1] M. Degiovanni, M. Marzocchi, "A critical point theory for nonsmooth functionals *Ann. Mat. Pura Appl.* 167 73–100 (1994).

[2] M. Degiovanni, S. Zani, "Euler equations involving nonlinearities without growth conditions *Potential Anal.* 5, 505–512 (1996).

[3] B. Ricceri, "Existence of three solutions for a class of elliptic eigenvalue problems *Math. Comput. Modelling* 32, 1485–1494 (2000).

The Directed Subdifferential and applications

Farkhi E.

elza@post.tau.ac.il

School of Math. Sci., Tel-Aviv University, Ramat Aviv, Tel-Aviv, 69978, Israel

This is a joint work with Robert Baier and Vera Roshchina.

The directed subdifferential of quasidifferentiable functions is introduced as the difference of two convex subdifferentials embedded in the Banach space of directed sets. Thus we are able to capture efficiently differential properties of a large variety of functions including amenable and lower/upper-C^k functions.

Preserving the most important properties of the quasidifferential, such as exact calculus rules, the directed subdifferential lacks major drawbacks of the quasidifferential: non-uniqueness and growing in size of the two convex sets representing the quasidifferential after applying calculus rules. Its visualization, which we call Rubinov subdifferential, is a non-empty, generally non-convex set in R^n.

Calculus rules for the directed subdifferentials of sum, product, quotient, maximum and minimum of quasidifferentiable functions are derived. The relations between the Rubinov subdifferential and the subdifferentials of Clarke, Dini, Michel-Penot, and Mordukhovich are discussed. Important properties implying the Ioffe's axioms for subdifferentials as well as necessary and sufficient optimality conditions for the directed subdifferential are obtained in terms of the directed and Rubinov subdifferential.

A bundle method for nonconvex nonsmooth minimization

Fuduli A., Gaudioso M., Nurminski E.A.
antonio.fuduli@unical.it, gaudioso@deis.unical.it, nurmi@dvo.ru
Dipartimento di Matematica - Universita' della Calabria - Via Pietro Bucci - Cubo 31B - Rende (CS) - Italia, Institute for Automation and Control Processes - Far East Branch, Russian Academy of Sciences - Vladivostok - Russia

We present a new bundle method for nonsmooth nonconvex minimization, where, at each iteration, the search direction is computed by solving a quadratic subproblem, based on the construction of a local lower approximation of the objective function of the cutting plane type. The main feature of the method is the way the negative linearization errors are treated.

The quadratic subproblem is solved by means of appropriate projection procedures which reduce to finding the minimum norm vector in a set given by the sum of a polyhedron plus a cone. In addition, inexact solution of the latter problem is also considered. Some numerical results are presented.

Cutting plane-based methods for convex minimization

Gaudioso M.
gaudioso@deis.unical.it
Dipartimento di Elettronica Informatica e Sistemistica - Universita' della Calabria - Via Pietro Bucci - Cubo 41C - Rende (CS) - Italia

Cutting plane is the very basic model for many algorithms (e.g. the entire bundle family) which have been designed in the last decades to deal with convex nonsmooth minimization.

We introduce a variant of the cutting plane model which is based on possible vertical shifting of some affine pieces of the polyhedral model. Our method stays somehow in between the classic Wolfe's conjugate subgradient method and the standard cutting plane.

In the talk we review also some possible extensions of the cutting plane method to the nonconvex setting.

Reproducing Kernel Banach Spaces and applications

Georgiev P.G., Sánchez-González L., Pardalos P.
pandogeorgiev@ufl.edu, lfsanche@mat.ucm.es, pardalos@ufl.edu
Center for Applied Optimization, University of Florida, Gainesville, FL 32601, Complutenza University, Madrid, Spain, Center for Applied Optimization, University of Florida, Gainesville, FL 32601, The National Research University - Higher School of Economics, Laboratory of Algorithms and Technologies for Networks Analysis, N.Novgorod, Russia

We extend the idea of reproducing kernels to Banach spaces and develop a theory of Reproducing Kernel Banach Spaces (RKBS), without the requirement for existence of semi-inner product (which requirement is already explored in another construction of RKBS). Several applications are presented: 1) we show how a special interpolation task with prescribed non-smooth functions from a RKBS can be performed by an algorithm involving a simple procedure for solving a linear system. However, such an algorithm needs a global optimization technique, which we address by reformulating the task as a mixed integer minimization problem; 2) In the terminology of statistical learning theory, we apply our construction of RKBS to the basic learning algorithms, including support vector machines, and generalize the kernel regression problem to a multi-class classification task; 3) The fundamental idea of embedding the domain set into a functional space (RKBS) and linearizing in such a way a non-linear problem, can be exploited further for some classes of global optimization problems and variational principles.

On the scalarization of some variational problems

Giannessi F., Mastroeni G.
fgiannessi3@gmail.com, mastroen@dm.unipi.it
Department of Mathematics,University of Pisa, L.go B. Pontecorvo, 5 56127 - PISA, Italy.

The image space analysis [2] has shown to be a powerful tool and a unifying scheme for studying both Vector Optimization Problems (VOP) and Vector Variational Inequalities (VVI). More generally, this approach can be applied to any kind of problem that can be expressed under the form of the impossibility of a parametric system.

In the present talk, exploiting separation arguments in the image space, we aim at developing the analysis of scalarization techniques for a generalized system, considering in particular the applications to VOP and VVI. We will show that the analysis in the image space allows one to recover most of the existing scalarization functions and to extend the applications to more general classes of optimization problems as vector equilibrium problems and set-valued optimization problems.

References

[1] *Giannessi F., Mastroeni G. and Pellegrini L.,* "On the Theory of Vector Optimization and Variational Inequalities. Image Space Analysis and Separation". In "Vector Variational Inequalities and Vector Equilibria" Kluwer Academic Publishers, Dordrecht, Boston, London, pp. 153-215, (2000).

[2] *Giannessi F., Constrained Optimization and Image Space Analysis.* Springer, London, (2005).

[3] *Mastroeni G.,* "Optimality Conditions and Image Space Analysis for Vector Optimization Problems". In *Recent Developments in Vector Optimization*, Q.H Ansari and J.-C. Yao (Ed.s), pp. 169-220, Springer, (2011).

Data Correcting Algorithms in Networks Optimization

Goldengorin B.

bgoldengorin@hse.ru

Laboratory of Algorithms and Technologies for Networks Analysis, National Research University Higher School of Economics, Nizhny Novgorod, Russia

Computer scientists have found that certain types of problems, called NP-hard problems, are intractable. Roughly speaking this means that the time it takes to solve any typical NP-hard problem seems to grow exponentially as the amount of input data (instance) increases. On the other hand, for many NP-hard problems we can provide provable analytic or algorithmic characterizations of the instances input data that guarantee a polynomial time solution algorithm for the corresponding instances. These instances are called *polynomially solvable special cases* of the combinatorial optimization problem.

Polynomially solvable special cases of combinatorial optimization problems have long been studied in the literature [1-6].

This talk is a step in the direction of incorporating polynomially solvable special cases into approximation and exact algorithms. We propose a *Data Correcting (DC) algorithm* — an approximation algorithm that makes use of polynomially solvable special cases to arrive at high-quality solutions. The basic insight that leads to this algorithm is the fact that it is often easy to compute an upper bound on the difference in cost between an optimal solution of a problem instance and any feasible solution to the instance. The results obtained with this algorithm are very promising.

The approximation in the DC algorithm is in terms of an *accuracy parameter*, which is an upper bound on the difference between the objective value of an optimal solution to the instance and that of a solution returned by the

DC algorithm. Note that this is not expressed as a fraction of the optimal objective value for this instance. In this respect, the algorithm is different from common ε-optimal algorithms, in which ε is defined as a fraction of the optimal objective function value.

Even though the algorithm is meant mainly for NP-hard combinatorial optimization problems, it can be used for functions defined on a continuous domain too. We will, in fact, motivate the DC approach by using a nondifferentiable function defined on a domain that has a finite range. We then show how this approach can be adapted for NP-hard optimization problems, using the Asymmetric TSP as an illustration. We conclude this talk by a summary and future research directions

References

[1] Goldengorin B. I. Correcting algorithm for the solution of some discrete optimization problems. Dokl. Akad. Nauk SSSR 270 (1983), no. 3, 525–528. MR0711845 (84k: 90073), 90C30 (65K10), (translated from Russian).

[2] Goldengorin, Boris; Tijssen, Gert A.; Ghosh, Diptesh; Sierksma, Gerard Solving the simple plant location problem using a data correcting approach. J. Global Optim. 25 (2003), no. 4, 377–406. (Reviewer: Zeng Kun Xu) MR1962793 (2003m:90054), 90B80 (90C59)

[3] Goldengorin, Boris; Ghosh, Diptesh A multilevel search algorithm for the maximization of submodular functions applied to the quadratic cost partition problem. J. Global Optim. 32 (2005), no. 1, 65– 82. (Reviewer: Hans-Ulrich Simon) MR2155564 (2006c:90115), 90C57 (90C27).

[4] D. Ghosh, B. Goldengorin, G. Sierksma. Data Correcting: A Methodology for Obtaining Near-Optimal Solutions. Operations Research with Economic and Industrial Applications: Emerging Trends. S.R. Mohan and S.K. Neogy (Eds.), Anamaya Publishers, New Delhi, India, 2005, pp. 119–127.

[5] Ghosh, Diptesh; Goldengorin, Boris; Sierksma, Gerard. Data correcting algorithms in combinatorial optimization. Handbook of combinatorial optimization. Supplement Vol. B, 1–53, Springer, New York, 2005. MR2114589, 90C27 (90C35).

[6] B. Goldengorin. Data Correcting Approach for Routing and Location in Networks. Handbook of combinatorial optimization. Pardalos, Panos M.; Du, Ding-Zhu; Graham, Ronald L. (Eds.), 2nd Edition, Springer, New York, 2013, 77 pp. http://www.springer.com/mathematics/book/978-1-4419-7996-4

Applications of Proximal Analysis to Regularity of a kind of Marginal Function

Goncharov V.V., Pereira F.F.
goncha@uevora.pt, fmfp@uevora.pt
CIMA-UE, Universidade de Évora, rua Romão Ramalho 59, 7000-671, Évora, Portugal

Continuing research in [3] and [4] on well-posedness of the optimal time control problem with a constant convex dynamics in a Hilbert space we adapt one of the regularity conditions obtained there to a slightly more general problem, where nonaffine additive term appears.

Namely, let H be a real Hilbert space, $C \subset H$ be an arbitrary nonempty closed subset, and $\theta : C \to \mathbb{R}$ be a lower semicontinuous function. Given a closed convex bounded set $F \subset H$ having the origin in the interior, we consider the following Mathematical Programming problem

$$\text{Minimize} \;\; \rho_{\mathrm{F}}(\mathrm{x}-\mathrm{y})+\theta(\mathrm{y}) \quad \text{on} \quad \mathrm{C}, \tag{1}$$

where $\rho_F(\cdot)$ is the *Minkowski functional* (*gauge function*) associated to F,

$$\rho_F(\xi) := \inf\{\lambda > 0 : \; \lambda^{-1}\xi \in F\}.$$

If $C = H \setminus \Omega$, where Ω is an open bounded region in H (for instance, in a finite dimensional space), and $\theta(\cdot)$ satisfies a *slope condition* w.r.t. F then the *value function* $\hat{u}(\cdot)$ in (1) can be interpreted as the (unique) *viscosity solution* of the *Hamilton-Jacobi equation*

$$\rho_{F^0}(-\nabla u(x)) - 1 = 0;$$

$$u(x) = \theta(x), \quad x \in \partial\Omega.$$

Here and in what follows F^0 means the *polar set* of F. Furthermore, if $\theta : H \to \mathbb{R}$ is of class $\mathcal{C}^1$ then $\mathfrak{T}_C^{F,\theta}(x) := \hat{u}(x) - \theta(x)$ gives the *minimal time* necessary to achieve the *target set* C from the point $x \in H$ by trajectories of the differential inclusion

$$\dot{x} \in \left(F^0 + \nabla\theta(x)\right)^0.$$

We are interested, in particular, in (*Fréchet*) *differentiability* of the value function $\hat{u}(\cdot)$ out of C. However, taking into account the interpretation as the viscosity solution (see above) we observe that it is not possible to have such differentiability everywhere inside Ω at once. This happens, e.g., already in the simplest case of the so called *eikonal equation* (F is a closed unit ball in H) and $\theta \equiv 0$ (its solution $\hat{u}(\cdot)$ is nothing else than the usual distance from point to the set C). Therefore, we are led to study the regularity of $\hat{u}(\cdot)$ in a neighbourhood of the target only.

On the other hand, the differentiability of $\hat{u}(\cdot)$ strongly relates with the existence, uniqueness and regularity of minimizers in the problem (1) (in the case of compact C one can see this, for instance, from the representation of the (*Clarke's*) *subdifferential* of a *marginal function* through Radon measures supported on the set of minimizers $\pi_S^{F,\theta}(x)$ as given in [1, Section 2.8]). So, we find first rather general conditions guaranteeing that $\pi_S^{F,\theta}(x)$ is a singleton (Lipschitz) continuous near a point $x_0 \in \partial C$. These conditions involve a ballance between the (proximal) subgradients of the restriction $\theta\,|_C$ and the normals to F. Then, under the same (local) hypotheses assuming, in addition, the smoothness either of $\theta\,|_C$ or of F we prove that the function $\hat{u}(\cdot)$ is (Fréchet) differentiable in $(x_0 + \delta\overline{B}) \setminus C$ for some $\delta > 0$ (with eventually Hölder continuous gradient $\nabla\hat{u}(\cdot)$). To achieve this goal we apply the mixed technique of Convex Analysis with some tools of Proximal Calculus (see [2, Chapter 1]) such as the *fuzzy sum rule* and the *Ekeland's variational principle*.

References

[1] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York (1983).

[2] F.H. Clarke, Yu.S. Ledyaev, R.J. Stern, P.R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer, New York (1998).

[3] V.V. Goncharov, F.F. Pereira, "Neighbourhood retractions of nonconvex sets in a Hilbert space via sublinear functionals", *J. Convex Anal.*, 18, 1-36 (2011).

[4] V.V. Goncharov, F.F. Pereira, "Geometric conditions for regularity in a time-minimum problem with constant dynamics *J. Convex Anal.*, to appear (2012).

Vector Optimization Problems with Non-solid Positive Cone: Scalarization and First and Second Order Optimality Conditions

Gorokhovik V.V.

gorokh@im.bas-net.by

Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, 220072, Belarus

Let X and Y be Banach spaces over the real field $\mathbb{R}$. The space Y is supposed to be ordered by a strict partial order $\prec$ such that for any $y_1,\, y_2 \in Y$ one has $y_1 \prec y_2$ if and only if $y_2 - y_1 \in P$, where $P \subset Y$ is an asymmetric convex cone with an empty interior.

The talk deals with a vector optimization problem

$$(VOP) \qquad\qquad \prec\text{–minimize}\, F(x) \text{ subject to } x \in Q,$$

where $F : X \to Y$ is a mapping from X into Y, Q is a subset of X.

A point $x^0 \in Q$ is called *a local $\prec$–minimizer* for (VOP) if there exists a neighborhood $N(x^0)$ of x^0 such that $F(x) \not\prec F(x^0)$ for all $x \in Q \cap N(x^0)$.

In the talk we present a unified approach to deriving first and second order local $\prec$–minimality conditions, both necessary and sufficient, for feasible solutions of vector optimization problems with non-solid positive cone. Our approach includes two stages.

At first we scalarize the vector optimization problem (VOP). To do it we assume that the vector subspace $P - P$, which coincides with the linear hull of P, is topologically complemented in Y and $\mathrm{ri}P \neq \varnothing$, where $\mathrm{ri}P$ is the relative interior of P. These assumptions allows us to choose a continuous projection operator $\pi : Y \to Y$ with $\ker\pi = P - P$ and a bounded sublinear function $\sigma : Y \to \mathbb{R}$ such that

$$\mathrm{ri}\, P_\prec = \{y \in Y \mid \sigma_\prec(-y) < 0,\, \pi_\prec(y) = 0\},$$

$$\mathrm{cl}\, P_\prec = \{y \in Y \mid \sigma_\prec(-y) \leqslant 0,\, \pi_\prec(y) = 0\}$$

and to prove the following scalarization theorem.

a) If a point $x^0 \in Q$ is a local $\prec$–minimizer of the mapping $F : X \to Y$ over the set $Q \subset X$ then there exists a neighborhood $\mathcal{N}(x^0)$ of the point x^0 such that

$$\sigma(F(x) - F(x^0)) \geqslant 0 \;\forall\, x \in Q \cap \mathcal{N}(x^0) \text{ such that } \pi_\prec(F(x) - F(x^0)) = 0. \quad (1)$$

b) If there exists a neighborhood $\mathcal{N}(x^0)$ of a point $x^0 \in Q$ such that

$$\sigma(F(x) - F(x^0)) > 0 \;\forall\, x \in Q \cap \mathcal{N}(x^0),\, x \neq x^0, \\ \text{such that } \pi_\prec(F(x) - F(x^0)) = 0, \quad (2)$$

then the point x^0 is a local $\prec$–minimizer of the mapping $F : X \to Y$ over the set $Q \subset X$.

Then we analyze (1) and (2) with variational (convex and nonsmooth) methods and derive both necessary and sufficient $\prec$–minimality conditions for local $\prec$–minimizers of (VOP).

Analyzing (1) and (2) we assume that the objective mapping F is differentiable in one or another sense, in particular, we admit that F is twice parabolic directionally differentiable. As local approximations for the feasible set Q and the set $E(x^0)$ we use first- and second-order tangent vectors. In the case when the objective mapping F is twice Frechet differentiable and the feasible set Q is the whole space X the $\prec$–minimality conditions are presented in the prime and dual forms. The first order dual necessary $\prec$–minimality condition has the form of the Lagrange multipliers rule while the second order dual necessary $\prec$–minimality condition asserts that the maximum of the family quadratic forms on the cone of critical vectors is nonnegative. Note that necessary $\prec$–minimality conditions are obtained under the additional assumption that a local $\prec$–minimizer satisfies the regularity condition.

This study was supported by Belarusian Republican Foundation for Basic Research (Project No. F10R - 025).

References

[1] *V.V. Gorokhovik, Convex and Nonsmooth Problems of Vector Optimization*, Navuka i Tekhnika, Minsk, (1990).

[2] *V.I. Bakhtin, V.V. Gorokhovik,* “First and second order optimality conditions for vector optimization problems on metric spaces”, *Proceedings of the Steklov Institute of Mathematics*, Suppl. 2, S28 – S39, (2010).

Primal exhausters of positively homogeneous functions

Gorokhovik V.V., Starovoitava M.A.
gorokh@im.bas-net.by, marvoitova@tut.by
Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, 220072, Belarus, Belarusian State University, Minsk, 220050, Belarus

In the talk we deal with positively homogeneous functions defined on a finite-dimensional vector space $\mathbb{R}^n$. Recall, that a function $p : \mathbb{R}^n \to \mathbb{R}$ is said to be positively homogeneous if $p(\lambda x) = \lambda p(x)$ for all $x \in \mathbb{R}^n$.

A positively homogeneous function $p : \mathbb{R}^n \to \mathbb{R}$ is said to be sublinear (superlinear) if it is convex (concave) or, equivalently, if it is subadditive (superadditive).

A family $\Phi := \{\varphi\}$ of sublinear functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ will be referred to as *a primal upper exhauster* of a positively homogeneous function $p : \mathbb{R}^n \to \mathbb{R}$ if

$$p(x) = \inf_{\varphi\in\Phi} \varphi(x) \text{ for all } x \in \mathbb{R}^n.$$

Due to the Minkowski duality each primal upper exhauster $\Phi := \{\varphi\}$ can be associated in a unique way with the family $\partial\Phi = \{\partial\varphi \,|\, \varphi \in \Phi\}$, where $\partial\varphi := \{u \in \mathbb{R}^n \,|\, \langle u, x\rangle \leqslant \varphi(x)\ \forall\, x \in \mathbb{R}^n\}$ is a subdifferential of the sublinear function φ (at 0). The family $\partial\Phi$ is called *the dual upper exhauster* of the positively homogeneous function $p : \mathbb{R}^n \to \mathbb{R}$.

Similarly, a family $\Psi := \{\psi\}$ of superlinear functions $\psi : \mathbb{R}^n \to \mathbb{R}$ is referred to as *a primal lower exhauster* of a positively homogeneous function $p : \mathbb{R}^n \to \mathbb{R}$ if

$$p(x) = \sup_{\psi\in\Psi} \psi(x) \text{ for all } x \in \mathbb{R}^n.$$

The family $\partial\Psi = \{\partial\psi \,|\, \psi \in \Psi\}$, where $\partial\psi := \{v \in \mathbb{R}^n \,|\, \langle v,\, x\rangle \geqslant \psi(x)\ \forall\, x \in \mathbb{R}^n\}$ is a superdifferential of the superlinear function ψ (at 0) is called *the dual lower exhauster* of the positively homogeneous function $p : \mathbb{R}^n \to \mathbb{R}$.

The notions of upper and lower exhausters were introdoced by Demyanov and Rubinov [1]. They proved that a positively homogeneous function $p : \mathbb{R}^n \to \mathbb{R}$ is upper (lower) semicontinuous on $\mathbb{R}^n$ if and only if it admits an upper (lower) exhauster. This implies that a positively homogeneous function $p : \mathbb{R}^n \to \mathbb{R}$ is continuous on $\mathbb{R}^n$ if and only if it admits both an upper exhauster and a lower one.

Other results concerning dual exhausters can be found in [2, 3, 4].

By $\mathcal{P}_C(\mathbb{R}^n)$ we will denote the Banach space of continuous positively homogeneous functions on $\mathbb{R}^n$ endowed with the norm $\|p\|_C := \max\limits_{\|x\|_{\mathbb{R}^n}=1} |p(x)|$. Considering primal upper and lower exhausters as subsets of the Banach space $\mathcal{P}_C(\mathbb{R}^n)$ we can perform various operations on them. In particular, passing to the topological closure of primal exhausters or the convex hull of ones we obtain also primal exhausters of the same positively homogeneous function.

In the next theorems we characterize various classes of positively homogeneous functions via the properties of their primal exhausters. The first theorem concerns with functions which satisfy the Lipschitz condition on the whole space $\mathbb{R}^n$.

Theorem 1 (cf. [4]). *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a positively homogeneous function. Then the following statements are equivalent:*

(i) p is Lipschitz continious on $\mathbb{R}^n$;

(ii) p admits a compact (as a subset of the Banach space $(\mathcal{P}_C(\mathbb{R}^n),\, \|\cdot\|_C)$) primal upper exhauster;

(iii) p admits a compact (as a subset of the Banach space $(\mathcal{P}_C(\mathbb{R}^n),\, \|\cdot\|_C)$) primal lower exhauster.

The next theorem characterize difference-sublinear functions, that is, the functions which may be represented as a difference of two sublinear functions.

Theorem 2. *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a positively homogeneous function. The following statements are equivalent:*

(i) p is a difference-sublinear function;

(ii) p admits a sublinear majorant $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that the family

$$\Phi_{p,\varphi} := \{x \to \varphi(x) - \langle v,\, x\rangle \,|\, v \in V_{p,\varphi}\},$$

where

$$V_{p,\varphi} = \{v \in \mathbb{R}^n \,|\, p(x) \leqslant \varphi(x) - \langle v, x\rangle\ \forall x \in \mathbb{R}^n\},$$

is a primal upper exhauster of the function p;

(iii) p admits a superlinear minorant $\psi : \mathbb{R}^n \to \mathbb{R}$ *such that the family*

$$\Psi_{p,\psi} := \{x \to \psi(x) + \langle u,\, x\rangle \,|\, u \in U_{p,\psi}\},$$

where

$$U_{p,\psi} = \{u \in \mathbb{R}^n \,|\, p(x) \geqslant \psi(x) + \langle u, x\rangle \ \forall x \in \mathbb{R}^n\}$$

is a primal lower exhauster of the function p.

Let $L(\mathbb{R}^n)$ be the vector space of linear functions on $\mathbb{R}^n$.

We say that a family Γ of positively homogeneous functions is $L(\mathbb{R}^n)$-*planar*, if the difference of any two functions of Γ belongs to $L(\mathbb{R}^n)$ or, equivalently, if the difference of any two functions of Γ is a linear function.

Corollary (cf. [4]). *The following statements are equivalent:*

(i) p is difference-sublinear;

(ii) p admits a $L(\mathbb{R}^n)$*-planar compact (as a subset of the Banach space* $(\mathcal{P}_C(\mathbb{R}^n), \|\cdot\|_C)$*) primal upper exhauster;*

(iii) p admits a $L(\mathbb{R}^n)$*-planar compact (as a subset of the Banach space* $(\mathcal{P}_C(\mathbb{R}^n), \|\cdot\|_C)$*) primal lower exhauster.*

In the last theorem we give a characterization of piecewise linear functions. Recall, that a function $p : \mathbb{R}^n \to \mathbb{R}$ is said to be piecewise linear if it is continuous and there exists a finite covering of the space $\mathbb{R}^n$ by convex cones $K_1, K_2, \ldots, K_k$ such that the restriction of p on each cone K_i coincides with some linear function.

Theorem 3. *A positively homogeneous function* $p : \mathbb{R}^n \to \mathbb{R}$ *is piecewise linear if and only if it admits both a finite primal (dual) upper exhauster and a finite primal (dual) lower exhauster.*

The research was supported by the National Program of Basic Researches of Belarus.

References

[1] *V.F. Demyanov, A.M. Rubinov, Constructive Nonsmooth Analysis*, Verlag Peter Lang, Frankfurt, (1995).

[2] *V.F. Demyanov,* "Exhausters of a positively hoomogeneous function", *Optimization*, **45**, No. 1, 13 – 29, (1999).

[3] *V.F. Demyanov,* "Exhausters and convexificators — new tools in nonsmooth analysis", In: Demyanov V.F. and Rubinov A.M. (Eds). *Quasidifferentiability and Related Topics*, Dordrecht, Kluwer Academic Publishers, 2000, 85 – 137.

[4] *V.F. Demyanov, V.A. Roshchina,* "Exhausters, optimality conditions and related problems" *Journal of Global Optimization*, **40**, No. 1 – 3, 71 – 85, (2008).

The problem of separating players into two coalitions with non-smooth analysis

Grigorieva X.
kseniya196247@mail.ru
Санкт-Петербургский Государственный Университет, Университетский пр., д.35, Санкт-Петербург, Петродворец, 198504, Россия

It is suggested to employ a linear criterion (with respect to given parameters) to identify players using a nonsmooth model.

Задача разделения игроков на две коалиции с помощью негладкого анализа

В прикладных задачах игрового типа возникает необходимость в использовании коалиционных моделей [1-4], суть которых заключается в следующем. Пусть имеется N игроков, разделенных на l коалиций, задано множество стратегий каждой коалиции и функция выигрыша для каждой коалиции на множестве ситуаций, образованных выбранными стратегиями коалиций. При этом предполагается, что коалиционное разбиение задано. Однако не в каждой задаче оговаривается, по какому критерию происходит разделение игроков на коалиции. Таким образом, формирование коалиционного разбиения - это отдельная задача, о которой ниже и пойдет речь.

Пусть задано множество стратегий каждого игрока и функция выигрыша для каждого игрока на множестве ситуаций, образованных выбранными стратегиями игроков. Требуется разделить игроков на две непересекающиеся коалиции. С этой целью вводится критерий идентификации игроков и возникает задача разделения множеств [5].

В работе предлагается применить линейный критерий идентификации игроков по набору параметров (гиперплоскость) с использованием негладкой модели [6]. Решая задачу идентификации относительно различных параметров, можно получить совокупность всевозможных разбиений множества N на два подмножества. В ряде случаев важно определить наиболее информативные (наиболее существенные) параметры для идентификации.

Рассмотрим, в частности, задачу о выборах, в которой N игроков — это избиратели, участвующие в альтернативных выборах. Целью является прогнозирование исхода выборов. В качестве параметров в этой задаче могут использоваться возраст, национальность, семейное положение, уровень доходов, политические предпочтения, идеологическая направленность и т. д. Требуется идентифицировать избирателей как потенциальный электорат каждой из партий, иными словами, разделить множество избирателей на две коалиции. Решение задачи протестировано на примере базы данных о голосовании 1984 года Конгресса США [7].

Сформулированы критерии идентификации, задача сведена к задаче разделения множеств с помощью гиперплоскости. Решение, полученное с использованием методов негладкого дискриминантного анализа, позволяет с некоторой точностью прогнозировать исход выборов и получить составы коалиций.

Работа выполнена при поддержке РФФИ, проект № 12-01-00752.

ЛИТЕРАТУРА

[1] Григорьева К. В., "Оптимизация в классе стохастических коалиционных игр", *Математическая теория игр и ее приложения*, 2, No. 1, 47-66 (2010).

[2] X. Grigorieva, S. Mamkina, "Solutions of Bimatrix Coalitional Games", *Contributions to game and management. Collected papers printed on the Second International Conference «Game Theory and Management» [GTM'2008], Edited by Leon A. Petrosjan, Nikolay A. Zenkevich*, 147-153 (2009).

[3] L. Petrosjan, S. Mamkina, "Dynamic Games with Coalitional Structures", *Intersectional Game Theory Review*, 8, No. 2, 295-307 (2006).

[4] Н. Зенкевич, Л. Петросян, Д. Янг, *Динамические игры и их приложения в менеджменте*, Высшая Школа Менеджмента, (2009).

[5] Григорьева К.В., "Суррогатные функционалы в задачах диагностики", *Вестник СПбГУ. Сер. 10*, No. 1, 34–44 (2009).

[6] Демьянов В.Ф., Васильев Л.В., *Недифференцируемая оптимизация*, М., (1981).

[7] http://mlearn.ics.uci.edu/databases/voting-records/.

Evolutionary Model of Tax Auditing

Gubar E., Kumacheva S.

alyona.gubar@gmail.com, s_kumach@mail.ru

St.Petersburg State University, Faculty of Applied Mathematics and Control Processes, Universitetskiy pr. 35, St.Petersburg, 198504, Russia

A model of tax auditing of the finite number of taxpayers is considered. It is supposed that each taxpayer has an income ξ and declares his level of income as η, $\eta \leqslant \xi$, after every tax period. To simplify the following arguments it is also supposed that taxpayers have only two levels of income — L and H, where $L < H$ (as it was done in [2, 8]). Therefore, there exist only one possibility of tax evasion, that is to declare $\eta = L$, when $\xi = H$.

The tax authority can audit taxpayers, declared $\eta = L$, with the probability $P\{A \,|\, \eta = L\}$, see ([1], [5]). If the evasion was revealed, the taxpayer must pay the level of his evasion and penalty $(\theta + \pi)(\xi - \eta)$, where θ and π are tax and marginal penalty rates correspondingly.

The tax authority assumed to get some statistical information about tax evasions from the previous tax periods. The mentioned information is called a signal s, as it was called in [5]. It is supposed that the probability of tax auditing in the current period depends on the signals.

Suppose that all considered taxpayers possess one of the three statuses, they can be Risk-Averse, Risk-Neutral and Risk-Preferred. Hence the total population of taxpayers is divided into three subgroups: Risk-Averse, Risk-Neutral and Risk-Preferred. An individual from each subgroup holds one programmed pure strategy. A population state in this model is described by shares of individuals with corresponding status.

We consider dynamical model in which individuals from each subgroup have a possibility to observe a situation (population state) after every tax period and at a given signal they can change their status. A decision about status changing depends on the payoffs that guaranteed by the pure strategies and the population's state.

Thereby during the long-time period we have a chain of changes of population states that can be described by imitations dynamics.

In our work we consider several modifications of dynamics such as dynamics of pairwise comparison, dynamics of imitation of successful agents and dynamics of pure imitation driven by dissatisfaction [4, 6, 7, 9]. To illustrate our model we are planning numerical simulations in each case.

References

[1] *Bure V.M., Kumacheva S.Sh.* A game theory model of tax auditing using statistical information about taxpayers, *Vestnik SPbGU*, series 10, vol. 4, P. 16–242, (2010).

[2] *Chander P., Wilde L.,* A General Characterization of Optimal Income Tax Enfocement, *Review of Economic Studies*, vol. 65, (1998).

[3] *Friedman D.* Evolutionary Games in Economics, *Econometrica*, 59: 637-666, (1991).

[4] *Gubar E.A. Construction Different Types of Dynamics in an Evolutionary Model of Trades in the Stock Market.* Contributions to Game Theory and Management. Vol 3. Eds. Leon A. Petrosyan, Nikolay A. Zenkevich - SPB.: Graduate School of Management, SPbU, pp. 162-171, (2010).

[5] *Macho-Stadler I., Perez-Castrillo J. D.* Auditing with signals, *Economica*, 02, P. 1–20, (2002).

[6] *Sandholm W.H., Durlauf S.N. and L.E. Blume.* Deterministic Evolutionary Dynamics, *The New Palgrave Dictionary of Economics*, 2nd edition, eds., Palgrave Macmillan, (2008).

[7] *Sandholm W. H.* Population Games and Evolutionary Dynamics, *The M.I.T.Press*, (2010).

[8] *Vasin A. A., Morozov V. V.* The Game Theory and Models of Mathematical Economics. *M.: MAKSpress*, P. 143–293, (2005).

[9] *Weibull J.* Evolutionary Game Theory, *Cambridge, MA: The M.I.T.Press*, (1995).

Approximation and Regularization Methods in Nonsmooth Mechanics

Gwinner J.

Joachim.Gwinner@unibw-muenchen.de

Institute of Mathematics, Department of Aerospace Engineering, Universität der Bundeswehr München, D-85577 Neubiberg, Germany

In this talk we report on recent progress in the numerical treatment of various nonsmooth boundary value problems in continuum mechanics, including unilateral contact with Tresca friction, see e.g. [1] and nonmonotone adhesion/delamination problems, see [7]. The contribution is based on the recent papers [4, 5] of the author and the recent PhD Thesis [6] of N. Ovcharova written under his guidance.

Firstly in an appropriate vectorial Sobolev function space V on a planar Lipschitz domain, we deal with nonsmooth variational problems in the form: Find $\mathbf{u} \in \mathbf{K}$ such that for all $\mathbf{v} \in \mathbf{K}$,

$$a(\mathbf{u}, \mathbf{v}-\mathbf{u}) + \mathbf{j}(\mathbf{v}) - \mathbf{j}(\mathbf{u}) \geqslant \mathbf{l}(\mathbf{v}-\mathbf{u}),$$

where K is the convex closed subset of kinematically admissible displacements,

$$K = \{\mathbf{v} \in \mathbf{V} \,|\, (\gamma_\mathbf{c}\, \mathbf{v})_\mathbf{n} \leqslant \mathbf{d}\},$$

γ_c denotes the trace map onto the boundary part Γ_c, and $d \geqslant 0$ is the initial gap between the body and the rigid foundation. Moreover, a is the bilinear form of strain energy and l the linear form of outer forces; on the boundary part Γ_c, we have the nonsmooth friction functional

$$j(\mathbf{v}) = \int_{\mathbf{\Gamma_c}} \mathbf{g}\, |(\gamma_\mathbf{c}\, \mathbf{v})_\mathbf{t}|\, \mathbf{ds}.$$

Here and in the following, the subscripts t, n stand for the tangential, respectively normal component of a vector field at the boundary.

Let $\mathcal{P}_N$ ($N \in \mathbb{N}$) be a shape regular sequence of meshes consisting of affine quadrilaterals $Q \in \mathcal{P}_N$ with diameter $h_{N,Q}$. Moreover, we introduce the set of edges on the contact boundary,

$$\mathcal{E}_{c,N} = \{E : \; E \subset \Gamma_c \text{ is an edge of } \mathcal{P}_N\}$$

and assume that g is piecewise constant with $g = g_E$ on each edge $E \in \mathcal{E}_{c,N}$. Further we associate a polynomial degree $p_{N,Q} \in \mathbb{N}$ to each $Q \in \mathcal{P}_N$. This leads to the finite element subspace V_N of continuous piecewise polynomial ansatz functions. We employ Gauss-Lobatto quadrature whose weights ω_j^q are positive for any quadrature order q. Choosing the Gauss-Lobatto quadrature points $G_{c,N}$ on Γ_c as control points of the unilateral constraint, we define

$$K_N := \{\mathbf{v_N} \in \mathbf{V_N} \,:\, (\gamma_\mathbf{c}\, \mathbf{v_N})_\mathbf{n} \leqslant \mathbf{d} \text{ on } \mathbf{G_{c,N}}\}\,.$$

Clearly, K_N is a convex closed subset of V_N. Note however, K_N is generally not contained in K for polynomial degree $\geqslant 2$ or for a non-concave obstacle d thus leading to a nonconforming approximation.

We also approximate the nonlinear nonsmooth functional j using the Gauss-Lobatto quadrature rule by

$$j_N(\mathbf{v}) = \mathbf{j_{c,N}}(\gamma_\mathbf{c}\, \mathbf{v})_\mathbf{t},\, \mathbf{j_{c,N}}(\psi) = \sum_{\mathbf{E}\in\mathcal{E}_{\mathbf{c},\mathbf{N}}} \mathbf{g_E} \sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{q_{N,E}}} \omega_\mathbf{j}^{\mathbf{q_{N,E}+1}} \left|\psi \circ \mathbf{F_E}(\xi_\mathbf{j}^{\mathbf{q_{N,E}+1}})\right|,$$

where F_E denotes the affine map on the reference interval $[-1, 1]$ to $\overline{E}$.

Thus we arrive at the following discretization of the above variational problem: Find $\mathbf{u_N} \in \mathbf{K_N}$ such that for all $\mathbf{v_N} \in \mathbf{K_N}$,

$$a(\mathbf{u_N}, \mathbf{v_N} - \mathbf{u_N}) + \mathbf{j_N}(\mathbf{v_N}) - \mathbf{j_N}(\mathbf{u_N}) \geqslant \mathbf{l}(\mathbf{v_N} - \mathbf{u_N})\,.$$

We show convergence of the hp-FEM approximations $\mathbf{u_N} \to \mathbf{u}$ $(\mathbf{N} \to \infty)$ for mechanically definite problems without imposing any regularity assumption. Moreover we treat the coercive case, when the body is fixed along some part of the boundary. Based on an abstract Céa-Falk estimate and operator interpolation arguments, we establish an a priori error estimate in the energy norm under a reasonable regularity assumption.

In the case of linear isotropic material when a fundamental solution to the Navier-Lamé system is available, potential methods apply. Then using singular boundary integral operators, equivalent variational inequalities on the boundary can be introduced leading to a reduction in dimension. For discretization then boundary element methods can be formulated and their convergence investigated; for results in a simplified scalar case of a variational inequality of the second kind we can refer to [4].

Secondly we are concerned with nonconvex nonsmooth variational problems in the form of a hemivariational inequality: Find $\mathbf{u}$ in some appropriate vectorial function space V defined on a Lipschitz domain Ω such that for all $\mathbf{v} \in \mathbf{V}$,

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + \int_{\mathbf{D}} \mathbf{f^0}(\mathbf{u}, \mathbf{v} - \mathbf{u})\ \mathbf{dx} \geqslant \mathbf{l}(\mathbf{v} - \mathbf{u})\,.$$

Here again, a is a bilinear form and l is a linear form, similar as above, but now f^0 stands for the Clarke subgradient of a locally Lipschitz functional f defined on D, where depending on the application, D is a subdomain of Ω or a part of the boundary of Ω. When lacking coerciveness, this problem is regularized as in [3] in the general framework of pseudomonotone bifunctions. To obtain convex substitute problems, another regularization is undertaken using well-known smoothing functions of nonsmooth finite dimensional optimization and using smoothing approximations via convolution. Particular attention is given to locally Lipschitz functions that can be represented as finite minimax functions. For these regularization methods strong convergence results are provided with respect to a regularization parameter ε. Furthermore to arrive at computable subproblems, similar to [2] for convex contact problems, finite element methods are employed using Newton-Cotes quadrature with piecewise linear and piecewise quadratic approximations. For the combined regularization-approximation procedures weak and norm convergence results are established with respect to mesh width h and regularization parameter ε.

References

[1] *C. Eck, J. Jarušek, M. Krbec,Unilateral Contact Problems*, Chapman & Hall, (2005).

[2] *J.Gwinner,* "Finite-element approximation for contact problems in plane linear elastostatics *Quart. Appl. Math.*,50,11-пïS25 (1992).

[3] *J.Gwinner,* "A note on pseudomonotone functions, regularization, and relaxed coerciveness *Nonlinear Anal.*,30,4217-пïS4227 (1997).

[4] *J.Gwinner,* "On the p-version approximation in the boundary element method for a variational inequality of the second kind modelling unilateral contact and given friction", *Appl. Numer. Math.*, 60,689-пïS704 (2010).

[5] *J.Gwinner,* "*hp*-FEM convergence for unilateral contact problems with Tresca friction in plane linear elastostatics",20 p.,2012,submitted.

[6] *N. Ovcharova, Regularization Methods and Finite Element Approximation of Some Hemivariational Inequalities with Applications to Nonmonotone Contact Problems*, Ph.D. Thesis, 2012.

[7] *P.D. Panagiotopoulos, Hemivariational Inequalities - Applications in Mechanics and Engineering*, Springer, (1993).

A Variational Approach of the Rank Function

Hiriart-Urruty J.-B.
jbhu@cict.fr
Institute of Mathematics, Paul Sabatier University, TOULOUSE (France)

Associated with (square) matrices are some familiar notions like the *trace*, the *determinant,* etc. Their study from the variational point of view, via the usual differential calculus, is easy and well-known. We consider here another function of (not necessarily square) matrices, the **rank function**. The rank function has been studied for its properties in linear algebra (or matrix calculus), semi-algebraic geometry... Here we consider it from the variational viewpoint. Actually, besides being integer-valued, the rank function is lower-semicontinuous, the only valuable topological property it enjoys.

If we are interested in the rank function from the variational viewpoint, it is because the rank function appears as an objective (or constraint) function in various modern optimization problems. The archetype of the so-called **rank minimization problems** is as follows:

$$(\mathcal{P}) \quad \begin{cases} \text{Minimize } f(A) := \text{rank of } A \\ \text{subject to } A \in \mathcal{C}, \end{cases}$$

where $\mathcal{C}$ is a subset of $\mathcal{M}_{m,n}(\mathbb{R})$ (the vector space of m by n real matrices). The constraint set $\mathcal{C}$ is usually rather "simple", the main difficulty lies in the objective function. A related (or cousin) problem, actually equivalent in terms of difficulty, stated in $\mathbb{R}^n$ this time, consists in minimizing the so-called counting function $x = (x_1, \ldots, x_n) \mapsto c(x) :=$ number of nonzero components x_i in x:

$$(\mathcal{Q}) \quad \begin{cases} \text{Minimize } c(x) \\ \text{subject to } x \in S, \end{cases}$$

where S is a subset of $\mathbb{R}^n$. Often $c(x)$ is denoted as $\|x\|_0$, although it is not a norm.

Problems $(\mathcal{P})$ and $(\mathcal{Q})$ share some bizarre and/or interesting properties, from the optimization or variational viewpoint. The first one, well documented and used, concerns the "relaxed" forms of them (determining the closed convex hull of the objective function, for example). We recall here some of these results and propose further developments:

- (*Relaxation*) Explicit forms of the (closed) convex hulls of the objective functions in $(\mathcal{P})$ or $(\mathcal{Q})$, restricted to appropriated balls, are available.
- (*Global optimization*) Every admissible point in $(\mathcal{P})$ or $(\mathcal{Q})$ is a local minimizer.
- (*Moreau-Yosida approximations*) The MOREAU-YOSIDA approximates (or regularized versions) of the objective functions in $(\mathcal{P})$ or $(\mathcal{Q})$, as well as

the associated proximal mappings, can be explicitly calculated (in spite of the inherent non-convexites and discontinuities of these bumpy functions).

- (*Generalized subdifferentials*) The generalized subdifferentials of the rank function (a lower-semicontinuous finite-valued function) can be determined. Actually, all the main ones (proximal, FRECHET, viscosity, limiting, CLARKE's) coincide and their common value is a vector subspace! We provide their common expression (in various forms) and also its dimension.

We conducted these reflections while supervising the Ph D Thesis of HAI YEN LE (to be completed in 2013).

References

[1] J.-B. HIRIART-URRUTY and H.Y. LE, *Convexifying the set of matrices of bounded rank. Applications to quasiconvexification and convexification of the rank function.* **Optimization Letters** (DOI: 10.1007/s11590-011-0304-4), 2011.

[2] J.-B. HIRIART-URRUTY, *When only global optimization matters.* Published in electronic form in **J. of Global Optimization**, 2012.

[3] H. Y. LE, *The generalized subdifferentials of the rank function.* To appear in **Optimization Letters**, 2012.

Constant-sign and nodal solutions for a differential inclusion via nonsmooth analysis

Iannizzotto A., Marano S.A., Motreanu D.

iannizzotto@dmi.unict.it, marano@dmi.unict.it, motreanu@univ.perp.fr

Università degli Studi di Catania, Viale A. Doria 6, 95125 Catania, Italy,

Départment de Mathématiques, Université de Perpignan, 52 Avenue P. Alduy, 66860 Perpignan, France

We deal with the following Dirichlet problem for a partial differential inclusion, depending on a parameter $\lambda > 0$:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u \in \lambda \partial j(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} .$$

Here $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a bounded domain with a C^2 boundary $\partial\Omega$, $p > 1$ and $\partial j(x,u)$ denotes the Clarke generalized subdifferential of a nonsmooth potential j, which is subject to the following assumptions:

$\mathbf{H}_j$ $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function s.t. $j(x,\cdot)$ is locally Lipschitz for a.e. $x \in \Omega$ and $j(\cdot,0) \in L^1(\Omega)$. Moreover, we assume:

(i) $j(x,\cdot)$ is even for a.e. $x \in \Omega$;

(ii) $|\xi| \leqslant a_1 \left(1 + |s|^{q-1}\right)$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $\xi \in \partial j(x,s)$ ($a_1 > 0$, $1 < q < p^*$);

(iii) $\limsup\limits_{s \to +\infty} \dfrac{j(x,s)}{s^p} \leqslant 0$ uniformly for a.e. $x \in \Omega$;

(iv) $a_2 \leqslant \liminf\limits_{s \to 0^+} \dfrac{\min \partial j(x,s)}{s^{p-1}} \leqslant \limsup\limits_{s \to 0^+} \dfrac{\max \partial j(x,s)}{s^{p-1}} \leqslant a_3$ ($0 < a_2 \leqslant a_3$).

Our main result is the following (by λ_2 we denote the second positive eigenvalue of the negative p-Laplacian in $W_0^{1,p}(\Omega)$):

Theorem 1. *If hypotheses* $\mathbf{H}_j$ *hold, then for all* $\lambda > \lambda_2/a_2 + 1$ *problem* (P_λ) *has at least three nonzero solutions, precisely a minimal positive solution* $u_\lambda \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$*, a maximal negative solution* $-u_\lambda \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$ *and a nodal solution* $u_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$.

The proof of Theorem 1 relies on a combination of nonsmooth variational methods and truncation techniques. In particular, we use the generalized differential calculus for locally Lipschitz functionals introduced by Clarke and the nonsmooth versions of the Mountain Pass Theorem and Deformation Theorem (see Gasiński & Papageorgiou [3]). Also, we employ the nonlinear maximum principle (see Vázquez [4]), techniques for finding extremal solutions for differential inclusions (see Carl, Le & Motreanu [1]) and the properties of the spectrum of the p-Laplacian (see Cuesta, de Figueiredo & Gossez [2]).

The study originates from a question posed by B. Ricceri.

References

[1] S. Carl, V.K. Le, D. Motreanu, *Nonsmooth variational problems and their inequalities*, Springer (2007).

[2] M. Cuesta, D. de Figueiredo, J.P. Gossez, “The beginning of the Fučik spectrum for the p-Laplacian”, *J. Differential Equations*, 159, 212-P238 (1999).

[3] L. Gasiński, N.S. Papageorgiou, *Nonsmooth critical point theory and nonlinear boundary value problems*, Chapman & Hall (2005).

[4] J.L. Vázquez, “A strong maximum principle for some quasilinear elliptic equations”, *Appl. Math. Optim.*, 12, 191–202 (1984).

Implementation of statistical methods of classification

Igolkin V.N., Mamochev V.A.

vnigolkin@mail.ru, vladpad@yandex.ru

St. Petersburg State University, Universitetskii prospekt 35, Petergof, 198504, St. Petersburg, Russia

The problem of implementation of statistical methods of classification is studied. Such problems as one- and two-dimensional ranking of parame-

ters and (based on this ranking) construction of discriminant functions for identification of classes of objects are discussed. Some numerical results are presented.

Реализация статистических методов классификаций

В задачах классификации нужно по наблюдаемому вектору признаков решить, к какому из возможных классов объект с данными признаками принадлежит. При статистической классификации предполагается, что наблюдаемые признаки являются случайными и каждому классу соответствует свое распределение признаков. Задача статистической классификации состоит в нахождении решающей функции, то есть в разбиении пространства X на непересекающиеся подмножества X_i такие, что $\bigcup_{i=1}^{m} X_i = X$, где m — количество классов. Если наблюдаемый вектор признаков $x \in X$, то принимается решение, что объект с данным вектором признаков из класса i. При $m = 2$ решающая функция имеет вид:

$$X_1 = \left\{ x \in X \mid \frac{f_1(x)}{f_2(x)} > k \right\}, X_2 = \left\{ x \in X \mid \frac{f_1(x)}{f_2(x)} < k \right\}.$$

Обычно функции плотности $f_i(x)$ не известны и заменяются на «выборочные», которые строятся по «обучающим последовательностям» (см. [1]). Среди наблюдаемых признаков есть более информативные. В методе Рао предлагается искать «наиболее информативный» признак в виде линейной комбинации исходных. После этого решается задача одномерной классификации. Рассматривается обобщение метода Рао, в котором ищутся два наиболее информативных признака l_1 и l_2, $l_1 \perp l_2$ в виде линейных комбинации исходных признаков. Если по каким-либо причинам одномерная классификация оказывается неудовлетворительной, то используется двумерная классификация (см. [2]). Оба метода опробованы на реальных базах данных.

В докладе представлены результаты анализа как тестовых баз данных, таких как Heart, Diabetes, LandRover, так и реальной базы с данными пациентов психиатрической больницы им. св. Николая Чудотворца.

Работа выполнена при финансовой поддержке Российского фонда фундаментальных исследований (грант № 12-01-00752).

ЛИТЕРАТУРА

[1] *Иголкин В.Н., Ковригин Ф.Б., Старшинов А.И.,Хохлов В.А.* Статистическая классификация, основанная на выборочных распределениях. Изд. ЛГУ, Ленинград, 1978, 102 с.

[2] *Мамочев В.А.* Применение статистического метода в задаче классификации // Процессы управления и устойчивость: Труды 42-й международной научной конференции аспирантов и студентов / Под ред. А.С. Ерёмина, Н.В. Смирнова. СПб.: Издат. Дом. С.-Петерб. гос. ун-та; 2011. С. 245-248.

Regularity in Variational Analysis

Ioffe A.
ioffe@math.technion.ac.il
Technion, Haifa 32000, Israel

Regularity theory in variational analysis is an extension to set-valued mappings of the classical regularity theory centered around two groups of results: the implicit function theorem and the Lusternik-Graves theorem on the one hand and the Sard-Smale theorem and the transversality theorem of Thom on the other. Not surprisingly, it occupies a central place also in variational analysis. In the talk I plan to survey the present state of the theory relating to various aspects of the theory itself and some its applications (e.g. optimization, fixed point theorems, generic behavior).

Modeling ductile fracture with DVDS

Ionescu I.R., Lupascu O., Oudet E.
ioan.r.ionescu@gmail.com, oana.lupascu@yahoo.com, edouard.oudet@imag.fr
LSPM , University Paris 13 (Sorbonne-Paris-Cite) 99 Av. J-B. Clement, 93430 Villetaneuse, France, IMAR, Romanian Academy, 21 Calea Grivitei Street, 010702 Bucharest, Romania, LJK, University Joseph Fourier, BP 53 51, rue des MathЋmatiques F-38041 Grenoble Cedex 9, France

Discontinuous velocity domain splitting method (DVDS) is a mesh free method which focuses on the strain localization and completely neglect the bulk deformations. It considers the kinematic variational principle on a special class of virtual velocity fields to get an upper-bound of the limit load. To construct this class of virtual velocity fields, the rigid-plastic body is splinted into simple connected sub-domains and on each such sub-domain a rigid motion is associated. The discontinuous collapse flow velocity field results in localized deformations only, located at the boundary of the sub-domains. In the numerical applications of the DVDS method we introduce a numerical technique based on a level set description of the partition of the rigid-plastic body and on genetic minimization algorithms. In the case of in-plane deformation of pressure insensitive materials, the internal boundaries of the sub-domains are parts of circles or straight lines, tangent to the collapse velocity jumps. In this case, DVDS reduces to the block decomposition method, which was intensively used to get analytical upper bounds of the limit loads. When applied to the two notched tensile problem of a von Mises material, DVDS gives excellent results with a low computational cost. Furthermore, DVDS was applied to model collapse in pressure sensitive plastic materials. Illustrative examples for homogenous and heterogeneous Coulomb and Cam-Clay materials shows that DVDS gives excellent prediction of limit loads and on the collapse flow.

Simulation of diffraction structure for wave fields in focusing areas

Ipatov E., Lukin D., Palkin E.
ipatoveb@mail.ru, palkin@rosnou.ru
Moscow Institute of Physics and Technology (State University),
Institutsky per., 9, Dolgoprudny, 141700, Russia, Russian New University,
Radio st., 22, Moscow, 105005, Russia

Focus area, in addition to their classic appearance in optical systems, observed in the propagation of electromagnetic waves in the atmosphere of the Earth and planets, the acoustic waves in the ocean and the earth's crust, and sanitation of the light signal in glass, visually in the wavelength range of visible light (sunlight glare, rainbow) etc. In such areas the wave field in an inhomogeneous medium is described by one or the system of linear partial differential equations with boundary or initial conditions given in the form of rapidly oscillating functions.

In this paper we consider the application of the canonical operator method by V.P. Maslov (COM) [1-4] to solve the differential equation

$$\sum_{|\beta|=0}^{m} \alpha_\beta(\mathbf{q}) \frac{\partial^{|\beta|} U(\mathbf{q})}{(i\Lambda\partial q_1)^{\beta_1} \times (i\Lambda\partial q_2)^{\beta_2} \times ...(\times i\Lambda\partial q_n)^{\beta_n}} = 0 \tag{1}$$

with coefficients $\alpha_\beta(\mathbf{q}) \in C^k(\mathbb{R}^n)$ and the initial or boundary conditions given on a surface G ($\mathbf{q} = \mathbf{q}\,^0(\alpha)$ on G where $\alpha \in \mathbb{R}^{n-1}$ are coordinates on G) and containing rapidly oscillating functions of the form $\exp(i\Lambda S_0(\mathbf{q}))$. Here $\Lambda \gg 1$ is a big parameter, $\mathbf{q} \in \mathbb{R}^n$ are spatially-temporal coordinates, β is a multi-index. We solve for (1) the system of differential equations

$$\frac{d\mathbf{q}}{d\tau} = \frac{1}{2}\frac{\partial H(\mathbf{q},\mathbf{p})}{\partial \mathbf{p}}; \quad \frac{d\mathbf{p}}{d\tau} = -\frac{1}{2}\frac{\partial H(\mathbf{q},\mathbf{p})}{\partial \mathbf{q}} \tag{2}$$

with Cauchy data

$$\mathbf{q}|_{\tau=0} = \mathbf{q}\,^0(\alpha), \quad \mathbf{p}|_{\tau=0} = \mathbf{p}\,^0(\alpha) = \frac{\partial S_0}{\partial \mathbf{q}}|_{\mathbf{q}=\mathbf{q}^0}$$

where $H(\mathbf{q},\mathbf{p}) = \sum\limits_{|\beta|=0}^{m} \alpha_\beta(\mathbf{q}) p_1^{\beta_1} \times \ldots \times p_n^{\beta_n}$ is a Hamiltonian of the problem, τ is a parameter along the solution, $\mathbf{p} \in \mathbb{R}^n$ are conjugate momentum coordinates.

After finding a solution $\{\mathbf{q} = \mathbf{q}(\tau,\alpha); \mathbf{p} = \mathbf{p}(\tau,\alpha)\}$ of the system (2), we fix α and search points of τ_j in which the Jacobian

$$J = \det \left\| \frac{\partial \mathbf{q}(\tau,\alpha)}{\partial(\tau,\alpha)} \right\| = 0.$$

The sets of such points form in $\mathbb{R}^n$ surfaces which are called caustics. Next, we select a new system of coordinates $\{\mathbf{p}_k, \mathbf{q}_{\overline{k}}\}$ $(k = 1,\ldots,\varrho;\ \overline{k} = \varrho{+}1,\ldots,n)$ so that in the neighborhood of (τ_j,α) the Jacobian

$$\widetilde{J} = \det \left\| \frac{\partial(\mathbf{p}_k, \mathbf{q}_{\overline{k}})}{\partial(\tau,\alpha)} \right\| \neq 0.$$

By the method of COM the asymptotic solution of (1) in the neighborhood of the caustic is constructed on the base of the integrals of rapidly oscillating functions

$$I(\eta,\Lambda) = \int\limits_{\Omega} g(\mathbf{t},\eta) \exp\{i\Lambda f(\mathbf{t},\eta)\} d\mathbf{t}. \tag{3}$$

Here the amplitude function $g(\mathbf{t},\eta)$ and the phase function $f(\mathbf{t},\eta)$ are defined by means of the solution of (2). The integral (3) is characterized by two types of parameters: the internal variables in $\mathbf{t} \in \mathbb{R}^n$ over which the integration is carried out and external vector of additional parameters and space-time coordinates $\eta \in \mathbb{R}^m$. Time for computing the integral can be greatly reduced, given the fact that for large Λ contribution to the integral is made by stationary points of the phase function. First, at Ω is constructed a covering $\{\omega_k\}, k = \overline{1,N}, \cup_{k=1}^{N} \omega_k = \Omega$. According to the theorem on partitions of unity for such coverage is always a set of functions $\varphi_{\omega_k}(\mathbf{t})$ such that

1. $\sum\limits_{k=1}^{N} \varphi_{\omega_k}(\mathbf{t}) \equiv 1; \quad \varphi_{\omega_k}(\mathbf{t}) \geqslant 0, \quad \mathbf{t} \in \Omega;$
2. φ_{ω_k} is a finite infinitely differentiable function, ie $\varphi_{\omega_k} \in C^\infty(\mathbb{R}^n)$ for any subset ω_k and its support $\operatorname{supp} \varphi_{\omega_k} \subset \omega_k$.

The amplitude function g and the phase function f can be represented as sums of functions $g_{\omega_k} = \varphi_{\omega_k} g$ and $f_{\omega_k} = \varphi_{\omega_k} f$. Indeed, $\sum\limits_{k=1}^{N} g_{\omega_k} = g$ and $\sum\limits_{k=1}^{N} f_{\omega_k} = f$. Next, we determine the existence of stationary points of the phase function f in each ω_k, f approximated by smoothing splines. According to the principle of localization [5], the main contribution when $\Lambda \to \infty$ in the original integral give only the subsets ω_k^* containing either the stationary points of the phase function f, or the boundary of Ω or removable singularities of g. This technique reduces the problem of calculating the integral (3) to the calculation of the sum of integrals of the form

$$I^* = \int\limits_{\omega_k^*} g_{\omega_k^*} \exp[i\Lambda f(x)] dx. \tag{4}$$

Since the size of a subdomain ω_k^* is small compared to the Ω the integrand is oscillating slightly above it and to compute (4) we can use formulas of direct numerical integration. Note that in the case when the region of integration in (3) is infinite it can be previously limited using the method of integration along the contour in the complex plane. The described algorithm allows the calculation time for the integral (3) to be little dependent on the value of Λ. Another advantage is absence of necessity in careful study of the integrand.

In the vicinity of caustics there is a significant intensification of the field compared to the average wave field outside the caustic zone. The spatial distribution of the field near the caustic contains lot of information about the characteristics of an inhomogeneous medium. Information about the state of the ionosphere that is in contact with the neutral atmosphere provide promising opportunities for diagnostic and real-time monitoring of processes in the neutral atmosphere and seismic processes in the crust. The sensitivity of the ionospheric plasma to terrestrial geophysical processes can be used to establish new physical laws, which would entail the solution of a number of applications.

The work was supported by the Federal Program “Scientific and Scientific-Educational Staff of Innovative Russia” for years 2009-2013.

References

[1] Maslov V.P., Fedoryuk M.V., *Quasiclassical approximation for the equations of quantum mechanics*, Moscow, "Nauka (1976).

[2] Lukin D.S., Palkin E.A., *A numerical canonical method in the problems of radiowaves diffraction and propagation in inhomogeneous media*, Moscow, MIPT, (1982).

[3] Ipatov E.B., Lukin D.S., Palkin E.A., “Numerical realization of the Maslov canonical operator method for problems of shortwave radio propagation in the ionosphere of the earth”, *Radiophysics and Quantum Electronics*, 33, No. 5, 411–420 (1990).

[4] Ipatov E.B., Lukin D.S., Palkin E.A., “Maslov canonical operator in problems of numerical simulation of diffraction and propagation of waves in inhomogeneous

media", *Soviet Journal of Numerical Analysis and Mathematical Modelling*, 5, No. 6, 465–488 (1990).

[5] Fedoryuk M.V., *Asymptotics. Integrals and Series*, Moscow, "Nauka (1987).

Identification Problems for a 3D Ocean Model

Ipatova V.

ipatval@mail.ru

Moscow Institute of Physics and Technology (State University), Institutsky per., 9, Dolgoprudny, 141700, Russia

In this paper we consider the solvability of the identification problems for the system of basic ocean dynamics equations under the available observational data (the data assimilation problems). A characteristic feature of 3D nonlinear hydrodynamic models is the non-uniqueness of solutions in those spaces where a solution exists. We define a class of weak solutions satisfying the additional energy inequality. The identification problem reduces to finding a set of input parameters and an admissible weak solution of the model which minimizes the cost functional. Solvability of large-scale ocean dynamics equations was studied in [1, 2, 3] under the assumption that the density of water $\rho = \rho(T, S)$ is a linear function of the temperature T and the salinity S. In this paper the density is a nonlinear Lipschitz continuous function, $|\rho(T_1, S_1) - \rho(T_2, S_2)| \leqslant L\sqrt{(T_1 - T_2)^2 + (S_1 - S_2)^2}$, $\forall T_1, T_2, S_1, S_2 \in \mathbb{R}$. Let Ω be an open submanifold of a sphere of radius R with a piecewise smooth boundary. By x, y, r we denote the spherical coordinates, $H(x, y)$ is a positive continuously differentiable function, $z = R - r$, $G = \{(x, y) \in \Omega, 0 < z < H(x, y)\}$, Σ is the lateral surface of the domain G, Ω_H is the botton boundary of G given by the condition $z = H(x, y)$, $\mathbf{n}_\Sigma = (\mathbf{n}, 0)$ and $\mathbf{n}_H$ are normal vectors to Σ and Ω_H respectively, $0 < t_1 < \infty$, $D = \Omega \times (0, t_1)$; $(u, v, w) \equiv (\mathbf{u}, w)$ is a velocity vector, $w = w(\mathbf{u}) = \operatorname{div} \int_z^{H(x,y)} \mathbf{u} dz'$, $\xi = \xi(x, y, t)$ is the elevation of the free ocean surface. Further the symbol φ is used as a generic term for functions u, v, T, S.

The system of ocean dynamics equations is written as [4, 5]

$$\frac{d\mathbf{u}}{dt} + (A + B(u))\mathbf{u} + g\nabla\xi + \frac{g}{\rho_0} \nabla \int_0^z \rho dz' = \mathbf{f}, \tag{1}$$

$$\frac{dT}{dt} + A_T T = f_T, \quad \frac{dS}{dt} + A_S S = f_S, \quad \frac{\partial \xi}{\partial t} + \operatorname{div} \int_0^H \mathbf{u} dz = Q_w, \tag{2}$$

where $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla + w(\mathbf{u})\partial/\partial z$, $B(u)\mathbf{u} = (2\omega \sin y + u \operatorname{tg} y/R)(-v, u)$, $A_\varphi = -\mu_\varphi \triangle - \nu_\varphi \partial^2/\partial z^2$, $g, \rho_0, \omega, \mu_\varphi, \nu_\varphi$ are positive constants, $A_u = A_v = A$, $\mathbf{f}, f_T, f_S$ are given functions.

Supplement system (1)-(2) with the initial conditions

$$\mathbf{u}=\mathbf{u}^0,\ T=T^0,\ S=S^0,\ \xi=\xi^0,\quad t=0 \tag{3}$$

and the boundary conditions

$$\nu\frac{\partial\mathbf{u}}{\partial z}=-\frac{\tau}{\rho_0}+\frac{w(\mathbf{u})}{2}\mathbf{u}\quad \nu_\psi\frac{\partial\psi}{\partial z}=\gamma_\psi(\psi-\psi_s)+\frac{w(\mathbf{u})}{2}\psi+Q_\psi,\quad z=0, \tag{4}$$

$$\mathbf{u}\cdot\mathbf{n}=\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\times\mathbf{n}=\nabla T\cdot\mathbf{n}=\nabla S\cdot\mathbf{n}=0\ \text{on}\ \Sigma;\ \left(\mu_\varphi\nabla\varphi,\nu_\varphi\frac{\partial\varphi}{\partial z}\right)\cdot\mathbf{n}_H=0\ \text{on}\ \Omega_H, \tag{5}$$

where ψ means T and S, γ_ψ are positive constants, τ,ψ_s,Q_ψ are given functions.

We denote by $(\cdot,\cdot)$ and $\|\cdot\|$, $(\cdot,\cdot)_0$ and $\|\cdot\|_0$, $(\cdot,\cdot)_D$ and $\|\cdot\|_D$ the scalar product and norm in $L^2(G)$, $L^2(\Omega)$ and $L^2(D)$ respectively. Let $H^k(G)=W_2^k(G)$ and $\mathbf{H}_0^k(G)=\{\mathbf{u}\in H^k(G)\times H^k(G)\,|\,\mathbf{u}\cdot\mathbf{n}=0\ \text{on}\ \Sigma\}$ be Sobolev spaces of functions and vector functions. Let $H^{-k}(G)$ and $\mathbf{H}_0^{-k}(G)$ be their dual spaces,

$$X=\left\{\mathbf{u}\in L^2(0,t_1;\mathbf{H}_0^1(G)),\ \frac{\partial\mathbf{u}}{\partial t}\in L^{4/3}(0,t_1;\mathbf{H}_0^{-2}(G))\right\},$$

$$Y=\left\{T\in L^2(0,t_1;H^1(G)),\frac{\partial T}{\partial t}\in L^{4/3}(0,t_1;H^{-2}(G))\right\},$$

$$Z=\left\{\xi\in L^2(D),\ \frac{\partial\xi}{\partial t}\in L^2(D)\right\},\quad V=X\times Y\times Y\times Z,$$

$$\mathcal{H}=(L^2(G))^4\times L^2(\Omega),\quad W=V\cap L^\infty(0,t_1;\mathcal{H}),\quad P=L^2(0,t_1;\mathbf{H}_0^{-1}(G));$$

$$U=\{\mathbf{u},T,S\},\ \Xi=\{\mathbf{u},T,S,\xi\},\quad \|\Xi\|^2=\|u\|^2+\|v\|^2+\|T\|^2+\|S\|^2+g\|\xi\|_0^2,$$

$$[\varphi,\varphi_1]=\mu_\varphi(\nabla\varphi,\nabla\varphi_1)+\nu_\varphi\left(\partial\varphi/\partial z,\partial\varphi_1/\partial z\right)+\gamma_\varphi(\varphi,\varphi_1)_0|_{z=0},$$

$$[U,U_1]=[u,u_1]+[v,v_1]+[T,T_1]+[S,S_1],\quad \gamma_u=\gamma_v=0,$$

$$F=(\mathbf{f},f_T,f_S),\quad F_0=(\tau/\rho_0,\gamma_T T_s-Q_T,\gamma_S S_s-Q_S).$$

We introduce the additional energy inequality

$$\frac{1}{2}\|\Xi(t)\|^2+\int_0^t[U,U]dt'\leqslant\frac{1}{2}\|\Xi^0\|^2+$$
$$\int_0^t\left((F,U)+\frac{g}{\rho_0}(\int_0^z\rho dz',\operatorname{div}\mathbf{u})+(F_0,U)_0|_{z=0}+g(Q_w,\xi)_0\right)dt'. \tag{6}$$

Note that (6) is valid for all smooth solutions to (1)-(5) and the strict equality holds in this case.

Theorem 1 [4]. *For all* $\Xi^0 = \{\mathbf{u}^0, T^0, S^0, \xi^0\} \in \mathcal{H}$, $\mathbf{f} \in P$, $f_T, f_S \in L^2(0, t_1; H^{-1}(G))$, Q_w, T_s, Q_T, S_s, Q_S *belonging to* $L^2(D)$ *and* $\tau \in (L^2(D))^2$ *the problem (1)-(5) has at least one weak solution* $\Xi \in W$ *satisfying the inequality (6) for almost all* $t \in [0, t_1]$.

It is assumed that the observations of the free ocean surface elevation $\xi = \xi_{obs}(x, y, t)$ and the observations of surface temperature $T|_{z=0} = T_{obs}(x, y, t)$ are available in the domain $D_1 \subset D$. For definiteness sake, ξ_{obs} and T_{obs} are extended by zero onto the set $D \backslash D_1$. The observation data should be used for determination of the moisture flow Q_w, the heat flow Q_T and the salinity flow Q_S, whereas all other input parameters of the model are fixed and correspond to the hypothesis of Theorem 1. It is assumed that $q = \{Q_w, Q_T, Q_S\}$ is sought for in the space $Q = E \times (L^2(D))^2$, where $E = L^p(0, t_1; W_2^1(\Omega)) \cap L^2(D)$, $1 < p \leqslant 2$. By $\mathcal{U}(q) \subset W$ we denote the set of all weak solutions to problem (1)-(5) corresponding to the given value of q and satisfying the inequality (6). In the space $Q \times W$ we consider the set M of all pairs $\{q, \Xi\}$ such that $q \in Q$, $\Xi \in \mathcal{U}(q)$. We define on M a cost functional measuring the discrepancy between observed values and the simulation results

$$J(q, \Xi) = \alpha(\|Q_w - Q_w^a\|_E^2 + \|Q_T - Q_T^a\|_D^2 + \|Q_S - Q_S^a\|_D^2) + \\ + \|\chi\xi - \xi_{obs}\|_D^2 + \|\chi T|_{z=0} - T_{obs}\|_D^2$$

where $\alpha = const \geqslant 0$ is a regularization parameter, χ is a characteristic function of D_1, $Q_w^a \in E$ and $Q_T^a, Q_S^a \in L^2(D)$ are given functions.

The following identification problem is studied: determine the element $\{q, \Xi\} \in M$ such that

$$J(q, \Xi) = \inf \{J(q', \Xi') | \{q', \Xi'\} \in M\}. \tag{7}$$

Using the following from (6) a priori estimate, we obtain the theorem.

Theorem 2. *For all* $\alpha > 0$ *and* $\xi_{obs}, T_{obs} \in L^2(D)$ *the problem (7) has a solution.*

Analogously we study the identification problem of reconstruction of the unknown initial state $\Xi^0 = \{\mathbf{u}^0, T^0, S^0, \xi^0\} \in \mathcal{H}$.

The work was supported by the Federal Program "Scientific and Scientific-Educational Staff of Innovative Russia" for years 2009-2013.

References

[1] Lions J.-L., Temam R., Wang S., “On the equations of the large-scale ocean”, *Nonlinearity*, No. 5, 1007–1053 (1992).

[2] Temam R., Ziane M., *Some mathematical problems in geophysical fluid dynamics. Handbook of Mathematical Fluid Dynamics*, Amsterdam: Elsevier, (2004).

[3] Agoshkov V.I., Ipatova V.M., "Existence Teorems for a Three-Dimentional Ocean Dynamics Model and a Data Assimilation Problem", *Doklady Mathematics*, 75, No. 1, 28–30 (2007).

[4] Ipatova V.M., "Solvability of the ocean hydrothermodynamics problem under a nonlinear state equation", *Russ. J. Numer. Anal. Math. Modelling*, 23, No. 2, 185–195 (2008).

[5] Ipatova V.M., "Data assimilation problems for basic equations of the ocean thermodynamics with a Lipschitz continuous density", *In: Modern Problems of Fundamental and Applied Mathematics*, MIPT, Moscow, 56–79 (2008).

On Well Posed Best Approximation Problems for a Nonsymmetric Seminorm

Ivanov G.E.

Moscow Institute of Physics and Technology, Institutski str. 9, Dolgoprudny, Moscow region, 141700, Russia

Let E be a real normed vector space. For a set $S \subset E$ by $\mathrm{int}S$, $\overline{S}$, and ∂S we denote the interior, the closure, and the boundary of S, respectively. We use $\langle p, x\rangle$ to denote the value of the functional $p \in E^*$ at the vector $x \in E$. For $R > 0$ and $c \in E$ we denote by $\mathfrak{B}_R(c)$ the closed ball with center c and radius R. Recall that a set $S \subset E$ is called *strictly convex* if for every different points $x, y \in \overline{S}$ we have $\frac{x+y}{2} \in \mathrm{int}S$. A set $S \subset E$ is called *Kadec* if for any sequence $\{x_k\} \subset \partial S$ weak convergence $x_k \to x_0 \in \partial S$ implies strong convergence of $\{x_k\}$. A subset S of a topological space X is called a *meagre set* or a *set of first category* if $S = \bigcup\limits_{n\in\mathbb{N}} S_n$ where S_n are nowhere dense in X. The complement of a meager subset is called a *residual set.* A subset S of a metric space X is said to be *porous* in X if there exist $r_0 > 0$ and $\vartheta \in (0, 1]$ such that for every $y \in X$ and $r \in (0, r_0]$ there exists $x \in X$ with $\mathfrak{B}_{\vartheta r}(x) \subset \mathfrak{B}_r(y) \setminus S$. A subset S of a metric space X is said to be *σ-porous* in X if $S = \bigcup\limits_{n\in\mathbb{N}} S_n$, where S_n are porous in X. Recall that the *Minkowski sum* of sets $A, B \subset E$ is defined as $A + B = \{a + b \mid a \in A,\ b \in B\}$.

We say that subset M of E is a *quasiball* if M is closed convex and $0 \in \mathrm{int}M$.

Let M be a quasiball. Recall that the *Minkowski functional* $\mu_M : E \to \mathbb{R}$ of M is defined by

$$\mu_M(x) = \inf\{t > 0 \mid x \in tM\} \qquad \forall x \in E.$$

Note that a functional $\mu : E \to \mathbb{R}$ is a Lipschitz continuous nonsymmetric seminorm iff it is the Minkowski functional of a quasiball.

For a closed set $A \subset E$ and a point $x_0 \in E$ we consider the problem to approximate x_0 by a point $a \in A$ best in the sense of the nonsymmetric seminorm:

$$\min_{a \in A} \mu_M(x_0 - a). \tag{1}$$

Through $\varrho_M(x_0, A)$ we denote *M-distance* from the point x_0 to the set A:

$$\varrho_M(x_0, A) = \inf_{a \in A} \mu_M(x_0 - a).$$

A sequence $\{a_k\} \subset A$ is called a *minimizing sequence* of the problem (1) if

$$\lim_{k \to \infty} \mu_M(x_0 - a_k) = \varrho_M(x_0, A).$$

The problem (1) is said to be *well posed* if every minimizing sequence of the problem (1) converges. We use $T_M(A)$ to denote the set of all $x_0 \in E$ such that the problem (1) is well posed.

Stechkin [5], Lau [3], Konjagin [2], Borwein and Fitzpatrick [1], and others considered the problem (1) for $M = \mathfrak{B}_1(0)$. It was proved that if the unit ball M is strictly convex and Kadec, a set A is closed, then $T_M(A)$ is a residual subset of E. Instead of the unit ball Li [4] considered a bounded quasiball and obtained a similar result for this more general case. In this paper we refuse from the assumption that the quasiball M is bounded and investigate some asymptotic properties (appearance far from the origin) of M which are necessary and/or sufficient for $S_M^{\text{int}}(A) \setminus T_M(A)$ to be a meagre or a σ-porous subset of *the intermidiate region*

$$S_M^{\text{int}}(A) = \left\{ x_0 \in E \,\middle|\, 0 < \varrho_M(x_0, A) < \sup_{x \in E} \varrho_M(x, A) \right\}.$$

Consider the barrier cone $b(M)$ and the cone $b_1(M)$:

$$b(M) = \left\{ p \in E^* \,\middle|\, \sup_{x \in M} \langle p, x \rangle < +\infty \right\},$$

$$b_1(M) = \left\{ p \in E^* \,\middle|\, \exists x_0 \in X : \langle p, x_0 \rangle = \sup_{x \in M} \langle p, x \rangle \right\}.$$

We say that a set $M \subset E$ is *equidistable* if there exist $\lambda > 1$, $\delta > 0$, and $d \in E$ such that

$$\forall w \in \mathfrak{B}_\delta(0)\ \forall \varepsilon > 0\ \exists x \in M : \quad (1+\varepsilon)x - w \notin \operatorname{int}\Big((M + \mathfrak{B}_\delta(0)) \bigcup (\lambda M - d) \Big).$$

Theorem 1. *For a quasiball M in a Banach space E the following statements are equivalent:*

(i) E is reflexive, M is strictly convex, Kadec and not equidistable, $b_1(M) = b(M)$;
(ii) for any closed $A \subset E$ the inclusion $S_M^{\mathrm{int}}(A) \subset \overline{T_M(A)}$ is valid;
(iii) for any closed $A \subset E$ the set $S_M^{\mathrm{int}}(A) \bigcap T_M(A)$ is a residual subset of $S_M^{\mathrm{int}}(A)$.

Remark that equidistableness is a rather complicated notion. The following substantially more simple notion provides a sufficient condition for $S_M^{\mathrm{int}}(A) \bigcap T_M(A)$ to be a residual subset of $S_M^{\mathrm{int}}(A)$, which is also necessary in finite-dimensional spaces. We say that a set $M \subset E$ is *parabolic* if it is closed convex and for every $d \in E$ the set $M \setminus (2M - d)$ is bounded. The term "parabolic" is due to the observation that the epigraph of the parabola $y = x^2$ is parabolic while the epigraph of the hyperbola $y = \frac{1}{x}$, $x > 0$ is not parabolic. Since parabolicity depends on the appearance of the set far from the origin, this property is asymptotic.

A set $M \subset E$ is called *boundedly uniformly convex* if $\delta_M^R(\varepsilon) > 0$ for all $R > 0$, $\varepsilon > 0$, where

$$\delta_M^R(\varepsilon) = \sup\left\{\delta \in \left(0, \frac{\varepsilon}{2}\right] \;\middle|\; \mathfrak{B}_\delta\left(\frac{x+y}{2}\right) \subset M \quad \forall x, y \in M \cap \mathfrak{B}_R(0) : \; \|x - y\| \geqslant \varepsilon\right\}.$$

Theorem 2. *Let M be a boundedly uniformly convex and parabolic quasiball in a reflexive Banach space E, $A \subset E$ be closed. Then $S_M^{\mathrm{int}}(A) \setminus T_M(A)$ is σ-porous in $S_M^{\mathrm{int}}(A)$.*

The work is partially supported by the Russian Foundation for Basic Research, grant 10-01-00139-a.

References

[1] *J. M. Borwein, S. Fitzpatrick*, Existence of nearest points in Banach spaces // Canad. J. Math. **41** (1989), 707–720.

[2] *S.V. Konjagin*, On approximation properties of closed sets in Banach spaces, Soviet Math. Dokl. **21** (1980), 418–422.

[3] *K. S. Lau*, Almost Chebyshev subsets in reflexive Banach spaces // Indiana Univ. Math. J. **27** (1978), 791–795.

[4] *C. Li*, On Well Posed Generalized Best Approximation Problems // Journal of Approximation Theory. **107** (2000), 96–108.

[5] *S.B. Stechkin*, Approximate properties of sets in linear normed spaces, Rev. Math. Pures Appl. **8** (1963) 5–18 (in Russian).

Nonsmoothness in the Identification and Evolution of Equilibrium

Jofré A.

ajofre@dim.uchile.cl

Center for Mathematical Modeling, Universty of Chile, Santiago

Processes in continuous time, virtual or real, can be contemplated in which market prices and the corresponding demands of agents gradually adjust. One version could be an information process aimed at generating a trajectory which would converge to equilibrium. Another version could be a dynamical system, perhaps even of control, in which equilibrium evolves over time in response to incremental additions to, or subtractions from, the goods held by the agents. Either way, reality demands that the quantities of some goods ought to be able to hit zero sometimes and later pull away. That, however, means nonsmoothness of the dynamics. The good news is that results in this direction are being obtained and they are surprisingly strong.

Generalized gradient method for control system stabilization

Kamenetskiy V.

vlakam@ipu.ru

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Profsouznaya 65, Moscow, 117997, Russia

Apart from ensuring the asymptotic stability of equilibrium, we provide three requirements on the closed-loop system obtained as a solution of the stabilization problem.

(1) The attraction domain of the closed-loop system must contain the desired set of the initial conditions H_0.

(2) System trajectories beginning in the set H_0 should not leave the given admissible set A (chosen as the domain of safe system operation and/or to prevent the overshoot).

(3) System trajectories beginning in the set H_0 must have a certain exponential degree of convergence.

It is understandable that the degree of success in satisfying the aforementioned requirements depends on the control system characteristics and the accepted trade-off between the requirements.

Within the framework of direct Lyapunov method, the sets bounded by the level surfaces of the Lyapunov functions serve as the main tool providing the aforementioned characteristics to the closed-loop system. The solution of the stabilization problem is based on a method of improving Lyapunov functions within a certain parametric class. This method is directed to enlarge the corresponding estimation for the attraction domain of the system $\dot{x} = f(x)$ and is shortly presented below.

The attraction domain increasing [1]. We assume that the Lyapunov functions $v(\alpha, x)$ belong to an arbitrary parametric class of smooth functions, $\alpha \in R^N$ is a parameter. We denote by $S(\alpha, r)$ the estimations of attraction domain corresponding to the Lyapunov functions $v(\alpha, x)$, where r is a level constant, and $S(\alpha, r)$ is the connected component within the set $= \{x \in R^n : v(\alpha, x) < r\}$ including zero. Because the sets $S(\alpha, r)$ have all properties of attraction domain, we also call them as attraction domains.

By a given Lyapunov function providing the stability in the small one can indicate the maximal atraction domain $S(\alpha, R(\alpha))$ corresponding to this function ($R(\alpha)$ is the critical value of the level constant). To improve this domain, we vary Lyapunov function parameters α along the trajectories of the differential inclusion

$$\dot{\alpha}(t) \in F(\alpha(t)). \tag{1}$$

Here $F(\alpha) \ : \ R^n \to P(R^n)$ is the set-valued map ($P(R^n)$ is the set of all subsets of R^n) such that the set $F(\alpha)$ consists of the vectors $w \in R^N$ under conditions

$$\left\langle \left(\frac{\partial v(\alpha, y_j)}{\partial \alpha} - \frac{\partial v(\alpha, x)}{\partial \alpha} + \gamma(\alpha, x) \frac{\partial \dot{v}(\alpha, x)}{\partial \alpha} \right), w \right\rangle_N \leqslant -\varepsilon(\alpha) \tag{2}$$

for all $x \in I(\alpha) = \mathrm{bnd} S(\alpha, R(\alpha)) \bigcap \{x \in R^n : \dot{v}(\alpha, x) = \langle\, \partial v(x)/\partial x, f(x)\, \rangle_n = 0\}$ and $y_j = \lambda_j(\alpha) a_j \in \mathrm{bnd} S(\alpha, R(\alpha)), j = \overline{1, J}$, where a_i are the desired directions of enlarging the domain $S(\alpha, R(\alpha))$. The coefficients $\gamma(\alpha, x) > 0$ in (2) are the Lagrange multipliers of the conditional minimum problem $R(\alpha) = \min\limits_{x \in \{\dot{v}(\alpha, x) = 0\}} v(\alpha, x)$, and $\varepsilon(\alpha) > 0$ is a continuous function. With varying the vector $\alpha = \alpha(t)$ according to (1) the relations $\lambda_j(\alpha(t_1)) a_j \in S(\alpha(t_2), R(\alpha(t_2))), t_1 < t_2, j = \overline{1, J}$, are satisfied.

We show that the decreasing of the function $v(\alpha, y_i) - R(\alpha)$ with changing α is sufficient to satisfy the above conditions. The function $R(\alpha)$ is not differentiable, and we need the Clarke generalized gradient [2] instead ordinary one in this case. In our case the Clarke generalized gradient is the convex hull of the "partial" gradients as it presented in (2).

The admissible set. Let the admissible set A be assigned by the nonlinear inequality (by the phase constraints),

$$A = \{x \in G \subseteq R^n : g(x) < 0\}, \tag{3}$$

where $g(x) \in C^{(2)}(G)$. To allow for the phase constraints like (3) we use, instead of $\dot{v}(\alpha, x)$, the function

$$M(\alpha, x) = \max\{\dot{v}(\alpha, x), g(\alpha, x)\}. \tag{4}$$

The time-optimality. The ordinary derivative of the Lyapunov function $\dot{v}(\alpha, x)$ is replaced by $v_\lambda(\alpha, x) = \dot{v}(\alpha, x) + 2\lambda v(\alpha, x)$ in order to ensure the rate of convergence.

Thus, the ordinary requirement $\dot{v}(\alpha, x) < 0$ with $x \in S(\alpha, R(\alpha))/\{0\}$ is replaced by $M(\alpha, x) < 0$ with $x \in S(\alpha, R(\alpha))/\{0\}$, where

$$M(\alpha, x) = \max\{v_\lambda(\alpha, x), g(\alpha, x)\}. \quad (5)$$

The affine system stabilization by the constrained feedback [3]. The control system is described in some domain $G \subset R^n$ by differential equations

$$\dot{x} = f_1(x) + u f_2(x), \quad (6)$$

where $x \in R^n$ is the vector of phase variables and $f_i(x) \in C^{(2)}(G)$, $i = 1, 2$. We look for the stabilizing control in the feedback form $u = u(x) : R^n \to R$, obeying the restriction $\mid u(x) \mid \leqslant 1$. For some $v(\alpha, x)$ its derivative along the system (6) trajectories has the form $\dot{v}(\alpha, x, u) = \langle \partial v(\alpha, x)/\partial x, f_1(x)\rangle_n + u\langle \partial v(\alpha, x)/\partial x, f_2(x)\rangle_n$. Minimizing $\dot{v}(\alpha, x, u)$ under $\mid u \mid \leqslant 1$, we obtain the discontinuous stabilizing feedback

$$\widehat{u}(x) = -\mathrm{sign}\langle \partial v(\alpha, x)/\partial x, f_2(x)\rangle_n,$$

defined by the parameters α, and the analog to the derivative $\dot{v}(\alpha, x)$ is the Lyapunov – Bellman function

$$L(\alpha, x) = \dot{v}(\alpha, x, \widehat{u}) = \langle \partial v(\alpha, x)/\partial x, f_1(x)\rangle_n - \mid \langle \partial v(\alpha, x)/\partial x, f_2(x)\rangle_n \mid . \quad (7)$$

It remains to modify the function $L(\alpha, x)$ by the way shown above to allow the phase constraints and the rate of convergence requirements, and to substitute it instead $\dot{v}(\alpha, x)$ into the algorithm (1)-(2).

Parametric stabilization [4] and the tuning of regulator parameters [5]. Let the system containing undefined parameters in the right-hand side be considered

$$\dot{x} = f(\alpha_1, x),\ f(\alpha_1, 0) = 0,\ \alpha_1 \in U_1 \subseteq R^{N_1}, \quad (8)$$

where $x \in G \subset R^n$ is the vector of phase coordinates, $\alpha_1 \in U_1 \subset R^{N_1}$ is the parameter vector, $f(\alpha_1, x) \in C^{(2)}(U_1 \times G)$, and $G \subseteq R^n$ is a certain domain. By the parametric stabilization we mean determination of the values of parameters so that the closed-loop system satisfies the aforementioned requirements. We also consider the interesting for applications system containing the nonlinear saturator

$$\dot{x} = f_1(x) + \mathrm{sat}(c(\alpha_1, x)) f_2(x), \quad (9)$$

in which the stabilizing parameters $\alpha_1 \in U_1 \subset R^{N_1}$ are to be determined to allow the same requirements on the transient performance. Here $x \in R^n$ is the phase vector, $c(\alpha_1, x)) \in C^{(2)}(U_1 \times G \to R)$, and $f_i(x) \in C^{(2)}(G \to R^n)$, $i = 1, 2$.

We rename the vector of Lyapunov function parameters by α_2 and combine the system parameters α_1 and Lyapunov function parameters α_2 into one vector $\alpha = (\alpha_1, \alpha_2)$ like in (1)-(2). The derivative $\dot{v}(\alpha, x)$ of Lyapunov function $v(\alpha_2, x)$ along the trajectories of system (8) or (9) depends on α. We modify the corresponding functions $\dot{v}(\alpha, x)$ like in (5) to allow the phase constraints and the rate of convergence requirements, and substitute them instead of $\dot{v}(\alpha, x)$ into the algorithm (1)-(2). The improvement of stabilizing parameters α_1 occurs by **simultaneous** varying this parameters α_1 and Lyapunov function parameters α_2 along the solution curves of differential inclusion (1)-(2).

Some examples of computer realization are considered.

References

[1] *V. A. Kamenetskiy,* "A method for construction of stability regions by Lyapunov functions" *Systems and Control Letters*, 26, 147-151 (1995).

[2] *F.H. Clarke, Optimization and Nonsmooth Analysis*, Wiley, (1983).

[3] *V.A. Kamenetskii,* "Synthesis of a Constrained Stabilizing Control for Nonlinear Control Systems", *Autom. Remote Control*, 56, No. 1, part 1, 35–45 (1996).

[4] *V.A. Kamenetskii,* "Parametric Stabilization of Nonlinear Control Systems under State Constraints", *Autom. Remote Control*, 57, No. 10, part 1, 1427–1435 (1996).

[5] *V.A. Kamenetskii,* "Parametric Stabilization of Automatic Control Systems", *Autom. Remote Control*, 72, No. 11, 2300–2314 (2011).

Exact penalties in a multipoint problem for ordinary differential equations

Karelin V.V.

vlkarelin@mail.ru

St. Petersburg State University, Universitetskii prospekt 35, Petergof, 198504, St. Petersburg, Russia

Exact penalty functions are used to reduce a multipoint problem for ordinary differential equations (which is a constrained optimization problem) to an unconstrained one.

Точные штрафные функции в многоточечной задачи для обыкновенных дифференциальных уравнений

Рассмотрим систему:

$$\dot{x} = f(x,t), \tag{1}$$

где $x = x(t)$ – неизвестная функция, подлежащая определению, $t \in [0,T]$. Выберем на этом отрезке k точек t_i $(i = 1,2,\dots,k)$. Будем рассматривать значения функции x(t) и ее производных до $(n-1)$ – го порядка включительно в указанных точках: $x(t_i), x'(t_i), \dots, x^{(n-1)}(t_i)$, $i = l,2,\dots,k$. Образуем теперь n уравнение, связывающие эти величины:

$$\Phi(x(t_1), x'(t_1), \dots, x^{(n-1)}(t_1), \dots, x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)) = \eta, \tag{2}$$

и поставим такую задачу: найти на отрезке $[0,T]$ решение $x(t)$ уравнения (1), которое удовлетворяло бы n поставленным условиям (2).

Рассмотрим функционал

$$I = (\Phi(x(t_1), x'(t_1), \dots, x^{(n-1)}(t_1), \dots, x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)) - \eta)^2$$

и $x(t,x_0)$ – решение уравнений (1) с $x_0 \in R^n$.

Введем множество

$$\Omega := \{[z,x_0] \,|\, z \in C[0,T], : \varphi(z,x_0) = 0\},$$

здесь

$$\varphi(z,x_0) = \left[\int_0^T \left(z(t) - f(x_0 + \int_0^t z(\tau)d\tau, x_0, t)\right)^2 dt\right]^{1/2}.$$

Заметим, что $\varphi(z,x_0) \geqslant 0 \;\; \forall z, x_0 \in P[0,T]$. Если z, $x_0 \in \Omega$, то функция

$$x(t) = x_0 + \int_0^t z(\tau)d\tau$$

удовлетворяет системе (1), и наоборот.

Таким образом, задача решения системы (1) для некоторого $x_0 \in R^n$ эквивалентна нахождению $z \in P[0,T]$ такого, что $\varphi(z,x_0) = 0$.

Теорема. Если φ липщицева на $C[0,T] \times R^n$ тогда найдется $\lambda_0 \geqslant 0$ такая, что для всех $\lambda \geqslant \lambda_0$ множество минимумов функционала I на множестве $\Omega = \{[z,x_0]|\varphi(z,x_0) = 0\}$ совпадает с множеством минимумов функции

$$\psi_\lambda(z,x_0) = I(z,x_0) + \lambda\varphi(z,x_0)$$

на всем множестве $C[0,T] \times R^n$.

Таким образом многоточечную задачу удалось свести к задаче оптимизации без ограничений.

Работа выполнена при поддержке РФФИ, проект № 12-01-00752.

ЛИТЕРАТУРА

[1] Demyanov V. F., Giannessi F., Karelin V. V., “ Optimal control problems via Exact Penalty Function ”, *of Global Optimiz.*, 12, No. 3, 215-223. (1998).

[2] Di Pillo G., Grippo L., *On the exactness of a class of Nondifferentiable Penalty functions,Optim. Theory Appl*, 57., 397-408. (1988).

Optimality of the statistical procedure of the market graph construction

Koldanov A., Koldanov P., Kalyagin V., Pardalos P.

alex.koldanov@gmail.com, kold@mail.ru, vkalyagin@hse.ru, pardalos@ufl.edu

National Research University Higher School of Economics, Bolshaya Pecherskaya 25/12, Nizhny Novgorod, 603155, Russia University of Florida, 401 Weil Hall P.O. Box 116595 Gainesville, FL 32611-6595, USA

The paper deals with the statistical analysis of the construction method of the market graph introduced in [1]. The main goal of the paper is the investigation of the optimality of the method of construction of the market graph from the statistical point of view. According to the classical approach by Wald the optimal statistical procedures is the statistical procedures with the minimal conditional risk in a fixed class. In our investigation we consider the class of unbiased statistical procedures. As a statistical model of the financial market we use the classical model by Markowitz. The market graph (true market graph) is the matrix with entries 0 and 1, where we put 0 if the associated correlation is less then given threshold and 1 otherwise. Sample market graph is the market graph constructed from the sample correlations. The main question discussed in this paper is the relation between true and sample market graphs. Let N be the number of the stocks on the financial market, n - number of observations. Denote by $P(t)$ the price of the stock i for the day t and define the daily return of the stock i for the period from $(t-1)$ to t by $R(t) = \ln(P(t)/P(t-1))$.

We suppose that the random variables $R_i(t)$ are independent for the fixed i, have all the same distribution as a random variable R_i , and vector of means and covariance matrix are known. This model is most appropriate in the case where the random vector $(R_1, R_2, \ldots, R_N)$ has a multivariate normal distribution. Denote its correlation matrix by $||\rho_{i,j}||$. Let

$$r_{i,j} = \frac{\sum(R_i(t) - \overline{R}_i)(R_j(t) - \overline{R}_j)}{\sqrt{\sum(R_i(t) - \overline{R}_i)^2}\sqrt{\sum(R_j(t) - \overline{R}_j)^2}}$$

be the sample correlation between the stocks i and j, and $\overline{R}_i = \frac{1}{n}\sum R_i(t)$ be the sample means of R_i. Note that sample correlation and sample mean are the statistical estimation of the correlation and mean. It is known, that for a multivariate normal vector both statistics are sufficient.

Matrix $||\rho_{i,j}||$ is a basic matrix for the construction of the true market graph and matrix $||r_{i,j}||$ is a basic matrix for the construction of the sample market graph. Each vertex of the graph corresponds to an stock of the financial market. The edge between two vertices i and j is included in the true market graph if $\rho_{i,j} > \rho_0$, and it is included in the sample market graph, if $r_{i,j} > r_0$ (where ρ_0 and r_0 are the thresholds). Two vertices are adjacent if they are connected by an edge. The construction method of the market graph introduced in [1] can be considered as a statistical procedure for the construction of the true market graph from the sample market graph. In the present paper we study the problem of optimality of the statistical procedure of the construction of the market graph.

This problem can be formulated as the problem of optimality of a multiple decision hypothesis testing:

$$\begin{array}{ll} H_1: & \rho_{i,j} > \rho_0 \\ H_2: & \rho_{1,2} \leqslant \rho_0,\ \rho_{i,j} > \rho_0,\ (i,j) \neq (1,2) \\ H_3: & \rho_{1,2} \leqslant \rho_0,\ \rho_{1,3} \leqslant \rho_0,\ \rho_{i,j} > \rho_0,\ (i,j) \neq (1,2),(1,3) \\ \multicolumn{2}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ H_L: & \rho_{i,j} \leqslant \rho_0 \end{array}$$

All together these hypothesis describe all possible true market graphs. We show that the method from [1] is optimal in the class of unbiased multiple decision statistical procedures. To prove this result we put the problem in the framework of Lehman theory of multiple decision statistical procedures [2] and precise the choice of generating hypothesis. The work is partly supported by Russian Federation Government Grant No. 11.G34.31.0057.

References

[1] 1.V. Boginski, S. Butenko, S., P. Pardalos, "On structural properties of the market graph". In: Nagurney, A. (Ed) *Innovations in financial and economic networks.* Northampton: Edward Elgar Publishing Inc., 2003, pp. 29-45.,

[2] E. Lehmann, J. Romano *Testing Statistical Hypotheses.* Springer, New York, 2005

The method of quadratic regularization in nonsmooth global optimization

Kosolap A.
anivkos@ua.fm
The Ukrainian State Chemical-Technological University Gagarin's avenue, 8, Dnipropetrovsk, 49005, Ukraine

In the paper [1] we offer the new method of quadratic regularization for searching the point of a global minimum in general quadratic problems. We prove, that this method is effective for finding the point of a global minimum in nonsmooth optimization.

Let's consider a problem

$$\min\{f_0(x)|f_i(x) \leqslant 0, i = 1, \ldots, m, x \geqslant 0, x \in E^n\}, \tag{1}$$

where all functions $f_0(x)$ can be nonsmooth. We use transformation

$$\min\{x_{n+1}|f_0(x) + s \leqslant x_{n+1}, f_i(x) \leqslant 0, i = 1, \ldots, m, x \geqslant 0, x \in E^n\}, \tag{2}$$

where the parameter s is chosen from a condition

$$f_0(x^*) + s \geqslant |x^*|(|x| = |x_1| + \ldots + |x_n|).$$

We use transformation $x = Az$ where matrix A of order $(n+1 \times n+1)$ equal

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & 1 \end{pmatrix}.$$

The problem (2) is transformed to a kind

$$\min\{e^T x \mid f_0(Iz) + s - e^T z \leqslant 0,\ f_i(Iz) \leqslant 0,\ i = 1, \ldots, m,\ z \geqslant 0\}, \tag{3}$$

where $e = (1, \ldots, 1)$ and I – unite matrix of order $(n \times n)$. For some classes of nonsmooth functions there is a constant $r > 0$ such that all functions

$$g_0(x) = f_0(Iz) + s + e^T z + r|z|,\ g_i(z) = f_i(Iz) + r|z|,\ i = 1, \ldots, m$$

are convex. Then the problem (3) is equivalent to the problem

$$\min\{e^T z \mid g_i(z) \leqslant d,\ i = 0, \ldots, m,\ r|z| = d,\ z \geqslant 0\}$$

or

$$\min\{d \mid g_i(z) \leqslant d,\ i = 0, \ldots, m,\ re^T z = d,\ z \geqslant 0\}. \tag{4}$$

The feasible set of the problem (4) is convex. Therefore it can be effectively solved. The point of a global minimum of the problem (1) is the first n components of the solution of the problem (4).

References

[1] *A. Kosolap,* Solution of general problem of quadratic optimization, Proceeding of the Toulouse Global Optimization Workshop 2010, 71–74 (2010).

Simplification of Integral Payoffs in Differential Games with Random Duration

Kostyunin S., Shevkoplyas E.

isg23@yandex.ru, ekaterina.shevkoplyas@gmail.com

St. Petersburg State University, Universitetskii prospekt 35, Petergof, Saint-Petersburg,198504, Russia

There are two players which participate in differential game $\Gamma(x_0)$. The game $\Gamma(x_0)$ with dynamics

$$\dot{x} = g(x, u_1, u_2), \quad x \in R^m,\, u_i \in U \subseteq \operatorname{comp} R^l, \tag{1}$$
$$x(t_0) = x_0.$$

starts from initial state x_0 at the time instant t_0. But here we suppose that the terminal time of the game is the random variable T with known probability distribution function $F(t)$, $t \in [t_0, \infty)$ [7].

Suppose, that for all feasible controls of players, participating the game, there exists a continuous at least piecewise differentiable and extensible on $[t_0, \infty)$ solution of a Cauchy problem (1).

So, we have that the expected integral payoff of the player i can be represented as the following Lebesgue-Stieltjes integral:

$$I_i(t_0, x_0, u_1, u_2) = \int_{t_0}^{\infty} \int_{t_0}^{t} h_i(\tau, x(\tau), u_1, u_2) d\tau dF(t), \quad i = 1, 2. \tag{2}$$

The calculation of the expected payoff by formula (2) causes some difficulties. The simple formula for the expected payoff is obtained in the following form:

$$I_i(t_0, x_0, u_1, u_2) = \int_{t_0}^{\infty} h_i(\tau)(1 - F(\tau)) d\tau, \quad i = 1, 2. \tag{3}$$

The necessary conditions for this simplification in a general form are prooved.

References

[1] Burness, H. Stuart "A Note on Consistent Naive Intertemporal Decision Making and an Application to the Case of Uncertain Lifetime", *The Review of Economic Studies*, 43, No. 3, 547–549, (1976).

[2] Boukas, E. K. and Haurie, A. and Michel, P. " An Optimal Control Problem with a Random Stopping Time", *Journal of optimizationa theory and applications*, 64, No. 3, 471–480, (1990).

[3] Chang, F.R. *Stochastic Optimization in Continuous Time.* Cambridge Univ. Press, (2004).

[4] B. Gelbaum, J. Olmsted. *Counterexamples in Analysis*, Holden-Day, (1964).

[5] Kostyunin S.Y., Shevkoplyas E.V. *On the simplificatin of the integral payoff in differential games with random duration* // Vestnik SPbGU. 2011. S.10. V.4, P. 47 – 56 (in Russian).

[6] J. Marin-Solano and E. V. Shevkoplyas, *"Non-constant discounting in differential games with random time horizon"*, *Automatica*, Vol. 47, 12, 2011, pp. 2626–2638.

[7] Petrosjan, L.A., Murzov, N.V. "Game-theoretic problems of mechanics", *Litovsk. Mat. Sb.*, 6, 423-433, (1966).

Maximally robust control under uncertainty conditions

Koulaguin V.V.
wkoula@gmail.com
Институт Проблем машиноведения РАН, Большой пр. В.О.,
Санкт-Петербург, 190178, Россия

A criterion of maximal robustness is formulated and discussed in the talk. The question of existence of solutions of maximally robust control problems is studied, an equivalent statement of the problem is given. An example of maximally robust nonlinear mechanical system is provided.

Максимально робастное управление в условиях неопределенности

Формулируется и обсуждается критерий максимальной робастности [3–6], рассматриваются вопросы существования решения задачи максимально робастного управления, эквивалентная постановка задачи, приводится пример максимально робастной нелинейной механической системы.

Введение. Известные подходы к управлению в условиях неопределенности [2] предполагают, что множество возможных значений параметра неопределенности задано, ограничено, и требуется построить управление, которое обеспечивает достижение цели управления (выполнение условий, накладываемых на траектории системы в фазовом пространстве) для каждого значения параметра из множества. Такое управление, в современной терминологии, называется робастным управлением. В данном сообщении рассматривается критерий максимальной робастности управления в условиях неопределенности, который применим для любых множеств возможных значений параметра неопределенности, в том числе и для неограниченных или равных всему пространству значений параметра, и устанавливает максимальное (в уточняемом ниже смысле) множество значений параметра неопределенности, для каждого из которых одним, называемым максимально робастным, управлением обеспечивается достижение цели управления . Максимальное множество робастности, если оно существует, не задается извне, оно строится как характеристика системы, ее предельных возможностей достигать цели управления при заданном типе неопределенности.

Постановка задачи. Рассматриваются: u – управление (управляющая функция заданного класса); v – параметр неопределенности; $x(t;u,v)$ – траектория динамической системы в фазовом пространстве; $\varphi_i(u,v)$, $\psi_j(u,v)$, $J(u,v)$ — функционалы, заданные на траекториях; ограничения, задающие множество допустимых пар (u,v) $A = \{(u,v) \mid \varphi_i(u,v) = 0, \psi_j(u,v) \leqslant 0\}$; множество допустимых управлений при фиксированном параметре v $U_v = \{u \mid (u,v) \in A\}$; множество допустимых значений параметра v при фиксированном управлении u $V_u = \{v \mid (u,v) \in A\}$; а также — для задач оптимального управления — условие

$$J_v(u) \to \min_{u \in U_v}, \tag{1}$$

где $J_v(u)$ – сечение функционала $J(u,v)$ по переменной v, и множество

$$A^+ = \{(u,v) \mid u = \arg\min_{u \in U_v} J_v(u)\}$$

всех пар (u,v), при которых достигается цель управления, а именно: соблюдение ограничений и условие оптимальности (1). Для отображения $u \to V(u)$, где $V(u) = \{v \mid (u,v) \in A^+\}$ называется множеством робастности управления u, ставится *задача максимально робастного управления*: найти управление u^*, такое что

$$V(u^*) \supseteq V(u) \quad \forall u; \tag{2}$$

$V(u^*)$ – максимальное множество робастности.

О существовании решения. У задачи (2) есть очевидное формальное решение, когда в качестве максимально робастного может быть взято управление u, зависящее от параметра v, а именно $u^*(v)$, где u^* есть решение задачи (1) для данного значения параметра v. Очевидное решение неудовлетворительно по двум причинам. Во-первых, управление $u^*(v)$ принадлежит другому классу функций по сравнению с исходным. Во-вторых, реализация этого управления требует знания параметра v, что противоречит статусу неопределенного параметра и сводит задачу управления в условиях неопределенности к параметрическому семейству задач (1) с полной определенностью.

Если управления u – функции времени, их множества робастности либо пусты, либо состоят как правило из одного элемента v, а множество решений задачи (2) пусто. Если управления u строятся по принципу обратной связи по фазовым координатам $u = u(\cdot \mid x)$, их множества робастности намного "богаче".

В качестве примера задачи максимально робастного управления можно привести известную проблему синтеза оптимального управления [1] и ее решение для задачи максимального быстродействия при неопределенности начального вектора движения в системе второго порядка. Максимально робастным управление там является поверхность над фазовой плоскостью с линиями переключений управления с $+1$ на -1 и с -1 на $+1$ в виде парабол, а максимальное множество робастности есть вся фазовая плоскость. Второй пример приводится в настоящем сообщении.

Эквивалентная постановка задачи. Для отображения $v \to U(v)$, где $U(v) = \{u \mid (u, v) \in A^+\}$ – множество управлений, достигающих цели управления для фиксированного параметра v, т.е. для случая отсутствия неопределенности, ставится *задача максимально робастного управления*: найти управление u^+, такое что

$$u^+ \in \bigcap_v U(v). \tag{3}$$

В предположении выпуклости множества A^+ справедлива следующая теорема.

Теорема. *Задачи (2), (3) эквивалентны.*

Пример задачи максимально робастного управления [5]. Рассматривается задача оптимального управления, состоящая в перемещении материальной точки вдоль отрезка прямой ограниченной по величине силой из состояния покоя в состояние покоя за заданное время с минимальным максимумом абсолютной величины скорости. Неопределенной является величина отрезка. Построены максимально робастное управление как функция фазовых координат и максимальное множество робастности. Говоря более строго, в задаче $\min\limits_{|u|\leqslant 1} \max\limits_{t} \mid \dot{x}(t; u) \mid$, решаемой при условиях $\ddot{x} = u$, $t \in [0, T]$, $x(0) = x_0$, $\dot{x}(0) = 0$, $x(T) = \dot{x}(T) = 0$, введена неопределенность начального положения

x_0 и построено управление $u^*(x, \dot{x})$ как поверхность над фазовой плоскостью со значениями $+1$, 0, -1 и с линиями переключений в виде соответствующих парабол. Множество робастности управления $u^*(x, \dot{x})$ – отрезок $[-0.25\,T^2,\ 0.25\,T^2]$ – содержит множества робастности любого другого управления вида $u(t, x, \dot{x})$, т. е. построенное управление является максимально робастным управлением.

Заключение. Предложен критерий выбора управления в условиях неопределенности, состоящий в том, что каждому управлению заданного класса сопоставляются все значения параметра неопределенности, такие, что достигается цель управления, и это множество максимизируется, в заданном смысле, на множестве всех управлений класса. Исследовались вопросы существования решения данной задачи для различных классов управляющих функций.

В случае неограниченности значений параметра (вектора или функции) неопределенности, а также в случае, когда известно только пространство возможных значений параметра неопределенности, применение критерия максимальной робастности позволяет установить предельные возможности управления как общую характеристику изучаемой динамической системы.

Предложенный подход к работе с неопределенностью может сделать возможным более конструктивную постановку проблемы синтеза оптимального управления [5].

ЛИТЕРАТУРА

[1] *Понтрягин Л.С., Болтянский В.Г., Гамкрелидзе Р.В., Мищенко Е.Ф., Математическая теория оптимальных процессов*, М.: Наука, (1961).

[2] *Куржанский А.Б., Управление и наблюдение в условиях неопределённости*, М.: Наука, (1977).

[3] *Кулагин В.В.,* "Максимальная робастность в задаче оптимального управления"// *Тезисы доклада. X международный семинар "Устойчивость и колебания нелинейных систем управления"*, Москва: Институт проблем управления РАН, (2008).

[4] *Кулагин В.В.,* "Задача математического программирования для функции с избыточным аргументом" *// Тезисы доклада. Конференция "Математическое программирование и приложения"*, Екатеринбург: Институт математики и механики УрО РАН, (2007).

[5] *Кулагин В.В., Слесарь Н.О.,* "К синтезу оптимального управления"// *Тезисы доклада. Международная конференция "Дифференциальные уравнения и топология. К 100-летию со дня рождения Л.С. Понтрягина"*, Москва: МГУ им. М.В. Ломоносова, (2008).

[6] *Кулагин В.В., Проурзин В.А.,* "Амортизатор, максимально робастный к изменению массы защищаемого объекта *Известия РАН, Механика твердого тела*, N1, С. 34-44, (2005).

About Extremality, Stationarity and Regularity of Collections of Sets

Kruger A.Y.
a.kruger@ballarat.edu.au
University of Ballarat, POB 663, Ballarat, Vic, 3350, Australia

Starting with the pioneering work by Dubovitskii and Milyutin [1], it has become natural when dealing with optimization and other related problems to reformulate optimality or some other property under investigation as a kind of extremal behaviour of a certain collection of sets. Considering collections of sets is a rather general scheme of investigating extremal problems. For instance, any set of extremality conditions leads to some optimality conditions for the original problem.

The concept of a finite *extremal collection of sets* was introduced and investigated in [2, 9, 10].

Definition 1 ([2, 11]). *A collection of sets $\{\Omega_i\}_{i\in I}$, $1 < |I| < \infty$, in a normed linear space X, is called locally extremal at $\bar{x} \in \bigcap_{i\in I}\Omega_i$ if there exists a $\rho > 0$ such that for any $\varepsilon > 0$ there are $a_i \in X$, $i \in I$, such that*

$$\max_{i\in I}\|a_i\| < \varepsilon \quad \textit{and} \quad \bigcap_{i\in I}(\Omega_i - a_i)\bigcap B_\rho(\bar{x}) = \varnothing. \tag{1}$$

A dual necessary extremality condition in terms of Fréchet ε-normal elements was established in [2, 10] (formulated without proof in [9]) for a collection of closed sets in the setting of a Banach space admitting an equivalent norm Fréchet differentiable away from zero. It was extended in [12] to general Asplund spaces and is now known as the *Extremal principle*.

Theorem 1 ([2, 9, 11, 12]). *If a collection of closed sets $\{\Omega_i\}_{i\in I}$, $1 < |I| < \infty$, in an Asplund space, is locally extremal at $\bar{x} \in \bigcap_{i\in I}\Omega_i$, then for any $\varepsilon > 0$ there exist $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ and $x_i^* \in N^F_{\Omega_i}(x_i)$ $(i \in I)$ such that*

$$\left\|\sum_{i\in I} x_i^*\right\| < \varepsilon\sum_{i\in I}\|x_i^*\|, \tag{2}$$

where $N^F_{\Omega_i}(x_i)$ is the Fréchet normal cone to Ω_i at x_i.

This result can be considered as a generalization of the convex *separation theorem* to collections of nonconvex sets and is recognized as one of the cornerstones of the contemporary variational analysis. It can substitute the latter theorem when proving optimality conditions and subdifferential calculus formulas. We refer the reader to [11] for other applications and historical comments.

Similar to the classical analysis, besides extremality, the concepts of *stationarity* and *regularity* have been introduced and investigated. It was established in [3, 4] that the conclusion (2) of the Extremal principle actually characterizes a much weaker than local extremality (1) property which can be interpreted as kind of stationary behaviour of the collection of sets.

Definition 2 ([7]). *A collection of sets* $\{\Omega_i\}_{i\in I}$, $1<|I|<\infty$, *is approximately stationary at* $\bar{x}\in\bigcap_{i\in I}\Omega_i$ *if for any* $\varepsilon>0$ *there exist* $\rho\in(0,\varepsilon)$; $\omega_i\in\Omega_i\cap B_\varepsilon(\bar{x})$ *and* $a_i\in X$ $(i\in I)$ *such that*

$$\max_{i\in I}\|a_i\|<\varepsilon\rho \quad \text{and} \quad \bigcap_{i\in I}(\Omega_i-\omega_i-a_i)\bigcap(\rho\mathbb{B})=\varnothing.$$

Replacing in the Extremal principle local extremality with approximate stationarity produces a stronger statement – the *Extended extremal principle*: approximate stationarity of a finite collection of closed sets in an Asplund space is equivalent to its *separability* (*Fréchet normal approximate stationarity* [6, 7]).

Theorem 2 ([3, 4]). *A collection of closed sets* $\{\Omega_i\}_{i\in I}$, $1<|I|<\infty$, *in an Asplund space, is approximately stationary at* $\bar{x}\in\bigcap_{i\in I}\Omega_i$, *if and only if it is Fréchet normally approximately stationary at* $\bar{x}$, *i.e., for any* $\varepsilon>0$ *there exist* $x_i\in\Omega_i\cap B_\varepsilon(\bar{x})$ *and* $x_i^*\in N^F_{\Omega_i}(x_i)$ $(i\in I)$ *such that* (2) *holds true.*

If a collection of sets is not approximately stationary, it is *uniformly regular.*

Definition 3 ([5, 6]). *A collection of sets* $\{\Omega_i\}_{i\in I}$, $1<|I|<\infty$, *is uniformly regular at* $\bar{x}\in\bigcap_{i\in I}\Omega_i$ *if there exists an* $\alpha>0$ *and an* $\varepsilon>0$ *such that*

$$\bigcap_{i\in I}(\Omega_i-\omega_i-a_i)\bigcap(\rho\mathbb{B})\neq\varnothing$$

for any $\rho\in(0,\varepsilon)$; $\omega_i\in\Omega_i\cap B_\varepsilon(\bar{x})$ *and* $a_i\in X$ $(i\in I)$ *satisfying* $\|a_i\|\leqslant\alpha\rho$.

The latter property is the direct analogue for collections of sets of the *metric regularity* of multifunctions. The corresponding dual property is called *Fréchet normal uniform regularity* ([6, 7]). The next theorem is a corollary of Theorem 2.

Theorem 3. *A collection of closed sets* $\{\Omega_i\}_{i\in I}$, $1<|I|<\infty$, *in an Asplund space, is uniformly regular at* $\bar{x}\in\bigcap_{i\in I}\Omega_i$, *if and only if it is Fréchet normally uniformly regular at* $\bar{x}$, *i.e., there exists an* $\alpha>0$ *and an* $\varepsilon>0$ *such that*

$$\left\|\sum_{i\in I}x_i^*\right\|\geqslant\alpha\sum_{i\in I}\|x_i^*\|$$

for any $x_i\in\Omega_i\cap B_\varepsilon(\bar{x})$ *and* $x_i^*\in N^F_{\Omega_i}(x_i)$ $(i\in I)$.

Given a collection of sets $\mathbf{\Omega} := \{\Omega_i\}_{i\in I} \subset X$, where $1 < |I| < \infty$, and a point $\bar{x} \in \bigcap_{i\in I} \Omega_i$, the discussed above extremality, stationarity and regularity properties can be equivalently defined in terms of certain nonnegative (possibly infinite) constants (see [5, 6, 7]):

$$\theta_\rho[\mathbf{\Omega}](\bar{x}) := \sup\Big\{r \geqslant 0 \Big| \bigcap_{i\in I}(\Omega_i - a_i)\bigcap B_\rho(\bar{x}) \neq \varnothing,\ \forall a_i \in r\mathbb{B}\Big\}, \quad \rho \in (0,\infty],$$

$$\theta[\mathbf{\Omega}](\bar{x}) := \liminf_{\rho\downarrow 0} \frac{\theta_\rho[\mathbf{\Omega}](\bar{x})}{\rho}, \quad \hat{\theta}[\mathbf{\Omega}](\bar{x}) := \liminf_{\rho\downarrow 0;\, \omega_i \xrightarrow{\Omega_i} \bar{x},\, i\in I} \frac{\theta_\rho[\{\Omega_i - \omega_i\}_{i\in I}](0)}{\rho},$$

$$\hat{\eta}[\mathbf{\Omega}](\bar{x}) := \liminf_{\substack{x_i \xrightarrow{\Omega_i} \bar{x},\, x_i^* \in N^F_{\Omega_i}(x_i)\ (i\in I) \\ \sum_{i\in I}\|x_i^*\|=1}} \Big\|\sum_{i\in I} x_i^*\Big\|.$$

Extensions of the above definitions and results to infinite collections of sets can be found in [8].

References

[1] *A.Y. Dubovitskii, A.A. Miljutin,* "Extremal problems with constraints", *USSR Comp. Maths. Math. Phys.*, **5**, 1–80 (1965).

[2] *A.Y. Kruger, Generalized differentials of nonsmooth functions*, Deposited in VINITI no. 1332-81. Minsk (1981). In Russian.

[3] *A.Y. Kruger,* "On Fréchet subdifferentials". *J. Math. Sci.*, **116**, No. 3, 3325–3358 (2003).

[4] *A.Y. Kruger,* "Weak stationarity: eliminating the gap between necessary and sufficient conditions", *Optimization*, **53**, No. 2, 147–164 (2004).

[5] *A.Y. Kruger,* "Stationarity and regularity of set systems", *Pac. J. Optim.*, **1**, No. 1, 101–126 (2005).

[6] *A.Y. Kruger,* "About regularity of collections of sets", *Set-Valued Anal.*, **14**, No. 2, 187–206 (2006).

[7] *A.Y. Kruger,* "About stationarity and regularity in variational analysis", *Taiwanese J. Math.* **13**, No. 6A, 1737–1785 (2009).

[8] *A.Y. Kruger, M.A. López,* "Stationarity and regularity of infinite collections of sets", to be published.

[9] *A.Y. Kruger, B.S. Mordukhovich,* "Extremal points and the Euler equation in nonsmooth optimization problems", *Dokl. Akad. Nauk BSSR*, **24**, No. 8, 684–687 (1980). In Russian.

[10] *A.Y. Kruger, B.S. Mordukhovich, Generalized normals and derivatives and necessary conditions for an extremum in problems of nondifferentiable programming*, Deposited in VINITI, I – no. 408-80, II – no. 494-80. Minsk (1980). In Russian.

[11] *B.S. Mordukhovich, Variational Analysis and Generalized Differentiation. I: Basic Theory*, Springer-Verlag, Berlin (2006).

[12] *B.S. Mordukhovich, Y. Shao,* "Differential characterizations of covering, metric regularity, and Lipschitzian properties of multifunctions between Banach spaces", *Nonlinear Anal.*, **25**, No. 12, 1401–1424 (1995).

Order subdifferentials and optimization

Kurc W.
wkurc@amu.edu.pl
Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, Poznań, 61-614, Poland

Let $f : X \longrightarrow \overline{R}$ be monotone over locally convex solid *Riesz* space X. f is said to be monotone if $f(x) = f(|x|)$ and $f(y) \leqslant f(x)$ whenever $0 \leqslant y \leqslant x$. $\vee-$ and $\wedge-$ subdifferentials $\partial^\vee f(x)$ and $\partial^\wedge f(x)$ of the functional f at the point $x \geqslant 0$ are defined by $\partial^\vee f(x) \equiv \{x^* - y^* : \langle x, y^* \rangle = f(x) + f^*(x^*),\ 0 \leqslant y^* \leqslant x^*\}$ ([1]) and $\partial^\wedge f(x) \equiv \{x^* - y^* : \langle x, y^* \rangle = f(0) + f^+(x^*),\ 0 \leqslant y^* \leqslant x^*\}$ where $f^+(x^*) = -(-f^*)(-x^*)$ for the Young conjugate f^* of f. We refere to these subdifferentials as to order (or lattice) subdifferentials. Let

$$\begin{aligned} f^\vee(x;y) &\equiv \lim_{\tau \searrow 0} \frac{1}{\tau} \left(f(x \vee \tau y) - f(x) \right), \\ f^\wedge(x;y) &\equiv \lim_{\tau \searrow 0} \frac{1}{\tau} \left(f(x \wedge \tau y) - f(0) \right), \end{aligned} \tag{1}$$

for $x, y \in X_+$, be $\vee-$ and $\wedge-$directional derivatives, respectively, of f at x in the direction y. If $f \in Conv(X)$ with $x \in dom(f)$ and $f \in Conc(X)$ with $0 \in dom(f)$ then $\vee-$ and $\wedge-$directional derivatives are well defined with inf and sup over $t > 0$ instead of $lim_{t \searrow 0}$, respectively.

$\vee-$ and $\wedge-$ subdifferentials can be expressed in terms of the usual subdifferential ∂. If $x \in dom(f)_+$ then $\partial^\vee f(x) = (\partial_+ f(x) - \partial_+ f(x))_+ \cap \{x\}^\perp$ for $f \in Conv(X)$. If $f \in Conc(X)$ and $0 \in dom(f)$ then $\partial^\wedge f(x) = ((x^\perp)_+ + \partial_+ f(0))_-$. Order subdifferentials are convex and $weakly^*$ compact if the above x and 0, respectively, are points of continuity for f.

There hold Moreau-Pshenichnii type theorems. Under suitable continuity assumptions, if $f \in Conv(X)$ and, respectively, $f \in Conc(X)$ then

$$\begin{aligned} f^\vee(x;y) &= \max\{< y, x^* >: x^* \in \partial^\vee f(x)\}, \\ f^\wedge(x;y) &= \min\{< y, x^* >: x^* \in \partial^\wedge f(x)\}. \end{aligned} \tag{2}$$

Next, Moreau-Rockafellar type theorems can be proved. Under suitable assumptions there hold

$$\begin{aligned} \partial^{\vee}(f_1+f_2)(x) &= \partial^{\vee} f_1(x) + \partial^{\vee} f_2(x), \\ \partial^{\wedge}(f_1+f_2)(x) &= \partial^{\wedge} f_1(x) + \partial^{\wedge} f_2(x). \end{aligned} \tag{3}$$

where f_1 and f_2 are in $Conv(X)$ and in $Conc(X)$, respectively.

Let us consider the following optimization problem

$$P_x(f) = \left\{ y_0 \in X_+ : f(x \vee y_0) = \min_{y \geqslant 0} f(x \vee y) \right\}$$

where $f \in Conv(X)$ and $x \geqslant 0$. In a sense we search $x + X_+$ in nonlinear way. Notice, $[0, x] \subset P_x(f)$ and $f(x \vee y_0) = f(x)$ for $y_0 \in P_x(f)$. Let f be continuous at $x \geqslant 0$ and let $y_0 \in P_x(f)$.

(a) Then $y_0 \perp \partial^{\vee} f(x)$ or, equivalently,

(b) For each $x^*, y^* \in \partial_+ f(x)$, such that $x^* \geqslant y^* \geqslant 0$, and $x^* - y^* \perp x$, there holds $x^* - y^* \perp y_0$.

The following statements are equivalent.

(a) $y_0 \in P_x(f)$,

(b) There exists $x^* \in X_+^*$ such that

  (*i*) $x^* \in \partial_+ f(x) \cap \partial_+ f(x \vee y_0)$

  (*ii*) $\langle y_0 - x, y^* \rangle \leqslant 0$, for all y^* such that $0 \leqslant y^* \leqslant x^*$

Similarly the optimization problem

$$Q_x(f) = \left\{ y_0 \in X_+ : f(x \wedge y_0) = \min_{y \geqslant 0} f(x \wedge y) \right\},$$

where $f \in Conc(X)$, can be characterized. Notice that $0 \in Q_x(f)$.

Mixed order subdifferentials $\partial^{\vee\wedge}(x)$, $\partial^{\wedge\vee}(x)$ appear in natural way. This is a consequence of the fact that there are formulae of the form

$$\frac{f(x \wedge ty) - f(0)}{t} = \sup_{x^* \geqslant 0} \inf_{x^* \geqslant y^* \geqslant 0} \left\{ < y, x^* - y^* > + \frac{< x, y^* > - f(0) - f^*(x^*)}{t} \right\}.$$

where, in proper way, f can be convex or concave, the operation $\vee$ can be considered instead of $\wedge$ (then 0 must be replaced by x), and the operations *inf*, *sup* can be in different combinations.

References

[1] *W. Kurc,* “Monotonicity, order smoothness and duality for convex functionals”, *Collect. Math.*, 48, No. 4-6, 635-–655 (1997).

On Computation Methods for Set-Valued solutions to Problems of Dynamics and Control

Kurzhanski A.B.
kurzhans@mail.ru, kurzhans@cs.msu.su
Lomonosov Moscow State University

The crucial elements of many solutions to problems of dynamics and control consist in describing related invariant sets and their dynamics. In practice this amounts to computation of forward and backward reachability tubes that turn out to be described by nonsmooth functions with values in nonconvex sets or in convex sets with non-smooth boundaries. The related theory may be based on modifications of Hamiltonian techniques to non-differentiable solutions while the computation relies on approximation of set-valued solution tubes through intersections and unions of parametrized arrays of ellipsoidal or polyhedral -valued tubes. This leads to a natural application of parallel computations. Indicated are classes of problems where the described approach appears effective.

Nonstandard Tools for Nonsmooth Analysis

Kutateladze S.S.
sskut@math.nsc.ru
Sobolev Institute, 4 Koptyug Avenue, Novosibirsk, 630090, Russia

This is an overview of a few possibilities that are open by model theory in applied mathematics. Most attention is paid to the present state and frontiers of the Leonid Kantorovich ideas in the Cauchy method of majorants, approximation of operator equations with finite-dimensional analogs, and the Lagrange multiplier principle in multiobjective decision making.

— Agenda

— The Art of Calculus

— Pure and Applied Mathematics

— Challenges of the 20th Century

— Enter New Mentality

— Enigmas of Economics

— Enter the Reals

— Scalarization

— Order Omnipresent

— Enter Fermat

— Enter Hahn–Banach

— Enter Kantorovich

— Kantorovich's Heuristics

— Canonical Operator

— Support Hull

— Hahn–Banach in Disguise

— Enter Boole

— Enter Descent

— The Reals in Disguise

— Norming Sequences

— Domination

— Enter Abstract Norm

— Exeunt Abstract Norm

— Approximation

— Enter Epsilon

— Pareto Optimality

— Approximate Efficiency

— Subdifferential Halo

— Exeunt Epsilon

— Discretization

— Hypoapproximation

— Hyperapproximation

— The Hull of a Space

— The Hull of an Operator

— One Puzzling Definition

— Enter Epsilon and Monad

— Exeunt Epsilon

— State of the Art

— Vistas of the Future

References

[1] *L. V. Kantorovich,* "Functional Analysis and Applied Mathematics," *Vestnik Leningrad Univ. Mat.*, 6, 3-–18 (1948).

[2] *L. V. Kantorovich,* "Functional Analysis and Applied Mathematics," *Uspekhi Mat. Nauk*, 3, 89–195 (1948).

[3] *L. V. Kantorovich,* "On Semiordered Linear Spaces and Their Applications in the Theory of Linear Operators," *Dokl. Akad. Nauk SSSR*, 4, No.1–2, 11–14 (1935).

[4] *L. V. Kantorovich,* "The Principle of Majorants and the Newton Method," *Dokl. Akad. Nauk SSSR*, 76, No. 1, 17–20 (1951).

[5] *L. V. Kantorovich, Selected Works. Part I. Descriptive Theory of Sets and Functions. Functional Analysis in Semi-ordered Spaces*, Gordon and Breach Publishers, Amsterdam (1996).

[6] *L. V. Kantorovich, Selected Works. Part II. Applied Functional Analysis. Approximation Methods and Computers*, Gordon and Breach Publishers, Amsterdam (1996).

[7] *A. G. Kusraev and S. S. Kutateladze, Introduction to Boolean Valued Analysis*, Nauka, Moscow (2005) [in Russian].

[8] *A. G. Kusraev and S. S. Kutateladze, Subdifferential Calculus: Theory and Applications*, Nauka, Moscow (2007) [in Russian].

[9] *E. I. Gordon, A. G. Kusraev, and S. S. Kutateladze, Infinitesimal Analysis: Selected Topics*, Nauka, Moscow (2008) [in Russian].

Some Approaches for Construction Exact Auxiliary Functions in Optimization Problems

Laptin Yu.P., Bardadym T.A.

laptin_yu_p@mail.ru, TBardadym@gmail.com

V.M.Glushkov Institute of Cybernetics of the NASU, Prospekt Academika Glushkova, 40, 03650 Kyiv, Ukraine

Two approaches to represent a constrained optimization problem as an unconstrained one are proposed. They can be treated in a framework of general notion of exact auxiliary functions proposed in [1]. The first approach is connected with exact penalty functions. The second one is considered for the case when a criterion function is not defined outside a feasible set. It is based on the idea of conical approximation.

A convex programming problem is considered: to find

$$f^* = \min\{f(x) : \quad x \in C\}, \tag{1}$$

where $C = \{x \in R^n : \quad h(x) \leqslant 0\}$, $\quad f, h : R^n \to R \cup \{+\infty\}$ are convex functions. The set $C \subseteq \mathrm{dom} f$ is supposed to be a convex compact, a feasible point $x^0 \in \mathrm{int} C$ is given, so $h(x^0) < 0$.

1. Let f, h be finite for any x. Denote $x^+ = \max\{0, \ x\}$. Consider a penalty function of the form

$$S(x, s) = f(x) + s \cdot h^+(x), \quad s \in R, \quad s \geqslant 0, \tag{2}$$

and optimization problem: to find

$$S^*(s) = \min\{S(x,s) : x \in R^n\}. \tag{3}$$

Penalty function $S(x,s)$ is exact, if $S^*(s) = f^*$ for a given penalty coefficient s, and solutions to the problems (1) and (3) coincide. To find such value of a penalty coefficient it is usually necessary to solve some auxiliary dual problems, this may lead either to overestimation of the values used, or to the need to solve the problem (3) several times to find a satisfactory penalty coefficient. An approach that allows to find automatically the value of penalty coefficient in optimization algorithm is proposed in [2]. Let $S'(x,s,p)$ be a directional derivative of function S at $x \in R^n$ in the direction p under fixed s, $p(x) = (x - x^0)/\left\|x - x^0\right\|$.

Lemma [2]. *If $\varepsilon > 0$ and for any $x \notin C$*

$$S'(x,s,p(x)) \geqslant \varepsilon, \tag{4}$$

then $S(x,s)$ is an exact penalty function.

Theorem 1 [2]. *Let C be a bounded closed set, then there exists a finite $\bar{s}$ such that condition of the Lemma above is satisfied for any $s > \bar{s}$.*

Let $g_f(x)$, $g_h(x)$ be subgradients of functions f and h respectively. Instead of (4) on can use an inequality

$$(g_f(x), p(x)) + s\,(g_h(x), p(x)) \geqslant \varepsilon. \tag{5}$$

Let the value of penalty coefficient s be fixed, and a globally convergent algorithm A for unconstrained minimization of convex functions is used to solve (3), x^k is a current point generated by the algorithm A at step k.

Theorem 2 [2]. *Suppose at each iteration k of the algorithm A the following condition is satisfied: if $x^k \notin C$, then for this point the inequality (5) is fulfilled. Then the algorithm A converges to the solution of problem (1).*

If inequalities (4) or (5) are violated at some iteration of the algorithm A, it is proposed to increase a penalty coefficient s to fulfill (5), but not less than on B, $B > 0$ is a given parameter. For $\bar{s}$ is finite, the number of such increments will also be finite. Then by Theorem 2 an algorithm converges to the solution of (1) .

2. Consider conical approximations to construct an exact auxiliary function. Let a number $E < f(x^0)$ be given, and F be an epigraph of function f on C: $F = \{(\lambda, x) \in R \times C : \lambda \geqslant f(x)\}$, $z = (\lambda, x)$, $z \in R \times R^n$. Let consider a conic hull $K(E)$ of an epigraph F with a vertex $z_E^0 = (E, x^0)$

$$K(E) = \{\nu \in R \times R^n : \quad \nu = z_E^0 + \alpha(z - z_E^0), \quad \alpha \geqslant 0, \quad z \in F\}. \tag{6}$$

The set $K(E)$ is convex (because F is convex) and can be considered as an epigraph of some convex function. Denote this function as $\gamma_E(x)$ and call it a conical approximation of function f on C.

It is easy to see that for an arbitrary point $x \in R^n$, $x \neq x^0$ there exists at least one point $\overline{x} \in C$ on a ray starting from x^0 and passing through x, that $f(\overline{x}) = \gamma_E(\overline{x})$. Let denote the nearest to x^0 point of this kind as $\mu_E(x)$. Denote

$$\varphi_E(x) = \begin{cases} f(x), & \text{if} \quad ||x - x^0|| \leqslant ||\mu_E(x) - x^0||, \\ \gamma_E(x), & \text{if} \quad ||x - x^0|| > ||\mu_E(x) - x^0||. \end{cases} \tag{7}$$

Consider a problem: to find

$$\varphi_E^* = \inf\{\varphi_E(x), \quad x \in R^n\}. \tag{8}$$

Theorem 3 [6]. *Let $E < f(x^0)$, than $\varphi_E(x) : R^n \to R$ is a convex function; if $E < f^*$, then $\varphi_E^* = f^*$.*

It is necessary to solve a special one-dimensional search problem to find a value of function $\gamma_E(x)$ in any point $x \in R^n$ and to find a point $\mu_E(x)$. Relations to calculate subgradients of function $\gamma_E(x)$ (so also $\varphi_E(x)$) are reported.

3. Operation of conical extension of function f from C to R^n, depending on a parameter E is considered. As a resul we get a function $\psi_E(x)$ coinsiding with f on C. Conditions when functions $\varphi_E(x)$ and $\psi_E(x)$ coinside on R^n are formulated.

The statements similar to theorems about nonsmooth penalty functions are formulated for conical extensions of functions and special classes of optimization problems.

If the problem (1) is reduced to the problem (8), one can use any unconstrained optimization algorithm to solve it. A special procedure to adjust a parameter E is used in calculations. r-Algorithm [3] was used as a tool for unconstrained optimization in a software implementation [5]. The developed programs are compatible with standard software environment AMPL. This allows to compare these programs with modern solvers (SNOPT, MINOS, LOQO etc.). Computational experiments were carried out on specially designed ill-conditioned test problems and demonstrated the advantages of this approach.

References

[1] *Yu.G. Evtushenko, V.G. Zhadan,* "Exact auxiliary functions in optimization problems", *J. Comp. Math. and Math. Physics*, 30, No. 1, 43–57 (1990).

[2] *Yu.P. Laptin,* "Some issues of the use of nonsmooth penalty functions", *Theory of Optimal Solutions*, No. 10, 127–135 (2011) (Russian).

[3] *N.Z. Shor, Nondifferentiable Optimization and Polynomial Problems*, Kluwer Academic Publishers, (1998).

[4] *Yu.P. Laptin,* "An approach to solve nonlinear constrained optimization problems", *Cybernetics and System Analysis*, No. 3, 182–187 (2009) (Russian).

[5] *Yu.P. Laptin, A.P. Lykhovyd,* "The use of convex extensions of functions for solution nonlinear optimization problems", *Control Mashines and Systems*, No. 6, 25–31 (2010) (Russian).

[6] *Yu.P. Laptin, T.A. Bardadym,* "Some approaches to regularization nonlinear optimization problems", *Journal of Automation and Information Sciences*, No. 3, 57–68 (2011)(Russian).

Subdifferential estimate of the directional derivative and optimality criterion

Lassonde M., Jules F.

marc.lassonde@univ-ag.fr, florence.jules@univ-ag.fr

Université des Antilles et de la Guyane, 97159 Pointe à Pitre, France

We provide an inequality relating the radial directional derivative and the subdifferential of proper lower semicontinuous functions, which extends the known formula for convex functions [1, 6]. We show that this property is equivalent to other subdifferential properties of Banach spaces, such as controlled dense subdifferentiability, optimality criterion, mean value inequality and separation principles. As an application, we obtain a first-order sufficient condition for optimality, which extends the known condition for differentiable functions in finite-dimensional spaces [2] and which amounts to the maximal monotonicity of the subdifferential for convex lower semicontinuous functions [5]. Finally, we establish a formula describing the subdifferential of the sum of a convex lower semicontinuous function with a convex inf-compact function in terms of the sum of their approximate ε-subdifferentials. Such a formula directly leads to the known formula relating the directional derivative of a convex lower semicontinuous function to its approximate ε-subdifferential [4]. This talk is based on our recent work [3].

References

[1] *J. M. Borwein,* "A note on ε-subgradients and maximal monotonicity", *Pacific J. Math.*, 103, No. 2, 307–314 (1982).

[2] *M. W. Botsko,* "The Teaching of Mathematics: A First Derivative Test for Functions of Several Variables", *Amer. Math. Monthly*, 93, No. 7, 558–561 (1986).

[3] *F. Jules and M. Lassonde,* "Subdifferential estimate of the directional derivative, optimality criterion and separation principles", *Optimization*, to appear.

[4] *R. T. Rockafellar,* "Characterization of the subdifferentials of convex functions", *Pacific J. Math.*, 17, 497–510 (1966).

[5] *R. T. Rockafellar,* "On the maximal monotonicity of subdifferential mappings", *Pacific J. Math.*, 33, 209–216 (1970).

[6] *P. D. Taylor,* "Subgradients of a convex function obtained from a directional derivative", *Pacific J. Math.*, 44, 739–747 (1973).

Nonsmooth Analysis on Smooth Manifolds

Ledyaev Y.S.
ledyaev@wmich.edu
Western Michigan University, Kalamazoo, MI 49008, USA

There are numerous instances of nonsmooth functions which arise naturally in many problems for smooth manifolds. For example, the well-known Ky Fan equality gives the following nonsmooth representation of the k-th largest eigenvalue of a symmetric matrix A on the manifold of symmetric $n \times n$ matrices

$$\lambda_k(A) = \max_{X \in S(k+1,n)} \operatorname{tr}(X^T AX) - \max_{X \in S(k,n)} \operatorname{tr}(X^T AX),$$

where the Stiefel manifold

$$S(k,n) := \{X \in \mathbf{R}^{n \times k} \ : \ X^T X = I_k\}$$

consists of real orthogonal $n \times k$ matrices.

The framework and some tools for studying nonsmooth semicontinuous functions on smooth manifolds were developed in [6]. In many aspects it was intended to provide the same wide range of applications as methods of monographs [1, 3, 4] do for nonsmooth optimization and control problems on linear spaces.

In this talk we'll discuss the following applications of the nonsmooth analysis on smooth manifolds:

Differential geometry. For Riemannian manifolds of nonpositive curvature (Hadamard-Cartan manifolds) we use nonsmooth analysis to prove a manifold's variant [7] of classical Helly's theorem on common point of a system of convex sets and classical Junge's theorem on the radius of circumscribed ball for a set on manifold.

Chow-Rashevskii Theorem. Classical Chow-Rashevskii's theorem provides sufficient conditions for existence of geodesic connecting arbitrary two points on finite-dimensional sub-Riemannian manifolds. Historically, it is also the first fundamental result for a finite-dimensional non-holonomic affine control system

$$\dot{q} = \sum_i u_i(t) X_i(q), \quad q \in M \tag{1}$$

which proves that the system (1) is globally controllable under the following condition

$$\mathrm{Lie}(X_1, X_2, \ldots)(q) = T_q(M) \quad \forall\, q \in M \tag{2}$$

where $\mathrm{Lie}(X_1, X_2, \ldots)$ denotes a linear span of all vector fields generated by $X_1, X_2, \ldots$ and their iterated Lie brackets, $T_q(M)$ denotes the tangent space at the point q.

A generalized version of Chow-Rashevskii's theorem for infinite-dimensional control system (1) and infinite-dimensional manifold M was suggested in [5]. We discuss nonsmooth analysis tools which can be used to demonstrate that conditions (2) imply global *approximate* controllability of the infinite-dimensional system (1) under more general assumptions than in [5].

Stabilization of control systems on manifolds.

Consider a problem of global stabilization of the control system $\dot{q} = X(q, u)$ on the n-dimensional manifold M. In accordance with Milnor's theorem if there exists a *continuous* stabilizing feedback $k(q)$ (such that $\dot{q} = f(q, k(q))$ is asymptotically stable) then the manifold M is diffeomorphic to the linear space $\mathbb{R}^n$.

In particular it implies that there is no continuous stabilizing feedback which provides a global satellite stabilization with the following dynamics

$$J\dot{\omega} = J\omega \times \omega + u,$$

$$\dot{R} = R\,\omega^{\times},$$

where $\omega \in \mathbb{R}^3$, $R \in \mathbb{SO}(3)$,

$$\omega^{\times} := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

It can be also shown for this problem that there is no *smooth* control Lyapunov function which can be used for finding a stabilizing feedback. Nevertheless it is possible to show that there exists a *nonsmooth* control Lyapunov function.

In the case of nonlinear control systems in $\mathbb{R}^n$ the appropriate concept of discontinuous feedback control was suggested in [2] which provides a precise and convenient model of digital computer-aided feedback control.

We demonstrate that a theory of nonsmooth analysis on smooth manifolds supplies useful tools for analysis of nonsmooth control Lyapunov functions and design of discontinuous feedback for control problems on manifolds.

References

[1] *F.H. Clarke, Optimization and nonsmooth analysis*, Wiley-Interscience, (1983).

[2] *F.H. Clarke, Yu.S. Ledyaev, E.D. Sontag, A.I. Subbotin,* "Asymptotic controllability implies feedback stabilization", *IEEE Trans. Automat. Control*, 42, 1394–1407 (1997).

[3] *F.H. Clarke, Yu.S. Ledyaev, R.J. Stern, P.R. Wolenski, Nonsmooth Analysis and Control Theory*, Springer-Verlag, (1998).

[4] *F.H. Clarke, Necessary conditions in dynamic optimization*, Amer. Math. Soc., (2005).

[5] *Yu.S. Ledyaev,* "On an infinite-dimensional variant of the Rashevskiĭ-Chow theorem", *Dokl. Akad. Nauk*, 398, 735-737 (2004).

[6] *Yu.S. Ledyaev, Q.J. Zhu,* "Nonsmooth analysis on smooth manifolds", *Trans. Amer. Math. Soc.*, 359 , 3687–3732 (2007).

[7] *Yu.S. Ledyaev, J.S. Treiman, Q.J. Zhu,* "Helly's intersection theorem on manifolds of nonpositive curvature", *J. Convex Anal.*, 13, 785-798 (2006).

Active sets and nonsmooth geometry

Lewis A.
adrian.lewis@cornell.edu
School of Operations Research and Information Engineering, Cornell University

The active constraints of a nonlinear program typically define a surface central to understanding both theory and algorithms. The standard optimality conditions rely on this surface; they hold generically, and then the surface consists locally of all solutions to nearby problems. Furthermore, standard algorithms "identify"the surface: iterates eventually remain there. A blend of variational and semi-algebraic analysis gives a more intrinsic and geometric view of these phenomena, attractive for less classical optimization models. A recent proximal algorithm for composite optimization gives an illustration.

Joint work with J. Bolte, A. Daniilidis, D. Drusvyatskiy, M. Overton and S. Wright.

Small survey on some recent contributions to subdifferential calculus

López-Cerdá M.
marco.antonio@ua.es
Alicante University, Ctra. San Vicente de Raspeig s/n, Alicante, 03080, Spain

The main objective of this talk is threefold. First, to provide a general formula for the optimal set of a relaxed minimization problem in terms of the approximate minima of the data function. Secondly, to derive explicit charac- terizations for the (convex) subdi¤erential mapping of the supremum function of an arbitrarily indexed family of functions, exclusively in terms of the data functions. Finally, to present alternative approaches and applications to subd- i¤erential calculus.

Numerical Methods for Nonsmooth Optimization

Mäkelä M.M., Karmitsa N., Bagirov A.
makela@utu.fi, napsu@karmitsa.fi, a.bagirov@ballarat.edu.au
University of Turku, Department of Mathematics and Statistics, FI-20014 Turku, Finland, Centre for Informatics and Applied Optimization, School of Science, Information Technology and Engineering, University of Ballarat, University Drive, Mount Helen, PO Box 663, Ballarat, VIC 3353, Australia

We consider unconstrained nonsmooth optimization problems of the form

$$\min_{x \in \mathbb{R}^n} f(x) \tag{1}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is supposed to be locally Lipschitz continuous. Note that no differentiability or convexity assumptions are made. Nonsmooth optimization problems of type (1) arise in many application areas: for instance, optimal shape design [4, 9], data mining and machine learning, economics, mechanics and engineering [5].

Most of methods for solving problems (1) can be divided into two main groups: subgradient [10] and bundle methods [8]. Both of these method groups have their own supporters. Usually, when developing new methods, researchers compare them with similar methods. Moreover, it is quite common that the test set used is rather concise.

Our aim is to compare different subgradient and bundle methods, as well as some of methods that lie between these two. A broad set of nonsmooth optimization test problems are used in numerical experiments. Rather than foreground some method over the others our aim is to get some insight on which method is suitable for certain types of problems.

Subgradient methods are the simplest methods in nonsmooth optimization. The idea behind them is to generalize smooth gradient based methods by replacing the gradient with an arbitrary subgradient. One of the most efficient subgradient method is the well-known *Shor's r-algorithm* with space dilations along the difference of two successive subgradients. The idea of method is to interpolate between the steepest descent and the conjugate gradient methods. The iteration formula is

$$x_{k+1} = x_k + \lambda_k d_k,$$

where the search direction $d_k = -B_k \xi_k$, B_k is so called space dilation matrix, $\lambda_k > 0$ is some pretermined step size and $\xi_k \in \partial f(x_k)$ is a subgradient of f at x_k.

The basic idea of bundle methods is to approximate the whole subdifferential of the objective function. In practice, this is done by gathering subgradients from the previous iterations into a bundle. In *proximal bundle method* [6] the search direction is calculated by

$$d_k = \operatorname*{arg\,min}_{d \in \mathbb{R}^n} \{\hat{f}_k(d) + \frac{1}{2} u_k d^T d\},$$

where the objective function is approximated by *cutting plane model*

$$\hat{f}_k(d) = \max_{j \in J_k} \{f(x_k) + \xi_j^T d\} \tag{2}$$

with $\varnothing \neq J_k \subset \{1, \dots, k\}$, $\xi_j \in \partial f(y_j)$ and $y_j \in \mathbb{R}^n$ for $j \in J_k$. The stabilizing term $\frac{1}{2} u_k d^T d$ guarantees the existence of the solution d_k and keeps the approximation local enough.

The *bundle-Newton method* [7] is a second order method, where instead of the piecewise linear cutting plane model (2), we utilize a piecewise quadratic model of the form

$$\tilde{f}_k(d) = \max_{j \in J_k} \{f(x_k) + \xi_j^T d + \frac{1}{2} \varrho_j d^T G_j d\}$$

where G_j is an approximation of the Hessian matrix at y_j and $\varrho_j \in [0, 1]$ is some damping parameter.

The *limited memory bundle method* [3] is developed for solving large-scale nonsmooth problems. The method is a hybrid of the variable metric bundle methods and the limited memory variable metric methods. The search direction is calculated using a limited memory approach

$$d_k = -D_k \tilde{\xi}_k,$$

where $\tilde{\xi}_k$ is some aggregate subgradient and D_k is the limited memory variable metric update that, in the smooth case, represents the approximation of the inverse of the Hessian matrix. The matrix D_k is not formed explicitly but the search direction is calculated using the limited memory approach.

The *discrete gradient method* [2] is a derivative free version of bundle methods approximating subgradients by discrete gradients using function values only. Similarly to bundle methods the previous values of discrete gradients are gathered into a bundle. The search direction is calculated as

$$d_k = \operatorname*{arg\,min}_{d \in S(x_k)} \|d\|^2,$$

where $S(x_k)$ is the convex hull of all the discrete gradients computed so far.

The *quasi-secant method* [1] can be considered as a hybrid of bundle methods and the gradient sampling method. The method builds up information about the approximation of the subdifferential using bundling idea, while subgradients are computed from the given neighborhood of the current iteration point like in the gradient sampling method. The procedure for finding search directions is pretty similar to that in discrete gradient method but we use here the quasi-secant instead of the discrete gradient.

References

[1] *A. M. Bagirov and A. N. Ganjehlou,* "A quasisecant method for minimizing nonsmooth functions", *Optim Method Softw*, No. 25(1), 3–18 (2010).

[2] *A. M. Bagirov, B. Karasozen and M. Sezer,* "Discrete gradient method: A derivative free method for nonsmooth optimization *J Optimiz Theory App*, No. 137, 317–334 (2008).

[3] *N. Haarala, K. Miettinen and M. M. Mäkelä,* "Globally convergent limited memory bundle method for large-scale nonsmooth optimization *Math Program*, No. 109(1), 181–205 (2007).

[4] *J. Haslinger and P. Neittaanmäki, Finite Element Approximation for Optimal Shape, Material and Topology Design*, 2nd ed., John Wiley & Sons, (1996).

[5] *N. Karmitsa, A. Bagirov and M. M. Mäkelä,* "Comparing different nonsmooth minimization methods and software *Optim Method Softw*, No. 27(1), 131–153 (20012).

[6] *K. C. Kiwiel,* "Proximity control in bundle methods for convex nondifferentiable minimization *Math Program*, No. 46, 105–122 (1990).

[7] *L. Lukšan and J. Vlček,* "A bundle-Newton method for nonsmooth unconstrained minimization *Math Program*, No. 83, 373–391 (1998).

[8] *M. M. Mäkelä,* "Survey of bundle methods for nonsmooth optimization *Optim Method Softw*, No. 17(1), 1–29 (2002).

[9] *M. M. Mäkelä and P. Neittaanmäki, Nonsmooth Optimization: Analysis and Algorithms with Applications to Optimal Control*, World Scientific Publishing Co., (1992).

[10] *N. Z. Shor, Minimization Methods for Non-Differentiable Functions*, Springer-Verlag, (1985).

Nonlinear Chebyshev approximations and nonsmooth optimization

Malozemov V.N.
Санкт-Петербургский Государственный Университет,
Университетская наб., д. 7-9, Санкт-Петербург, 198504, Россия

In the lecture, a theory of Nonlinear Chebyshev approximations in its historical evolution is presented, starting from the famous seminal Chebyshev paper where necessary optimality conditions for functions least deviating from zero were first stated.

Нелинейные чебышёвские приближения и негладкая оптимизация

В докладе будет представлена теория нелинейных чебышёвских приближений в её историческом развитии, начиная со знаменитой работы П. Л. Чебышёва [1], в которой впервые были указаны необходимые условия оптимальности в задаче о функциях, наименее уклоняющихся от нуля. Говоря современным языком, необходимым условием оптимальности является наличие *альтернанса*, полного или неполного. Поиск альтернанса составляет одну из увлекательнейших особенностей задач равномерного приближения.

В нелинейном случае имеются две принципиальные конструкции, обладающие полным альтернансом: дроби Е. И. Золотарёва и функции В. М. Тихомирова (совершенные сплайны, наименее уклоняющиеся от нуля). В докладе будет рассказано об этих функциях, их обобщениях и приложениях.

Задачи чебышёвского приближения относятся к классу задач негладкой оптимизации, в которых условия оптимальности записываются в форме Куна-Таккера. В работе [2] удалось так расширить понятие альтернанса и так уточнить условия Куна-Таккера, что для задач чебышёвского приближения альтернансные условия и условия Куна-Таккера стали эквивалентными. Это позволило развить общий метод получения альтернансных условий оптимальности (как при отсутствии ограничений на параметры, так и при их наличии).

В докладе мы коснемся также численных методов нелинейных чебышёвских приближений.

ЛИТЕРАТУРА

[1] Чебышев П. Л. “Вопросы о наименьших величинах, связанных с приближённым представлением функций”. Полное собрание сочинений. Том 2. М.-Л.: Изд-во АН СССР, 1947. С. 151–235.

[2] Даугавет В. А., Малозёмов В. Н. "Нелинейные задачи аппроксимации". В сб.: Современное состояние теории исследования операций. М.: Наука, 1979. С. 336–363.

On Existence Theorems for Monotone and Nonmonotone Variational Inequalities

Maugeri A., Raciti F.
maugeri@dmi.unict.it, fraciti@dmi.unict.it
Dipartimento di Matematica e Informatica, Università di Catania
Viale A. Doria 6–I, 95125, Catania, Italy

We make a comparison among various existence theorems for variational inequalities in reflexive Banach spaces. We first study the case where the operator has only some type of continuity (Brezis-pseudomonotonicity, Fan-hemicontinuity, hemicontinuity along line segments) and then consider the case of monotone operators and of Karamardian-pseudomonotone operators. The role of coercivity is analyzed in both cases and some new results are presented in the case of linear operators in Hilbert spaces. We also present a new theorem in the functional setting (L^∞, L^1) together with its application to the time-dependent Walras' problem.

References

[1] *A. Causa, F. Raciti, Some Remarks on the Walras Equilibrium Problem in Lebesgue Spaces*, Optimization Letters, 5, pp.99–112, 2011.

[2] *E. Ernst, M. Théra. A converse to the Lions-Stampacchia Theorem*, ESAIM: Optimisation, control and calculus of variations,15,810–817, 2009.

[3] *A. Maugeri, F. Raciti, On Existence Theorems for Monotone and Nonmonotone Variational Inequalities*, Journal of Convex Analysis, Vol.16, n.3&4, pp. 899-911, 2009.

Optimality Conditions via Exact Penalty Functions

Meng K.W., Yang X.Q.
mkwfly@126.com, mayangxq@polyu.edu.hk
School of Economics and Management, Southwest Jiaotong University, Chengdu 610031, China, Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong

In this paper, we study KKT optimality conditions for constrained nonlinear programming problems and strong and Mordukhovich stationarities for mathematical programs with complementarity constraints using l_p penalty functions with $0 \leqslant p \leqslant 1$. We introduce some optimality indication sets by using contingent derivatives of penalty function terms. Some characteri-

zations of optimality indication sets by virtue of the original problem data are obtained. We show that KKT optimality condition holds at a feasible point if this point is a local minimizer of some l_p penalty function with p belonging to the optimality indication set. Our result on constrained nonlinear programming includes some existing ones in the literature as special cases.

This work is supported by the Research Grants Council of Hong Kong (PolyU 533408/08E).

A first step in designing a VU-algorithm for nonconvex minimzation

Mifflin R., Sagastizabal C.
mifflin@math.wsu.edu
Washington State University, Mathematics 3113, Pullman WA, 99164, USA,
CEPEL, Rio de Janeiro, Brasil

This talk is concerned with laying the ground work for a future VU-type minimization algorithm to run on locally Lipschitz functions for which only one Clarke generalized gradient is known at a point. This entails development of a bundle method sub-algorithm that has provable convergence to stationary points for semismooth functions and can make adequate estimates of bases for "V-subspaces" in the presence of nonconvexity. Ordinary bundle methods generate consecutive "null steps" from a fixed "bundle center" until a "serious step" point is found, which then becomes the next center. A VU-algorithm is similar except that its serious descent point is "very serious" (called serious in [1]) which means it defines a good "V-step" and also generates a good "U step" to add to its very serious point in order to define the next center.

For an objective function of one variable the desired VU-algoritm exists, but it does not extend directly to functions of n variables. With regard to convex functions about 20 years worth of proximal point and VU theory had to be developed before a rapidly convergent method for the multivariable case could be defined. For the nonconvex case there are some ideas associated with the single variable algorithm which can be adapted for developing an n variable method. One is to use second derivative estimates from differencing generalized gradients to give better "V-model" approximation functions when negative curvature is detected. Another is to employ a certain form of a safeguard to guarantee desired convergence even when the negative curvature estimates are not good enough for proving stationarity in the limit.

The new bundle algorithm described here aims at keeping as many as possible properties of an ordinary bundle method for convex minimization. This leads to basing convergence proofs on first showing that $\{\delta_\ell\}$, a sequence

of V-model proximal point subproblem objective values, converges to zero. Associated with this is the desire to preserve the concepts of null and serious steps in the sense that, under reasonable assumptions, there is convergence of a bundle algorithm generated sequence to a stationary point for the problem objective function if there are either
(1) an infinite number of serious step iterations
or
(2) a finite number of serious steps followed by an infinite number of consecutive null step iterations.

Here the definition of an affine V-model subfunction can be different from that in the convex case. However, if the objective is convex and a certain safeguard parameter is set to zero then a V-model subfunction is exactly the same as in the convex case. In general, a difficulty that arises, due to not having the subgradient inequality from convex analysis, is that in order to find a null or serious step point, a sophisticated line search needs to be performed. This search is along a direction determined by the difference between a serious point and its corresponding center. Such a search must, either generate a sequence of stepsizes going to infinity with corresponding objective values going to minus infinity, or find either a null or serious point with a finite number of objective evaluations. The latter requirement is where a semismoothness assumption comes into play. Such an assumption specifies a certain kind of limiting consistency between directional derivatives and generalized gradients.

For the null step point definition we have chosen a condition that is parameterized in such a way as to allow the weakest condition we know of, whose satisfaction implies that $\{\delta_\ell\}$ converges to zero sublinearly under assumption (2) above. For the serious step point definition there are two conditions to satisfy. One is an Armijo-type objective descent condition, familiar from smooth or convex minimization, but modified to fit in with the form of the null point condition. The other condition is one to make serious stepsizes sufficiently large in order to have desirable convergence results under assumption (1). Its form is chosen to fit in with the forms of the other conditions in such a way as to be able to show the existence of the type of line search specified above. Because of all of the above requirements, the form of this last condition does not look like a familiar Wolfe-type condition that lower bounds a directional derivative estimate. It is of interest to note that the null point condition does imply a strong lower bound on such an estimate. Also, nonsatisfaction of a Wolfe condition is useful for calling for extrapolation when the first line search stepsize tried satisfies an Armijo condition and, hence, there will be a sufficiently long serious step.

The null point definition and the serious point Armijo-type condition each have two parameters, one of which is common to both conditions. If the common parameter is relatively large, then it is relatively easy (difficult) to obtain a null (serious) step. Future research is needed to see if a large enough valid parameter value exists to generate a sufficiently large number of null steps before ending an iteration with a very serious descent step. If so, the V-model would be accurate enough to generate a good U-step to add to the very serious point and define the next bundle center for a VU-algorithm.

This research is supported by grants NSF DMS 0707205 and AFOSR FA9550-11-1-0139.

References

[1] R. Mifflin and C. Sagastizabal, "A VU-algorithm for convex minimization", *Mathematical Programming*, 104, No. 2-3, 583-–608 (2005).

On Differential Properties of Value Functions

Minchenko L.
Belarussian State University of Informatics and Radioelectronics, Minsk, 220027, Belarus

Directional derivatives of value functions play an essential role in the sensitivity analysis of optimization problems, in studying min-max problems, in quasidifferentiable calculus.

Consider the parametric nonlinear programming problem P(x):

$$f(x,y) \to \min_{y},$$

$$y \in F(x) = \{y \in R^m \,|\, h_i(x,y) \leqslant 0 \;\; i \in I\,,\;\; h_i(x,y) = 0 \;\; i \in I_0\}$$

which depends on a parameter $x \in R^n$, where $I = \{1, \ldots, s\}$, $I_0 = \{s+1, \ldots, p\}$ or $I_0 = \varnothing$.

Let us introduce the optimal value function $\varphi(x) = \inf\{f(x,y)|y \in F(x)\}$ and the set of optimal solutions

$$\omega(x) = \{y \in F(x) | f(x,y) = \varphi(x)\}\,.$$

Let $\bar{x}, \bar{x}_1, \bar{x}_2 \in R^n$. The goal of the paper is to study the directional derivatives

$$\varphi'(x_0;\bar{x}) = \lim_{t\downarrow 0} t^{-1}(\varphi(x_0+t\bar{x}) - \varphi(x_0)),$$

$$\varphi''(x_0;\bar{x}_1,\bar{x}_2) = \lim_{t\downarrow 0} 2t^{-2}(\varphi(x_0+t\bar{x}_1+t^2\bar{x}_2) - \varphi(x_0) - t\varphi'(x_0;\bar{x}_1)).$$

In spite of seminal results [1-3] in this area, the problem above remains not sufficiently studied for the case of non-singletone set of optimal solutions

$\omega(x)$. Our approach is based on the development of the method of the first order approximations by V.F.Demyanov and A.M.Rubinov [1,2]. Throughout this paper we denote by $F$and ω the multivalued mapping $x \mapsto F(x)$ and $x \mapsto \omega(x)$ and assume that the functions $f, h_i i = 1, ..., p$ are twice differentiable, $F(x_0) \neq \varnothing$ for some point x_0, the multivalued mapping F is uniformly bounded near x_0 and satisfies the Mangasarian-Fromovitz regularity condition at all points $z_0 = (x_0, y_0)$ such that $y_0 \in \omega(x_0)$.

The solution mapping ω is said to be upper pseudolipschitzian (or calm) at a point $z_0 = (x_0, y_0) \in gr\omega$ if there exist neighborhoods $V(x_0)$ and $V(y_0)$ of the points x_0, y_0 and a number $l > 0$ such that $\omega(x) \cap V(y_0) \subset \omega(x_0) + l\,|x - x_0|\,B$ for all $x \in V(x_0)$.

The assertion below gives one of sufficient conditions of upper pseudolipschitzian continuity of the solution mapping ω. Note that unlike [1-4] this assertion don‘t use strong second order sufficient optimality condition based on second order derivatives of Lagrange function.

Proposition 1. Let in the problem $P(x)$ the functions $f(x_0, \cdot)$ and $h_i(x_0, \cdot)$ $i \in I$ be concave and the functions $h_i(x_0, \cdot)$ $i \in I_0$ be affine. Then the solution mapping ω is upper pseudolipschitzian at all points $z_0 = (x_0, y_0) \in gr\omega$.

Let $z_0 = (x_0, y_0) \in grF$, $\bar{z} = (\bar{x}, \bar{y})$, $\bar{z}_1 = (\bar{x}_1, \bar{y}_1), \bar{z}_2 = (\bar{x}_2, \bar{y}_2)$, $I(z_0) = \{i \in I \mid h_i(z_0) = 0\}$, $\quad I^2(z_0, \bar{z}_1) = \{i \in I(z_0) \mid \langle \nabla h_i(z_0), \bar{z}_1 \rangle = 0\}$,

$$\Phi(z_0, \bar{z}_1, \bar{z}_2) = \langle \nabla f(z_0), \bar{z}_2 \rangle + \frac{1}{2} \langle \bar{z}_1, \nabla^2 f(z_0), \bar{z}_1 \rangle,$$

$$\Gamma(z_0; \bar{x}) = \{\bar{y} \in R^m |\, \langle \nabla h_i(z_0), \bar{z} \rangle \leqslant 0 \;\; i \in I(z_0), \langle \nabla h_i(z_0), \bar{z} \rangle = 0, \;\; i \in I_0\},$$

$$\Gamma^2(z_0, \bar{z}_1; \bar{x}_2) = \{\bar{y}_2 \in R^m |\, \langle \nabla h_i(z_0), \bar{z}_2 \rangle + \tfrac{1}{2} \langle \bar{z}_1, \nabla^2 h_i(z_0) \bar{z}_1 \rangle \leqslant 0 i \in I^2(z_0, \bar{z}_1), \langle \nabla h_i(z_0), \bar{z}_2 \rangle + \tfrac{1}{2} \langle \bar{z}_1, \nabla^2 h_i(z_0) \bar{z}_1 \rangle | = 0\, i \in I_0\},$$

$$\Gamma^*(z_0; \bar{x}) = \{\bar{y}_1 \in \Gamma(z_0; \bar{x}) |\, \langle \nabla f(z_0), (\bar{x}, \bar{y}_1) \rangle = \min_{\bar{y} \in \Gamma(z_0; \bar{x})} \langle \nabla f(z_0), (\bar{x}, \bar{y}) \rangle\}.$$

Theorem 1. Let $t_k \downarrow 0$, $x_k = x_0 + t_k \bar{x}_1 + t_k^2 \bar{x}_2 + o(t_k^2)$, $y_k \in \omega(x_k)$ and $y_k \to y_0 \in \omega(x_0)$ as $k \to \infty$. If the solution mapping ω is upper pseudolipschitzian at $z_0 = (x_0, y_0)$, then there exist a sequence $y_{0k} \in \omega(x_0)$ and bounded sequences $\bar{y}_{1k} \in \Gamma^*(z_{0k}; \bar{x}_1)$ and $\bar{y}_{2k} \in \Gamma^2(z_{0k}, \bar{z}_{1k}; \bar{x}_2)$ such that $y_{0k} \to y_0$ and beginning with some $k = k_0$ the following expansions hold

$$y_k = y_{0k} + t_k \bar{y}_{1k} + t_k^2 \bar{y}_{2k} + o(t_k^2),$$

$$\varphi(x_k) - \varphi(x_0) = t_k \langle \nabla f(z_{0k}), \bar{z}_{1k} \rangle + t_k^2 \Phi(z_{0k}, \bar{z}_{1k}, \bar{z}_{2k}) + o(t_k^2),$$

where $z_{0k} = (x_0, y_{0k})$, $\bar{z}_{1k} = (\bar{x}_1, \bar{y}_{1k})$, $\bar{z}_{2k} = (\bar{x}_2, \bar{y}_{2k})$.

Denote

$$\omega(x_0, \bar{x}_1) = \{(y, \bar{y}_1) | y \in \omega(x_0), \quad \bar{y}_1 \in \Gamma((x_0, y); \bar{x}_1),$$

$$\varphi'(x_0; \bar{x}_1) = \langle \nabla f(x_0, y), (\bar{x}_1, \bar{y}_1) \rangle\}.$$

Corollary 1. Let the set $\omega(x_0, \bar{x}_1)$ be non-empty and the solution mapping ω be upper pseudolipschitzian at all points $z_0 = (x_0, y_0)$ such that $y_0 \in \omega(x_0)$. Then for all directions $\bar{x}, \bar{x}_1, \bar{x}_2 \in R^n$ there exist the directional derivatives $\varphi'(x_0; \bar{x})$ and $\varphi''(x_0; \bar{x}_1, \bar{x}_2)$ and

$$\varphi'(x_0; \bar{x}) = \inf_{y_0 \in \omega(x_0)} \min_{\bar{y} \in \Gamma(z_0; \bar{x})} \langle \nabla f(z_0), \bar{z} \rangle,$$

$$\varphi''(x_0; \bar{x}_1, \bar{x}_2) = \inf_{(y_0, \bar{y}_1) \in \omega(x_0, \bar{x}_1)} \min_{\bar{y}_2 \in \Gamma^2(z_0, \bar{z}_1; \bar{x}_2)} 2\Phi(z_0, \bar{z}_1, \bar{z}_2).$$

References

[1] V.F.Demyanov, *Minimax: differentiability in directions.* Leningrad University Publ., 1974.

[2] V.F.Demyanov and A.M.Rubinov, *Foundations of nonsmooth analysis and quasidifferential calculus.* Nauka, Moscow, 1990.

[3] J.F.Bonnans and A.Shapiro, *Perturbations analysis of optimization problems.* Springer-Verlag, New York, 2000.

[4] B.Luderer, L.Minchenko and T.Satsura, *Multivalued Analysis and Nonlinear Programming Problems with Perturbations.* Kluwer Acad Publ., Dordreht, 2002.

Second-Order Variational Analysis And Stability In Optimization

Mordukhovich B.S., Rockafellar R.T.

boris@math.wayne.edu, rtr@uw.edu

Wayne State University, Department of Mathematics, Detroit, USA, Department of Mathematics, University of Washington, Seattle

This talks concerns the second-order generalized differentiation theory of variational analysis and new applications of this theory to some problems of constrained optimization in finite-dimensional spaces. The main attention is paid to the full and partial second-order subdifferentials (or generalized Hessians) of extended-real-valued functions, which are dual-type constructions generated by coderivatives of first-order subdifferential mappings. We develop an extended second-order subdifferential calculus and analyze the basic second-order qualification condition ensuring the fulfillment of the principal second-order chain rule for strongly and fully amenable compositions.

We also calculate the second-order subdifferentials for some major classes of piecewise linear-quadratic functions. These results are applied to the study of tilt and full stability of local minimizers for important classes of problems in constrained optimization that include, in particular, problems of nonlinear programming and certain classes of extended nonlinear programs described in composite terms.

Implicit Function Theorem in the Nonsmooth Analysis

Murzabekova G.Y.
guldenmur07@gmail.com
Kazakh Agrotechnical University, 62, Pobeda Ave, Astana 010011, Kazakhstan

The paper is related to Implicit Functions in Nonsmooth Analysis. Recently Implicit Functions were treated by means of upper and lower exhausters as new tools of Nonsmooth analysis.

Let $f_i(x,y)$ $(i \in 1:n)$ be continuous on $S = S_1 \times S_2 \subset \mathbb{R}^m \times \mathbb{R}^n$, where $S_1 \subset \mathbb{R}^m$ and $S_2 \subset \mathbb{R}^n$ are open sets. Put $f = (f_1, \ldots, f_n)$. Consider the system

$$f_i(x,y) = 0 \quad \forall\, i \in 1:n.$$

For the smooth case, if f_i are continuously differentiable functions in the neighborhood of $z_0 = [x_0, y_0]$, and determinant of the matrix from $f'_{iy}(x_0,y_0)$ is not equal to zero, then a unique vector function $y(x)$ exists in the neighborhood of x_0 such that $f(x,y(x)) = 0_n$ and this function is continuous and differentiable in the neighborhood of x_0. For the nonsmooth case, different generalizations of gradient are used. Demyanov (1999) introduced the notion of exhauster (see [1]), which is helpful in solving various problems in nonsmooth analysis.

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous and positively homogeneous function of the first degree, i.e. $h(\lambda g) = \lambda h(g) \ \ \forall\, \lambda > 0$. Then (see [2]) the function h can be represented as

$$h(g) = \inf_{C \in E^*} \max_{v \in C} (v, g) \quad \forall\, g \in \mathbb{R}^n$$

or

$$h(g) = \sup_{C \in E_*} \min_{w \in C} (w, g) \quad \forall\, g \in \mathbb{R}^n,$$

where E^* and E_* are families of convex compact sets, $E^* = E^*(h)$ being an upper exhauster h and $E_* = E_*(h)$ — a lower one.

In the nonsmooth case it makes sense to introduce a directional implicit function. Fix a direction $g \in \mathbb{R}^m$, $g \neq 0$ and consider the system

$$f_i(x_0 + g, y) = 0 \quad \forall\, i \in 1:n.$$

We say that an implicit function in the direction g exists if $\alpha_0 > 0$ and a vector function $y(\alpha)$ given on $[0, \alpha_0]$ exists such that

$$y(\alpha) \underset{\alpha \downarrow 0}{\rightarrow} y_0, \quad f(x_0 + \alpha g, y(\alpha)) = 0_n \quad \forall\, \alpha \in [0, \alpha_0].$$

Let functions $f_i(z) \,\forall\, i \in 1 : n$ be differentiable at $z_0 = [x_0, y_0]$ in all directions and the directional derivatives $h_i(q) = f_i'(z_0, \eta)$ be continuous as functions of q, where $\eta = [g, q]$, $q \in \mathbb{R}^n$. Then the expansion

$$f_i(z_0, \eta) = f_i(z_0) + \alpha h_i(\eta) + o_{\eta i}(\alpha)$$

holds, where E_i^* – upper exhauster h_i,

$$h_i(\eta) = \inf_{C_i \in E_i^*} \max_{v \in C_i} (v, \eta), \quad \frac{o_{\eta i}(\alpha)}{\alpha} \to 0 \quad \forall\, \eta \in \mathbb{R}^{m+n}.$$

In order to solve the problem of existence and to study properties of an implicit function in the direction g one should find (see [3]) all solutions of the following system

$$\min_{C_i \in E_i^*} \max_{v \in C_i} [v_1 g + v_2 q] = 0 \quad \forall i \in 1 : n. \tag{1}$$

Assume $f = (f_1, \ldots, f_n)$, $h = (h_1, \ldots, h_n)$. Let $q_0 \in \mathbb{R}^n$ be the solution of (1).

Introduce the set of matrices

$$\mathcal{L}(q_0) = \left\{ a = \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix} \middle| \; a_i \in C_i, \quad C_i \in E_i^*, \quad \forall i \in 1 : n \right\},$$

where E_i^* is an upper exhauster of the function h_i. Here T denotes the transposition.

Theorem. If

$$|\det A| \geqslant c > 0 \quad \forall A \in \mathcal{L}(q_0),$$

then, for any $\varepsilon > 0$ there exist $\alpha_0 > 0$ and $q(\alpha) \in \mathbb{R}^n$ such that

$$\|q(\alpha) - q_0\| \leqslant \varepsilon,\; f(x_0 + \alpha g, y_0 + \alpha q(\alpha)) = 0_n \quad \forall \alpha \in [0, \alpha_0].$$

Note. It is possible to prove that such a solution exists also for the function $h(\eta) = \max\limits_{i \in 1:2} \min\limits_{w \in C_i} [v_1 g + v_2 q]$, in this case one should use a lower exhauster E_* of the function $h(\eta)$.

The work is supported by the Russian Foundation for Basic Research (RFFI) under Grant No 12-01-00752.

References

[1] V.F. Demyanov, "Exhausters and Convexificators - New Tools in Nonsmooth Analysis", *Quasidifferentiability and related topics*, Dordrecht: Kluwer Academic Publishers, (2000).

[2] M. Castellani, "A dual characterization of proper positively homogeneous functions", *Technical Report*, 3.235.1111, Department of Mathematics, University of Pisa, (1998).

[3] G. Murzabekova, "Exhausters and implicit functions in nonsmooth systems", *Optimization*, 59, No. 1, 105-–113 (2010)

Minimax solution of a singular perturbed control problem under incomplete information

Myshkov S.K.

skmyshkov@mail.ru

St. Petersburg State University, Universitetskii prospekt 35, Petergof, 198504, St. Petersburg, Russia

A singularly perturbed control problem under incomplete information with a minimax criterion function is stated. Problems of existence and construction of a minimax solution are discussed.

Минимаксное решение сингулярно-возмущенной задачи управления при неполной информации

Рассматривается следующая задача управления

$$\dot{x}_1 = P_{11}(\mu)x_1 + P_{12}(\mu)x_2 + Q_1(\mu)u, \quad x_1(0) = x_{10}, \tag{1}$$

$$\mu\dot{x}_2 = P_{21}(\mu)x_1 + P_{22}(\mu)x_2 + Q_2(\mu)u, \quad x_2(0) = x_{20}, \tag{2}$$

$$z = H_1(\mu)x_1, \tag{3}$$

$$u = Mz, \tag{4}$$

$$J(u) = \int_0^\infty (x^* A(\mu)x + x^* B(\mu)u + u^* B^*(\mu)x + u^* C(\mu)u)dt. \tag{5}$$

Здесь $x_1 \in R^{n_1}, x_2 \in R^{n_2}$, $n_1 + n_2 = n$, $x = col(x_1, x_2)$ — вектор координат состояния; $u \in R^r$ –управление; $z \in R^m$ – измерение, $m \leqslant n_1$; μ-малый параметр, $0 < \mu \ll 1$; t–время, $t \geqslant 0$. В (1)-(5) все матрицы от t не зависят, но как функции от μ допускают разложение в ряд по степеням μ; например,

$$P_{ij}(\mu) = \hat{P}_{ij} + \sum_{k=1}^{\infty} \frac{\mu^k}{k!} P_{ij}^k, \quad P_{ij}(0) = \lim_{\mu \to 0} P_{ij}(\mu) = \hat{P}_{ij}.$$

Аналогичные соотношения имеют место и для остальных матриц в (1)-(5). Предполагается, что все эти ряды – абсолютно сходящиеся в области $|\mu| < \bar{\mu}$, где $\bar{\mu}$ – положительная константа. Предполагается также, что в области $|\mu| < \bar{\mu}$ выполняются следующие условия: подинтегральная квадратичная форма от x, u – положительно полуопределенная, причем $C(\mu) > 0$; матрица $P_{22}(\mu)$-гурвицева; матрица $H_1(\mu)$ — максимального ранга: $rank(H_1(\mu)){=}m$. Из (3),(4) следует, что управление осуществляется исключительно по "медленным модам причем при $m{<}n_1$ информация о доминирующих координатах неполная.

Через $\mathcal{M}{\subset}R^{r\times m}$ обозначим множество матриц $M{\in}R^{r\times m}$ таких, что система (1)-(2), замкнутая управлением (4) с $M{\in}\mathcal{M}$, асимптотически устойчивая. Пусть для всех $\mu \in (0, \bar{\mu})$ множество $\mathcal{M} \neq \varnothing$, т.е. система (1)-(4) стабилизируема. Управление (4) с матрицей $M{\in}\mathcal{M}$ называется допустимым управлением. При любом допустимом управлении функционал (5) принимает следующее значение

$$J(u) = x_0^* \Theta(M) x_0 \overset{def}{=} J(M), \tag{6}$$

где $\Theta(M)$-решение уравнения Ляпунова для задачи (1)-(5) (см. ниже уравнение (8)). По причинам, изложенным в [1, 2], вводится управление, оптимальное в следующем смысле:

$$\lambda_{max}(\Theta(M)) {\Longrightarrow} \min_{M \in \mathcal{M}} . \tag{7}$$

Поскольку $\lambda_{max}(\Theta) = \max\{\lambda_1(\Theta), \ldots, \lambda_n(\Theta)\}$, где $\lambda_k(\Theta), k \in [1:n]$ – собственные значения матрицы Θ, то управление $u_0 = M_0\, z$, $M_0{\in}\mathcal{M}$, решающее задачу (7), называется минимаксным управлением. При $\mu \in (0, \bar{\mu})$ необходимые условия оптимальности имеют вид:

$$\Theta(P + QMH) + (P + QMH)^* \Theta + W(M) = 0, \tag{8}$$

$$L(P + QMH)^* + (P + QMH)L + v * v^* = 0, \tag{9}$$

$$\Theta v = \lambda v, \tag{10}$$

$$CMHLH^* + (Q^*\Theta{+}B^*)LH^* = 0. \tag{11}$$

При этом (11)–это уравнение для определения искомого значения матрицы минимаксного управления (4). Если матрица HLH^* – неособая, то M из (11) определяется однозначно:

$$M = -C^{-1}(Q^*\Theta+B^*)LH^*(HLH^*)^{-1}. \tag{12}$$

В соотношениях (8)-(12) P, Q-матрицы размера, соответственно, $(n\times n)$ и $(n\times r)$ для системы (1)-(2) следующего вида

$$P = \begin{pmatrix} P_{11} & P_{12} \\ \frac{1}{\mu}P_{21} & \frac{1}{\mu}P_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 \\ \frac{1}{\mu}Q_2 \end{pmatrix};$$

величины λ, v — это максимальное собственное значение и соответствующий собственный вектор матрицы Θ; $H{=}[H_1\, 0]$; L — положительно полуопределенная n-матрица; матрица $W(M) = A + BMH + (BMH)^* + (MH)^*CMH$. Для исследования уравнений (8)-(11) используется метод асимптотических представлений, который позволяет выделить главные члены искомых величин. Известно [3], что при этом ключевую роль играет решение редуцированной задачи управления, которая в рассматриваемом случае имеет такой вид:

$$\dot{x}_s \;=\; P_s x_s \;+\; Q_s\, u, \tag{13}$$

$$z(t) \;=\; H_s\, x_s(t), \tag{14}$$

$$u \;=\; M_s\, z, \tag{15}$$

$$J(u) = \int_0^\infty [x_s^* A_s x_s{+}x_s^* B_s u{+}u^* B_s^* x_s{+}u^* C_s u]dt. \tag{16}$$

Здесь $x_s{\in}R^{n_1}$, а матрицы P_s, Q_s, H_s, A_s, B_s, C_s явным образом выражаются через матрицы P_{11}, P_{12}, $\ldots, C$ системы (1)–(5) в точке $\mu = 0$. Условия разрешимости редуцированной задачи имеют вид, аналогичный соотношениям (8)–(11), но размерность задачи будет $n_1 < n$, а также отсутствует зависимость от параметра μ [3].

Доказывается, что имеет место следующее утверждение.

Теорема. Пусть при $\mu \in (0, \bar{\mu})$ матрица P_{22}–гурвицева, квадратичная форма функционала (5)–положительно определенная и редуцированная система (13)–(15) стабилизируема. Тогда минимаксная задача (1)-(7) разрешима и асимптотические представления искомых решений имеют вид:

$$M_o{=}M_s{+}O_1(\mu), J(u_o){=}x_{10}\Theta_s x_{10}{+}O_2(\mu),$$

где M_s, Θ_s — решение редуцированной задачи (13)–(16), а $O_1(\mu)$, $O_2(\mu)$ — соответственно, матричный и скалярный степенные ряды, начинающиеся с членов порядка μ^1.

Для практического использования данного подхода весьма существенным обстоятельством может оказаться то, что редуцированная задача имеет порядок меньше, чем исходная задача (1) – (5), и не зависит от параметра μ. Отметим также, что разработаны эффективные численные методы решения задачи оптимизации (13)-(16) [1,2].

Работа выполнена при поддержке РФФИ, проект 12-01-00752.

ЛИТЕРАТУРА

[1] Мышков С.К., “Линейные управляемые системы с неполной информацией о координатах состояния”, *Негладкие задачи теории оптимизации и управления. Под ред. В.Ф.Демьянова*, Л.: Изд-во Ленингр. ун-та, 248-272 (1982).

[2] Мышков С.К., Полякова Л.Н., Тарасова В.В., “О применимости численных методов негладкого анализа к решению линейной квадратичной задачи оптимального управления с неполной информацией”, *Вестник С.-Петерб. ун-та*, 10, No. 4, 130-137 (2005).

[3] Мышков С.К., “Минимаксное управление сингулярно возмущенной системой с неполной информацией”, *Вестник С.-Петерб. ун-та*, 10, No. 4, 284-291 (2009).

Strong graphical LLN for random outer semicontinuous mappings and its applications in stochastic variational analysis

Norkin V., Wets R.

norkin@i.com.ua

Institute of Cybernetics, Glushkov avenue 40, Kiev, 03187, Ukraine, University of California, Davis, 95616, CA

In the report we give sufficient conditions providing strong graphical law of large numbers (LLN) for random outer semi-continuous mappings, where convergence of set valued mappings is understood as convergence of their graphs and generally it is not uniform. These results extend LLN known for random sets [1], [2] to the case of random set valued mappings.

In case of integrably bounded random mappings we show that graphical convergence is equivalent to uniform convergence to some fattened mappings. In case of unbounded mappings we give a number of sufficient conditions for the fulfillment of the LLN. In particular, they cover the case of a sum of bounded and cone random outer semi-continuous mappings.

The study is motivated by applications of the set convergence and the graphical LLN in stochastic variational analysis [3], including approximation and solution of stochastic generalized equations, stochastic variational inequalities and stochastic optimization problems. The nature of these applications consists in sample average approximation of the problem mappings, application of the graphical LLN and obtaining from here a graphical approximation of the set of solutions.

References

[1] Z. Artstein and S. Hart, Law of large numbers for random sets and allocation processes, *Mathematics of Operations Research*, 6, No. 4, 485–492 (1981).

[2] H.Attouch and R.J-B Wets, Epigraphical processes: Law of large numbers for random lsc functions, *Seminaire d'Analyse Convexe*, Montpellier (1990), pp. 13.1–13.29.

[3] R.T. Rockafellar and R.J-B Wets, *Variational Analysis*, Grundlehren der mathematischen Wissenschaften. Springer (1998) (3rd Printing in 2009).

Quasidifferential Calculus and Minimal Pairs of Compact Convex Sets

Pallaschke D., Urbański R.

diethard.pallaschke@kit.edu

Institute of Operations Research, University of Karlsruhe (KIT), Kaiserstr. 12, D-76128 Karlsruhe, Germany

This is a survey lecture on common work with Jerzy Grzybowski from Poznań.

The quasidifferential calculus developed by V.F. Demyanov and A.M. Rubinov provides a complete analogon to the classical calculus of differentiation for a wide class of non-smooth functions. Although this looks at the first glance as a generalized subgradient calculus for pairs of subdifferentials it turns out that, after a more detailed analysis, the quasidifferential calculus is a kind of Fréchet-differentiations whose gradients are elements of a suitable Minkowski–Rådström–Hörmander space. Since the elements of the Minkowski–Rådström–Hörmander space are not uniquely determined, we mainly focused our attention to smallest possible representations of quasidifferentials, i.e. to minimal representations.

Therefore let $X = (X, \tau)$ be a topological vector space and $\mathcal{B}(X)$ (resp. $\mathcal{K}(X)$) the family of all nonempty bounded closed (resp. compact) convex subsets of X. For nonempty $A, B \subset X: \quad A + B$ denotes the algebraic Minkowski and $A \dot{+} B$ the closure of $A + B$. For $A, B \in \mathcal{K}(X): \quad A \dot{+} B = A + B$. Since $\mathcal{B}(X)$ satisfies the order cancellation law, i.e, for $A, B, C \in \mathcal{B}(X)$ the inclusion $A \dot{+} B \subset B \dot{+} C$ implies $A \subset C$, the set $\mathcal{B}(X)$ endowed with the sum $\dot{+}$ and $\mathcal{K}(X)$ with the Minkowski sum are commutative semigroups with cancellation property. For $A, B, C \in \mathcal{B}(X)$ we put $A \vee B = \text{cl conv}(A \cup B)$ "cl conv" denotes the closed convex hull.

A equivalence relation on $\mathcal{B}^2(X) = \mathcal{B}(X) \times \mathcal{B}(X)$ is given by $(A, B) \sim (C, D)$ iff $A \dot{+} D = B \dot{+} C$ and a partial ordering by the relation: $(A, B) \leqslant (C, D)$ iff $A \subset C$ and $B \subset D$. With $[A, B]$ the eqivalence class of (A, B) is denoted.

A pair $(A, B) \in \mathcal{B}^2(X)$ is called *minimal* if there exists no pair $(C, D) \in [A, B]$ with $(C, D) < (A, B)$. For any $(A, B) \in \mathcal{K}^2(X)$ exists a minimal pair $(A_0, B_0) \in [A, B]$, but this is not true for $\mathcal{B}^2(X)$. There exists a class $[A, B] \in \mathcal{B}^2(c_0)$ which contains no minimal element, where c_0 is the Banach space of all real sequences which converge to zero. For the 2-dimensional case, equivalent minimal pairs of compact convex sets are uniquely determined up to translation. For the 3-dimensional case, this is not true.

Let $A, B, S \in \mathcal{B}(X)$, then we say that S *separates* the sets A and B if for every $a \in A$ and $b \in B$ we have $[a, b] \cap S \neq \varnothing$.

The following statements are equivalent:
i) $A \cup B$ is convex, *ii)* $A \cap B$ separates A and B, *iii)* $A \vee B$ is a summand of $A \dot{+} B$.

The condition that for $A, B, S \in \mathcal{B}(X)$ the inclusion $A + B \subset (A \vee B) \dot{+} S$ implies that S separates the sets A and B is called the *separation law.* It is equivalent to the order cancellation law.

We consider conditional minimality: A pair $(A, B) \in \mathcal{K}^2(X)$ is called *convex* if $A \cup B$ is a convex set and a convex pair $(A, B) \in \mathcal{K}^2(X)$ is called *minimal convex* if for any convex pair $(C, D) \in [A, B]$ the relation $(C, D) \leqslant (A, B)$ implies that $(A, B) = (C, D)$.

It is possible to consider the problem pairs of convex sets in the more general frame of a commutative semigroup S which is ordered by a relation $\leqslant$ and which satifies the condition: if $as \leqslant bs$ for some $s \in S$, then $a \leqslant b$. Then $(a, b) \in S^2 = S \times S$ corresponds to a *fraction* $a/b \in S^2$ and *minimality* to a relative prime representation of $a/b \in S^2$.

Approximations for an NP-hard Min-Max Coverage Problem in Wireless Sensor Networks

Pardalos P.

pardalos@ufl.edu

Department of Industrial and Systems Engineering, University of Florida 303 Weil Hall, Gainesville, FL 32611 USA, Laboratory of Algorithms and Technologies for Networks Analysis (LATNA) National Research University, Higher School of Economics 20 Myasnitskaya st. Moscow 101000 Russia

Consider a set of sensors and a set of targets. Suppose each sensor has unit lifetime. Given a time period, we want to find a sleep/active schedule to maximize the minimum coverage during the time period. The problem has been known to be NP-hard for a long time. In this talk, some new approximation algorithms will be presented.

Cooperation in Dynamic Games with Stochastic Payoffs

Parilina E.

elena.parilina@gmail.com

Saint-Petersburg State University, Faculty of Applied Mathematics and Control Processes, Universitetskii prospekt 35, Petergof, Saint-Petersburg, 198504, Russia

There are many types of dynamic games where players' payoffs are stochastic. It happens because the real dynamic players' cooperation is influenced by many factors, and these influences often have random nature. That is why, the total payoffs in the whole game are more often stochastic. There is a natural question: how should we allocate stochastic total payoff among the players who take place in cooperation? The second important task is to find the scheme of payments to the players at each stage of dynamic game so that their expected payoffs could be not less than the expected value of appropriate components of stochastic allocation.

Stochastic games were introduced by Shapley in 1953 [5]. He considered two-players zero-sum stochastic games with finite state space and finite strategy spaces. Shapley proved that the players have optimal stationary strategies if they maximize the expectation of the discounted payoff. In the work the stochastic games in pure stationary strategies are considered. We are limited by the set of the pure strategies because of the computational difficulties of stationary equilibrium deriving [1, 2].

In the earlier works on cooperative stochastic games the expectation of the player's payoff is considered as a measure of his payoff in the stochastic game. This approach to cooperative stochastic games were studied by Petrosjan and Baranova in [3, 4], but they did not suppose that players' payoffs are random variables, i.e. players allocated the sum of expected payoffs. Suijs et al. [7] introduced a new class of cooperative games arising from cooperative decision making problems in a stochastic environment. They introduced some types of utilities to order the stochastic payoffs. In [7] Suijs et al. determined the core of the cooperative game with stochastic payoffs and found the necessary and sufficient conditions for the non-emptiness of the core. Suijs in [7] introduced the deterministic equivalent of stochastic payoff with some properties. The Shapley-like solutions and nucleolus for the cooperative game with stochastic payoffs were found by Timmer et al. [8] and Suijs [6]. In the work new solutions of the cooperative stochastic games with some utilities on stochastic payoffs of the players are introduced.

References

[1] M. Breton, “Algorithms for stochastic games”, in: T.E.S. Raghavan, T.S. Ferguson, T. Parthasarathy, O.J. Vrieze (Eds.), *Stochastic Games and Related Topics: In Honor of Professor L.S. Shapley, Theory and Decision*, Library C: Game Theory, Mathematical Programming and Operations Research, Vol. 7, Kluwer Academic Publishers, Dordrecht, 45–57 (1991).

[2] J.A. Filar, T.A. Schultz, F. Thuijsman, O.J. Vrieze, “Nonlinear programming and stationary equilibria in stochastic games”, *Math. Prog.*, 50, 227–237 (1991).

[3] L.A. Petrosjan, “Cooperative Stochastic Games”, *Advances in Dynamic Games, Annals of the International Society of Dynamic Games, Application to Economics, Engineering and Environmental Management*, ed. by A. Haurie, S. Muto, L. A. Petrosjan, T.E.S. Raghavan, 139–146 (2006).

[4] L.A. Petrosjan, E.M. Baranova, “Cooperative Stochastic Games in Stationary Strategies”, *Game theory and Applications*, Nova Science Publishers, 11, 7–17 (2006).

[5] L.S. Shapley, “Stochastic games”, *Proc. Nat. Acad. Sci. USA*, 39, 1095–1100 (1953).

[6] J. Suijs, “A Nucleolus for Stochastic Cooperative Games”, Discussion Paper 1996-90, Tilburg University, Center for Economic Research (1996).

[7] J. Suijs, P. Borm, A. De Waegenaere, S. Tijs, “Cooperative Games with Stochastic Payoffs”, *European Journal of Operational Research*, 113, No. 1, 193–205 (1999)

[8] J. Timmer, P. Borm, S. Tijs “On three Shapley-like solutions for cooperative games with random payoffs”, *International Journal of Game Theory*, 32, No. 4, 595–613 (2003).

A Complementarity Partition Result for Multifold Conic Systems

Peña J., Roshchina V.

jfp@andrew.cmu.edu, vera.roshchina@gmail.com

Tepper School of Business Carnegie Mellon University 5000 Forbes Avenue Pittsburgh, PA 15213, USA, CIMA, University of Évora Colégio Luís Verney Rua Romão Ramalho, 59 Évora, 7000-671, Portugal

Consider a homogeneous multifold convex conic system

$$Ax = 0,\ x \in K_1 \times \cdots \times K_r$$

and its alternative system

$$A^{\mathrm{T}} y \in K_1^* \times \cdots \times K_r^*,$$

where $K_1, \ldots, K_r$ are regular closed convex cones. We show that there is a canonical partition of the index set $\{1, \ldots, r\}$ determined by certain complementarity sets associated to the most interior solutions to the two systems. Our results are inspired by and extend the Goldman-Tucker Theorem for linear programming. The talk is based on our recent work available in preprint [1].

References

[1] J. Peña, V. Roshchina, "A Complementarity Partition Theorem for Multifold Conic Systems", *arXiv:1108.0760v1.*

An algorithm for constructing the characteristic function in a assymetric network game

Petrosyan L., Seryakov I.

spbuoasis7@peterlink.ru, ilya.seryakov@gmail.com

Saint-Petersburg State University, 7-9 Universitetskaya nab., Saint-Petersburg, 199034, Russia

A model of network game in which n players wish to rich a fixed node of the network $G = (X, D)$ with minimal costs is considered. It is assumed that the trajectories of the players do not have common edges, i.e. should not intersect. The latter condition considerably complicates the problem, since sets of the strategies are mutually dependent.

Additionally assumed that the cost of transportation along the edge of the network are different for all players. Conditionally cooperative equilibrium and cooperative equilibrium are introduced. On this basis, the corresponding characteristic functions are constructed. To calculate the characteristic function the Bellman equation is used.

Corresponding example is considered.

References

[1] Петросян Л.А., “Одна транспортная теоретико-игровая модель на сети”, *Математическая теория игр и ее приложения*, 4, No. 4, 34-–92 (2011).

[2] Петросян Л.А., Серяков И.А., “Об условно оптимальном перемещении n игроков по непересекающимся путям.”, *Процессы управления и устойчивость*, No. 42, 34-–92 (2011).

On Differentiation of Set-Valued Functions and Differential Inclusions

Polovinkin E.

polovinkin@mail.mipt.ru

Moscow Institute of Physics and Technology (State University),

Institutsky per., 9, Dolgoprudny, 141700, Russia

The report provides a direct method for investigating optimization problems with differential inclusions in Banach spaces. The method consists in the fact that any differential inclusion in a neighborhood of the test trajectory can be approached by a simpler differential inclusion, whose graph of the right side is a closed convex cone, which is measurably time-dependent. In contrast to other approximation research methods for such non-smooth optimization problems, the given direct method allows to obtain the necessary conditions with more precise conjugate (polar) cones.

1. Let E, E_1, E_2 be a separable Banach spaces. We denote by $\mathcal{P}(E)$ ($\mathcal{F}(E)$) the family of all nonempty (closed) subsets of E. There are a lot of known concepts of tangent cones to a given nonconvex set A at a point $a \in A$ of E. Among them it should be noted the lower tangent cone (also called: simple tangent cone) $T_H(A;a)$, the upper tangent cone (also called: contingent cone) $T_B(A;a)$ (see [1]), and the Clarke tangent cone $T_C(A;a)$ (see [2]). Besides, using the concept of the Minkowski difference of the sets of the form $A \overset{*}{-} B \doteq \{x \in E \mid x + B \subset A\}$, and following the work of [3], we define another tangent cones. For example, this is the asymptotically lower tangent cone $T_{AH}(A;a) \doteq T_H(A;a) \overset{*}{-} T_H(A;a)$ and the asymptotically upper tangent cone $T_{AB}(A;a) \doteq \overline{T_{AH}(A;a) + T_B(A;a) \overset{*}{-} T_B(A;a))}$ (see [3]). Obviously, the cones $T_{AH}(A;a)$, $T_{AB}(A;a)$ are convex, closed, and the following inclusions are valid: $T_C(A;a) \subset T_{AH}(A;a) \subset T_{AB}(A;a) \subset T_B(A;a)$.

Definition 1 (see [1, 3]) *The L- derivative (where $L \in \{H, B, C, AH, AB\}$) of set-valued mapping $F\colon E_1 \to \mathcal{P}(E_2)$ at point $z_0 \in \overline{\text{graph}F} \subset Z \doteq E_1 \times E_2$ is the set-valued mapping $D_L F(z_0) : E_1 \to \mathcal{P}(E_2)$ defined by*

$$D_L F(z_0)(u) \doteq \{v \in E_2 \mid (u, v) \in T_L(\text{graph}F; z_0)\}, \quad u \in E_1.$$

2. Denote by T the interval $[t_0, t_1] \subset \mathbb{R}^1$ and by $AC(T, E)$ the Banach space of absolutely continuous functions $f : T \to E$ with the norm $\|f(\cdot)\|_{AC} \doteq \|f(t_0)\|_E + \|f'(\cdot)\|_{L_1}$. Let a subset $C_0 \subset E$ and mapping F: $T \times E \to \mathcal{P}(E)$ be given. We consider the Cauchy problem for differential inclusion

$$x'(t) \in F(t, x(t)), \quad x(t_0) \in C_0, \quad t \in T. \tag{1}$$

The set of all Caratheodory solutions $x(\cdot) \in AC(T, E)$ of problem (1) we denote by $\mathcal{R}_T(F, C_0)$.

Let the solution $\hat{x}(\cdot) \in \mathcal{R}_T(F, C_0)$ be fixed. Following Definition 1, you can differentiate the right side of the differential inclusion (1) about the decision, but you can differentiate mapping $x \to \mathcal{R}_T(F, x)$ and compare them. To do this we denote for every $t \in T$ the $L-$ derivatives of mapping $x \to F(t, x)$ at the point $(\hat{x}(t), \hat{x}'(t))$ as follows

$$F'_L(t, u) \doteq D_L F(t, \hat{x}(t), \hat{x}'(t))(u).$$

Then the set of all Caratheodory solutions $u(\cdot) \in AC(T, E)$ of problem $u'(t) \in F'_L(t, u(t))$ with $u(t_0) = u_0$ we denote by $\mathcal{R}_T(F'_L, u_0)$. Also the lower derivative of set-valued mapping $\mathcal{R}_T(F, \cdot) : \quad E \to \mathcal{P}(AC(T, E))$ at the point $(\hat{x}(t_0), \hat{x}(\cdot))$ we denote as

$$D_H(u) \doteq \liminf_{\lambda \to 0} (\limsup_{x \to u} \lambda^{-1}(\mathcal{R}(F, \hat{x}(t_0) + \lambda x) - \hat{x}(\cdot))), \quad u \in E.$$

Theorem 1.(see [3]) *Let $\hat{x}(\cdot) \in \mathcal{R}_T(F, C_0)$ and the mapping F be measurable in t and pseudo- Lipschitz in x around $(\hat{x}(t), \hat{x}'(t)) \in \operatorname{graph} F(t, \cdot)$ with modulus $l(t) \in L_1(T, \mathbb{R}^1_+)$ (see [4]). Then for any $u_0 \in E$ and any solution $u(\cdot) \in \mathcal{R}_T(F'_H, u_0)$ that has $u'(\cdot) \in L_\infty(T, E)$, the following inclusion hold $u(\cdot) \in D_H(u_0)$.*

3. An existence theorem for solutions of inclusion.

Theorem 2. *Let F: $T \times E \to \mathcal{P}(E)$, $y(\cdot) \in AC(T, E)$, $\rho(\cdot) \in L_1(T, \mathbb{R}^1_+)$, $\delta > 0$ and $x_0 \in E$ be given such that $\|y(t_0) - x_0\| \leqslant \delta$, $d(y'(t),\ F(t, y(t))) \leqslant \rho(t)$ for a.e. $t \in T$. Let $\beta > 0$ exist such that F is measurable- pseudo-Lipschitz around $(\hat{x}(t), \hat{x}'(t))$ for the neighborhood $V_\beta \doteq \{(t, x, z) \in T \times E \times E \mid \|x - y(t)\| < \xi_\beta(t),\ \|z - y'(t)\| < \eta_\beta(t),\ t \in T\}$ with modulus $l(\cdot) \in L_1(T, \mathbb{R}^1_+)$, where the functions m, ξ_β, $\eta_\beta : T \to \mathbb{R}^1_+$ defined as*

$$m(t) \doteq \int_{t_0}^{t} l(\tau)\, d\tau, \quad \xi_\beta(t) \doteq e^{m(t)+\beta}(\delta + \int_{t_0}^{t} e^{-m(\tau)} \rho(\tau)\, d\tau (1 + \beta)),$$

$$\eta_\beta(t) \doteq l(t)\xi_\beta(t) + \rho(t)(1 + \beta).$$

Then there is a solution $x : T \to E$ of problem (1) such that

$$x(t_0) = x_0, \quad \|x(t) - y(t)\| \leqslant \xi_\beta(t), \quad \|x'(t) - y'(t)\| \leqslant \eta_\beta(t), \quad \forall t \in T.$$

4. A theorem on the polar of a convex process.

Theorem 3. Let K_0 be a closed convex cone in a separable Banach space E. Let $F\colon T \times E \to \mathcal{F}(E)$ be such that $F(t,x) \doteq \{y \in E \mid (x,y) \in K(t)\}$, where the set $K(t)$ is a closed convex cone in space $E \times E$ and it's measurably time-dependent (i.e. $F(t,\cdot) : E \to E$ is a closed convex process). Suppose that there is a function $\gamma(\cdot) \in L_1(T, \mathbb{R}^1_+)$ such that $\|F(t,\cdot)\| \leqslant \gamma(t), \quad t \in T$. Let K_0^0 and $K^0(t)$ be polar cones to the cones K_0 and $K(t)$ respectively. Then the polar cone $(\mathcal{R}_T(F,K_0))^0$ to the set of solutions $\mathcal{R}_T(F,K_0)$ of the differential inclusion consists of pairs of points $b^* \in E^*$ and functions $y^*(\cdot) \in L_\infty(T,E^*)$ such that for each pair there is a function $x^*(\cdot) \in L_1(T,E^*)$, for which the following inclusions hold

$$b^* - \int_{t_0}^{t_1} x^*(s)\,ds \in K_0^0; \quad \left(x^*(t), y^*(t) - \int_t^{t_1} x^*(s)\,ds\right) \in K^0(t), \quad \forall t \in T.$$

5. The optimization problem for a differential inclusion.

In the interval $T \doteq [t_0, t_1]$, we consider the following problem (see [5])

$$\text{Minimize}\,\{\varphi(x(t_1)) \mid x(\cdot) \in \mathcal{R}_T(F, C_0)\}. \tag{2}$$

Let the function $\varphi\colon E \to \mathbb{R}^1$ satisfies a local Lipschitz condition, and the set $C_0 \subset E$ is closed.

Suppose we are given a solution $\hat{x}(\cdot) \in \mathcal{R}_T(F, C_0)$ of the differential inclusion. Let the mapping $F\colon T \times E \to \mathcal{P}(E)$ be measurable- pseudo-Lipschitz around $(\hat{x}(t), \hat{x}'(t))$. Assume that the closed convex cone $K(t) \subset E \times E$ measurably depends on $t \in T$ and satisfies the inclusion

$$K(t) \subset T_H(\mathrm{graph} F(t,\cdot); (\hat{x}(t), \hat{x}'(t))) \quad \forall t \in T.$$

An example of such cone $K(t)$ is any cone of $T_L(\mathrm{graph} F(t,\cdot); (\hat{x}(t), \hat{x}'(t)))$ at $L \in \{C, AH1, AH2\}$. The necessary optimality conditions in (2) take the following form:

Theorem 4. Let $\hat{x}(\cdot)$ be a local (in $AC(T,E)$) solution to the problem (2). Then there is a function $p(\cdot) \in AC(T,E^*)$ such that

$$p(t_0) \in T^0_{AH}(C_0, \hat{x}(t_0)), \quad p(t_1) \in -\partial^+_{AB}\varphi(\hat{x}(t_1)),$$

$$(p'(t), p(t)) \in K^0(t), \quad \forall t \in [t_0, t_1].$$

6. Similarly, the direct method can be used to obtain the necessary optimality conditions for optimization problems with differential inclusions in Banach space with the conditions at the endpoints, for time optimal control problem, etc.

The work was supported by the Federal Program "Scientific and Scientific-Educational Staff of Innovative Russia" for years 2009-2013 and by RFBR № 10-01-00139-a.

References

[1] Aubin J.P., "Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions," *Advances in Math. Suppl. Studies,* Acad. Press, 160-272 (1981).

[2] Clarke F.H., *Optimization and nonsmooth analysis.* – Wiley–Interscience, New York, (1983).

[3] Polovinkin E.S., *Theory of set-valued mapping.* -Moscow (in Russian) (1983).

[4] Aubin, J.P. "Lipschitz behavior of solutions to convex minimization problems,"*Math. of Oper. Res.*, 9, 87-111 (1984).

[5] Polovinkin E.S., "Necessary Conditions for Optimization Problems with Differential Inclusion."*Set-valued Analysis and Differential Inclusions.* Progress in Systems and Control Theory. Vol. 16.—Birkhäuser, 157–170 (1993).

One problem of global optimization

Polyakova L.

lnpol07@mail.ru

St. Petersburg State University, Universitetskii prospekt 35, Petergof, 198504, St. Petersburg, Russia

Let us consider the following optimization problem: find

$$\inf_{x\in\mathbb{R}^n} f(x), \quad f(x) = f_1(x) - f_2(x), \quad x \in \mathbb{R}^n,$$

where f_1 and f_2 are polyhedral functions defined on $\mathbb{R}^n$, i.e.

$$f_1(x) = \max_{i\in I} f_{1i}(x),\ f_{1i} = \langle a_i, x\rangle + b_i,\ I = \{1, \ldots, m\},$$

$$f_2(x) = \max_{j\in J} f_{2j}(x),\ f_{2j}(x) = \langle c_j, x\rangle + d_j,\ J = \{1, \ldots, p\},$$

$a_i,\ c_j \in \mathbb{R}^n,\ b_i,\ d_j \in \mathbb{R},\ i \in I,\ j \in J.$

The solution of the given problem can be reduced to solving a finite number of minimax problems (which, in turn, are reduced to linear programming problems). Therefore this problem can be solved in a finite number of iterations. If at some step the function is unbounded from below, then, obviously, and the initial problem is also unbounded from below.

The function f is quasidifferentiable on $\mathbb{R}^n$ and $\mathcal{D}f(x) = [\partial f_1(x), -\partial f_2(x)]$ is its quasidifferential at a point $x \in \mathbb{R}^n$, where $\partial f_i(x)$ are the subdifferentials of convex functions $f_i(x),\ i = 1, 2,$ at the point $x \in \mathbb{R}^n$ in the sense of the definition of Convex Analysis [1]. In our case the sets ∂f_1 and ∂f_2 are convex polyhedra at any $x \in \mathbb{R}^n$.

Theorem 1. [2]. *For a point $x^* \in \mathbb{R}^n$ to be a minimizer of the function f on $\mathbb{R}^n$, it is necessary that*

$$\partial f_2(x^*) \subset \partial f_1(x^*).$$

For the first time necessary and sufficient conditions for a global minimum of the difference of convex functions were received by Hiriart-Urruty [3]. In the proof of these conditions he used ε-subdifferentials.

Theorem 2.[3] *For ȧ point $x^* \in \mathbb{R}^n$ to be a global minimizer of the function f* on $\mathbb{R}^n$, *it is necessary and sufficiently that the inclusion*

$$\partial_\varepsilon f_2(x^*) \subset \partial_\varepsilon f_1(x^*) \quad \forall \varepsilon \geqslant 0,$$

be hold, where $\partial_\varepsilon f_i(x^*)$ are ε- subdifferentials of the convex functions $f_i,\ i = 1, 2$, at the point x^*.

Consider some optimization properties of the function f.

Formulate conditions of the unboundedness of the function f on $\mathbb{R}^n$.

Theorem 3. *For the function f to be unbounded from below in $\mathbb{R}^n$, it is necessary and sufficient that there exist $j^* \in J$ and a vector c_{j^*}, such that the condition*

$$c_{j^*} \notin \operatorname{co}\left\{\bigcup_{i\in I} a_i\right\}$$

holds.

The function f on $\mathbb{R}^n$ is also codifferentiable [4] and $Df(x) = [df_1(x), -df_2(x)]$ is its codifferential at a point $x \in \mathbb{R}^n$, where $df_i(x),\ i = 1, 2$, are the hypodifferentials of convex functions $f_i(x),\ i = 1, 2$, at the point $x \in \mathbb{R}^n$. In this case one can take the following polyhedra

$$df_1(x) = \operatorname{co}\left\{\bigcup_{i\in I}\begin{pmatrix} a_i \\ \langle a_i, x\rangle + b_i - f_1(x)\end{pmatrix}\right\} \subset \mathbb{R}^n \times \mathbb{R},$$

$$df_2(x) = \operatorname{co}\left\{\bigcup_{j\in J}\begin{pmatrix} c_j \\ \langle c_j, x\rangle + d_j - f_2(x)\end{pmatrix}\right\} \subset \mathbb{R}^n \times \mathbb{R},$$

as hypodifferentials of the functions $f_i(x),\ i = 1, 2$, at any $x \in \mathbb{R}^n$. The given hypodifferentiable mappings

$$df_i : \mathbb{R}^n \longrightarrow 2^{\mathbb{R}^{n+1}},\ i = 1, 2,$$

are continuous in the Hausdorff metric. It is obvious that the set $df_i(x) \subset \mathbb{R}^{n+1}$ for each $i = 1, 2$, is also a convex polyhedron contained in the half-space

$$H = \{z = (z_1 ..., z_n, z_{n+1})^T \in \mathbb{R}^n \times \mathbb{R} \mid \ z_{n+1} \leqslant 0\},$$

where T denotes transposition.

Let the function f be bounded from below in $\mathbb{R}^n$. Formulate necessary and sufficient conditions for the global minimum of the function f in $\mathbb{R}^n$.

Theorem 4. *For a point $x^* \in \mathbb{R}^n$ to be a global minimizer of the function f on $\mathbb{R}^n$, it is necessary and sufficient, that the condition*

$$df_1(x^*) \bigcap \text{co} \left\{ \begin{pmatrix} c_j \\ f_{2j}(x^*) - f_2(x^*) \end{pmatrix}, \begin{pmatrix} c_j \\ 0 \end{pmatrix} \right\} \neq \varnothing \quad \forall j \in J, \tag{1}$$

hold.

Corollary 1. *The condition* (1) *is equivalent to the following condition*

$$0_{n+1} \in \left[df_1(x^*) - \text{co} \left\{ \begin{pmatrix} c_j \\ f_{2j}(x^*) - f_2(x^*) \end{pmatrix}, \begin{pmatrix} c_j \\ 0 \end{pmatrix} \right\} \right] \quad \forall j \in J.$$

Corollary 2. *The condition* (1) *is equivalent to the condition*

$$0_{n+1} \in \bigcap_{j \in J} \left[df_1(x^*) - \text{co} \left\{ \begin{pmatrix} c_j \\ f_{2j}(x^*) - f_2(x^*) \end{pmatrix}, \begin{pmatrix} c_j \\ 0 \end{pmatrix} \right\} \right].$$

Corollary 3. (A sufficient condition for a global minimum of the function f on $\mathbb{R}^n$) *If at a point $x^* \in \mathbb{R}^n$ the inclusion*

$$df_2(x^*) \subset df_1(x^*)$$

holds then the point x^* is a global minimizer of the function f on $\mathbb{R}^n$.

The work is supported by the Russian Foundation for Basic Research (RFFI) under Grant No 12-01-00752.

References

[1] R.T. Rockafellar *Convex analysis*, Princeton: Princeton Univ., (1970).

[2] L.N. Polyakova,"Necessary conditions for an extremum of quasidifferential functions", *Vestn. Leningrad university*, No. 13. 57–62. (1980).

[3] J.-B. Hiriart-Urruty, "From convex minimization to nonconvex minimization: Necessary and sufficient conditions for global optimality", *Nonsmooth optimization and related topics.* Eds. F. N. Clarke, V. F. Demyanov, and F. Giannessi. New York: Plenum, 219–240 (1989).

[4] V.F. Demyanov, A.M. Rubinov, *Constructive nonsmooth analysis*, Frankfurt a/M: Peter Lang Verlag. (1995).

To a Problem of Equivalent Substitution of a Control System

Proudnikov I.
pim_10@hotmail.com
St.Petersburg University, St.Petersburg, Petergof, Universitetskii Prospekt, 35, 198504, Russia

A system of differential equations with control is considered. A rule for equivalent substitution of this system is given. The system of differential equations is changed for two systems with upper and lower approximation functions of a function on the right side of the initial system in a region of attainability.

Calculation of a Biexhauster for a Lipschitz Function

Proudnikov I.
pim_10@hotmail.com
St.Petersburg University, St.Petersburg, Petergof, Universitetskii Prospekt, 35, 198504, Russia

It was found an upper(lower) Exhaustive family for any Lipschitz function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ at any point. We use for this purpose a set of curves defined in [1], [2].

A special class of curves and the average values E of gradients of $f(\cdot)$ are considered. Using the set E and the proved Theorem, a BiExhauster is constructed. An algorithm for construction of the BiExhauster is given.

The work is supported by the Russian Foundation for Basic Research (RFFI) under Grant No 12-01-00752.

References

[1] Proudnikov I.M., New constructions for local approximation of Lipschitz functions.I,*Nonlinear analysis*, Vol. 53. N 3. 2003. P. 373-390.

[2] Proudnikov I.M., New constructions for local approximation of Lipschitz functions.II,*Nonlinear analysis*, Vol. 70. N 2. 2007.

Third-order longitudinal correlation moments in the intermittency model

Pyrkova O.
opyr@mail.ru
Moscow Institute of Physics and Technology (State University), Institutsky per., 9, Dolgoprudny, 141700, Russia

In this paper we study the time dependence for third-order longitudinal correlation moments $B_{LL,L}$ in the intermittency model [1, 2], i.e. we use the following model: flow is considered as a mixture of turbulent and viscous regimes. We discuss the behavior of the function $\frac{B_{LL,L}}{u^3}(\xi)$ in terms of Lytkin and Chernykh self-similar solutions [3], where λ is the Taylor dissipation-length parameter.

The behavior of $\frac{B_{LL,L}}{u^3}(\xi)$ is in good agreement with the experimental data of Stewart and Townsend. But one cannot observed the self-similarity in the variables $\frac{B_{LL,L}}{u^3}$ and ξ [4].

Accounting intermittency amends value of the function $B_{LL,L}$ [2]:

$$B_{LL,L} = \gamma \cdot (B_{LL,L})_T + (1-\gamma)(B_{LL,L})_\nu,$$

where γ is coefficient of intermittency (fraction of the turbulent regime). Index $()_T$ refers to the purely turbulent Kolmogorov regime, index $()_\nu$ refers to the purely viscous regime.

We have $B_{LL,L} = \gamma \cdot (B_{LL,L})_T$ in the hypothesis that the third-order longitudinal correlation moments are negligible in a purely viscous regime. Influence of intermittency affects the behavior of the curves at small values of ξ.

Fundamental Karman-Howarth equation relates the magnitude of the two-point second-order and third-order longitudinal correlation moments:

$$\frac{\partial B_{LL}}{\partial t} = \frac{1}{r^4} \cdot \frac{\partial}{\partial r}\left(r^4\left(B_{LL,L} + 2\nu\frac{\partial B_{LL}}{\partial r}\right)\right).$$

In the absence of viscosity $\nu = 0$ it takes the following form in the purely turbulent Kolmogorov regime

$$\frac{\partial B_{LL}}{\partial t} = \frac{1}{r^4} \cdot \frac{\partial}{\partial r}\left(r^4 B_{LL,L}\right). \tag{1}$$

Equation (1) is initially not closed, because it contains two unknown functions. Gradient hypothesis of Lytkin and Chernykh [3] is used to make it closed by the expression of the two-point third-order correlation moment

$B_{LL,L}$ through the two-point second-order correlation moment B_{LL} in the regime of Kolmogorov turbulence in the inertial range:

$$B_{LL,L} = 2K_\tau \frac{\partial B_{LL}}{\partial r}, \tag{2}$$

where K_τ is the turbulent diffusion (viscosity)coefficient. This provides a self-similar solution for purely turbulent motion.

Considering von Karman self-similarity

$$(B_{LL})_\tau = u_\tau^2 f_\tau(\xi_1), \tag{3}$$

equation (2) takes the form

$$-Bf_\tau = \sqrt{1-f_\tau}\frac{\partial f_\tau}{\partial \xi_1}. \tag{4}$$

Here $\xi_1 = \dfrac{r}{\Lambda}$, Λ is an integral correlation scale, $B = \dfrac{13.2}{Re_\lambda}\dfrac{\Lambda}{\lambda}$ [2], $Re_\lambda = \dfrac{u\lambda}{\nu}$.

One can find a solution of (2)

$$B\,\xi_1 = -2\sqrt{1-f_\tau} + \ln\left(1+\sqrt{1-f_\tau}\right) - \ln\left(1-\sqrt{1-f_\tau}\right), \tag{5}$$

by integrating (4) with the boundary conditions $f_\tau|_{\xi_1=1} = 1,\ f_\tau|_{\xi_1=\infty} = 0$ [1, 3].

The value of $\dfrac{\Lambda}{\lambda} = \dfrac{\Lambda}{\lambda}(Re_\lambda)$ is defined by the interpolation function of the generalized von Karman model for the spectrum [5]

$$\frac{\Lambda}{\lambda} = 1.23 + 0.0351\sqrt{Re_\lambda}\left(\sqrt{Re_\lambda}-1\right).$$

Thus we have found that the value of B depends on the Reynolds number, taken at the initial time.

The expression for the turbulent diffusion (viscosity) coefficient takes on the basis of approximation between the expressions for one-point case of the inertial range $\left(\xi_1 = \dfrac{r}{\Lambda} = \infty\right)$ and the limit for $(\xi_1 \to 0)$ on the basis of semi-empirical model developed [1, 2]. in this case $K_\tau = æ\, r\sqrt{D_{LL}}$, where æ is empirical constant, $D_{LL} = 2{u_\tau}^2(1-f_\tau)$ is the longitudinal structure function.

From (2), (3), (4), (5) we find

$$\left(\frac{B_{LL,L}}{u^3}\right)_\tau = \frac{2æ\, r}{u_\tau^3}\sqrt{2u_\tau^2(1-f_\tau)}\frac{\partial(u_\tau^2 f_\tau(\xi_1))}{\partial r} =$$

$$= 2\sqrt{2}æ\xi_1\Lambda\sqrt{1-f_\tau}\tfrac{\partial f_\tau(\xi_1)}{\partial \xi_1}\tfrac{\partial \xi_1}{\partial r} = 2\sqrt{2}æ\xi_1\sqrt{1-f_\tau}\tfrac{\partial f_\tau(\xi_1)}{\partial \xi_1} = -2\sqrt{2}æ\xi_1 B f_T.$$

Considering that $f_T = f_T(\xi_1)$ at the same time, we obtain

$$\frac{B_{LL,L}}{u^3} = \frac{B_{LL,L}}{u^3}(\xi_1).$$

It was found that the curves of the variables $\frac{B_{LL,L}}{u^3}$ and ξ_1 are the same at different times. This fact confirms the hypothesis about the similarity of the generated turbulence from grid.

References

[1] O.A. Pyrkova, A.A. Onufriev and A.T. Onufriev, "Initial time behavior of the velocity in a homogeneous and isotropic turbulent flow", [in Russian] *Proceedings of MIPT*, 3, No. 1, 127–131 (2011).

[2] A.A. Onufriev, A.T. Onufriev and O.A. Pyrkova,"The problem of the decay of homogeneous isotropic turbulence, taking into account the phenomenon of intermittency", [in Russian] *In: Actual problems of fundamental and applied mathematics*, MIPT, Moscow, 124–135 (2009).

[3] Yu.M. Lytkin and G.G. Chernykh,"One method of closing the Kŕarmŕan-Howarth equation", [in Russian] *In: Dynamics of Continuous Media*, Inst. Hydrodynamics, Sib. Div., Acad of Sci. of the USSR, 27, 124–130 (1976).

[4] O.A. Pyrkova,"Behavior of the third-order longitudinal correlation functions in the intermittency model ", *function theory, its applications and related problems – Materials tenth international summer school of the Kazan conference*, [in Russian] 43, 295–296 (2011).

[5] Driscoll R.J., Kennedy L.A.,"A model for the turbulent energy spectrum", *Phys. Fluids*, 26, No. 5, 1228-1233 (1983).

On Some Major Trends in Mathematics

Rassias T.M.

trassias@math.ntua.gr

Department of Mathematics, National Technical University of Athens, Zografou Campus, GR-15780 Athens, Greece

We present here three basic directions of research in Mathematics and its applications based on recent progress and the general trends of science.

Applying Variational Analysis to Stablility Issues in Economics

Rockafellar R.T.
rtr@uw.edu
Department of Mathematics, University of Washington, Seattle

Does the concept of equilibrium as defined in economics reflect the kinds of stability that should go with it, or is equilibrium too delicate or subject to pathology to be satisfactory without further qualifications? Discouraging examples in the literature have pushed opinion toward the latter, but it seems now that questions may not have been asked in the right way. Variational inequality modeling and the perturbation theory for solutions to variational inequalities have led to new results indicating that true stability of equilibrium can be counted on in circumstances even broader than typically examined. Moreover, these results deal effectively with the kinds of inequality constraints that researchers had turned their backs on.

The Kelly Cutting Plane Method in Solving Minimax Problems

Romanovsky J.V.
jsephromanovsky@gmail.com
St-Petersburg University, St-Petersburg, Russia

The cutting plane method, proposed by J. Kelley in 1960 [1] is remaining its popularity and even actively developed now (cf. for example, [2].

I am planning to talk about our applications of this method in minimax problems arising in a problem of optimal maintenance [3], where we are finding the optimal answer to the least favorable distribution selected by the Nature (following the terms of A. Wald [4]).

We also considered some antagonistic games (i. e. zero-sum two person games) with random information [5]. In the game problems the idea of column generation was very efficient. Our own experience with this method was reviewed in [6].

It was a game when the first player selected its move $i \in I$, then the Nature selected a parameter ξ with a known distribution function, and then the second player selects its move $j \in J$. The payoff to the first player is given by a known function $K(i, \xi, j)$. It is known that if the distribution of ξ is continuous then there exist a pure optimal strategy of the second player [7], [8].

References

[1] J. E. Kelley, jr., “The cutting-plane method for solving convex programs”, *J. Soc. Indust. and Appl. Math.*, 8, No. 4, 703-–712 (1960).

[2] O. du Merle, J.-Ph. Vial, *Proximal ACCPM, a cutting plane method for column generation and Lagrangian relaxation: Application to the p-median problem.* Technical report, Logilab, HEC. University of Geneva (2002).

[3] J. V. Romanovsky, M. I. Freiman< V. I. Sharapov, “Application of operations research in solving problems of technical diagnostics”, In *Operations Research and Statictical Modelling*, No. 1, 152-–159, Leningrad University (1972).

[4] A. Wald, *Statistical Decision Functions*, John Wiley & Sons, New York (1950).

[5] M. A. Genin, “Algorithms to solve some concave-convex and linear-convex games”, In *Metody vychisleniy*, No. 6, 96-–106, Leningrad University (1970).

[6] L. V. Kantorovich, J. V. Romanovsky, ”The column generation in the simplex method”, *Economic-Math. Methods and Models*, 21, No. 1, 128-–138 (1985).

[7] A. Dvoritsky, A. Wald, J. L. Wolfowitz, ”Elimination of randomization in certain problems of statistics and of the theory of games”, *Proc. Nat. Acad. Sci. U.S.A.*, 36, 256-–260 (1950).

[8] J. V. Romanovsky, ”Derandomization of optimal strategies in antagonistic games with bluffing”, *Doklady AN SSSR*, 157, 1066-–1068 (1964).

Optimality conditions in nondifferentiable fuzzy optimization

Ruziyeva A., Dempe S.

ruziyeva@student.tu-freiberg.de, dempe@tu-freiberg.de

TU Bergakademie Freiberg, Department of Mathematics and Computer Science, Freiberg, D-09596, Germany

In practical situations, data are often not known exactly, i.e. only (subjective) estimations are pro- vided. One commonly used approach to deal with these problems is to model them as fuzzy optimization problems [1]. This approach proved very useful in many applied sciences, such as economics, physics, etc.

An early approach for solving a fuzzy optimization problem is the extension principle due to Bellman and Zadeh [2]. Nevertheless, in the talk the fuzzy optimization problem is solved by its reformulation into the biobjective optimization problem and application of methods of the multiobjective optimization problem's scalarization technique [3], where modern solution algorithms based on the minimization of the α-cut on the feasible set are used (see e.g. [4, 5, 6, 7, 8]). Elements of the Pareto set of each biobjective optimization problem are interpreted as solutions fuzzy optimization problem on a certain level-cut.

An involvement of adapted to the fuzzy case the tangent cone, directional derivative and the Hadamard derivatives permits to derive with the aforesaid approach necessary and sufficient optimality conditions for the optimal solution of the nondifferentiable fuzzy optimization problem.

References

[1] L.A. Zadeh, "Fuzzy Sets Information and Control, 8, 338-353 (1965)

[2] R.E. Bellman and L.A. Zadeh, "Decision making in a fuzzy environment Management Science, 17, 141-164 (1970).

[3] S. Dempe and A. Ruziyeva, "The Karush-Kuhn-Tucker optimality conditions in fuzzy optimization Technical Report: TU Bergakademie Freiberg, Freiberg, Germany, 2010-06.

[4] S. Chanas and D. Kuchta, Fuzzy Optimization: Linear programming problem with fuzzy coefficients in the objective function, Physica-Verlag, Wurzburg, 148-157 (1994).

[5] S. Chanas and D. Kuchta, "Multiobjective programming in optimization of interval objective functions - a generalized approach European Journal of Operational Research, 94, 594-598 (1996).

[6] M. Ehrgott, Multicriteria Optimization, Springer Verlag, Berlin, (2005).

[7] H. Rommelfanger, R. Hanuscheck and J. Wolf, "Linear programming with fuzzy objectives Fuzzy Sets and Systems, 29, 31-48 (1989).

[8] H.-J. Zimmermann, Fuzzy Set Theory and its Applications, Kluwer Academic Publishers, Dordrecht, (1991).

On variational description of the trajectories of averaging dynamical semigroups

Sakbaev V.

fumi2003@mail.ru

Moscow Institute of Physics and Technology (State University),

Institutsky per., 9, Dolgoprudny, 141700, Russia

We study the dynamics of quantum system with degenerated Hamiltonian. For this aim we consider the approximating sequence of regularized Hamiltonians and corresponding sequence of dynamical semigroups acting in the space of quantum states. The limit points set of the sequence of regularized semigroups is obtained as the result of averaging by finitely additive measure on the set of regularizing parameters. We establish that the family of averaging dynamical maps does not possess the semigroup property and the injectivity property. We define the functionals on the space of maps of the time interval into the quantum states space such that the maximum points of this functionals coincide with the trajectories of the family of averaging dynamical maps.

The work was supported by the Federal Program "Scientific and Scientific-Educational Staff of Innovative Russia"for years 2009-2013.

Dynamic Network Games with Changing Link Costs

Sedakov A.

a.sedakov@yahoo.com

St. Petersburg University, 35 Universitetski prospekt, Petrodvorets, 198504, Russia

In the paper an n-person network game is considered. A strategy of a player is a profile consisting of zeros and ones. Zero means that the player does not want to form a link, and one means that the player does.

It is assumed that players choose their strategies simultaneously, and after that a network is formed. A link connecting two players is formed if only one of them wants it to be formed. Players payoffs from being connected depend on problem parameters and current network and can be either positive or negative.

A dynamic model is also considered. In dynamic case payoffs are changed from stage to stage.

The cooperative approach of the dynamic network model is studied. As a solution it is considered such evolution of the network, which converges to efficient network. And the problem that we face, is to define the optimal in some sense allocation rule of joint maximal payoff among all players in accordance with some imputation. For this purpose a characteristic function is constructed in a special way.

References

[1] Ule A., Goeree J. K., Riedl A. “In search of stars: network formation among heterogeneous agents”, *Games and Economic Behavior*, 67, No. 2, 445-–466 (2009).

[2] Petrosyan L., Sedakov A. “On A Multistage Link Formation Game”, *Contributions to Game Theory and Management*, Vol. 4, 354–363 (2011).

[3] Petrosjan L. A., Zenkevich N. A. *Game Theory*, World Scientific Publ., (1996).

Choice of service model in the presence of various policies and desired delays in the customer order fulfillment

Sergeeva A., Bure V.
sergeeva_a_a@mail.ru, vlb310154@gmail.com
St. Petersburg State University, Universitetskaya nab. 7/9, St. Petersburg, 199034, Russia

We consider the cases of different policies of customer order fulfillment schemes in company which provide some kind service for customers. The game-theoretic model of choosing order service is constructed.

We consider company which services to build customer orders and provides various ways to make orders. Customers, in turn, refer to the company for the service, while trying to minimize the total cost of implementing the order. Each ordering device has its own scheme of service: the first device serves all customers in a queue and takes a fixed cost for customer order fulfillment, the second device serves all clients at ones but it takes a fixed cost for customer order fulfillment and also a cost for unit service time and the third device serves all customers in a queue and takes only cost for unit service time.

Except cost of order fulfillment customers also bear costs for waiting time as a costs for missed opportunities and penalty for late. We assume that after a certain amount of time T customers pay heavy fine R for having to delay. Customers choose the ordering scheme in company trying to minimize its operational costs.

The model of n-person game with perfect information is suggested.

Define the non-antagonistic game in normal form: $\Gamma =< N, \{p_i^j\}_{i\in N}, \{H_i\}_{i\in N} >$, where

$N = \{1, \dots, n\}$ - set of players,

$\{p_i^{(j)}\}_{i\in N}$ - set of strategies, $p_i^{(j)} \in [0,1]$, $j = 1, 2, 3$,

$\{H_i\}_{i\in N}$ - set of payoff functions.

$$H_i = -(p_i^{(1)} Q_{1i} + (1 - p_i^{(1)} - p_i^{(3)}) Q_{2i} + p_i^{(3)} Q_{3i}) =$$
$$= -(p_i^{(1)}(Q_{1i} - Q_{2i}) + p_i^{(3)}(Q_{3i} - Q_{2i}) + Q_{2i}),$$

where $p_i^{(1)}$ is the probability of player i choose service scheme 1, $p_i^{(3)}$ - is the probability of player i choose service scheme 3, $p_i^{(2)} = 1 - p_i^{(1)} - p_i^{(3)}$ - is the probability of player i choose service scheme 2.

Let c_{j1} - fixed cost of customer order fulfillment and c_{j2} - cost of - unit service time, $\tau_i^{(j1)}$ - time of waiting service, $\tau_i^{(j2)}$ - time of service for the device j, $j = 1, 2, 3$ and player i, $i = 1, \dots, n$. For the first and second devices we have $\tau_i^{(1)} = \tau_i^{(11)} + \tau_i^{(12)}$ and $\tau_i^{(3)} = \tau_i^{(31)} + \tau_i^{(32)}$ respectively, but for the second device $\tau_i^{(22)} = 0$ because we don't have a queue in the second device, so we have $\tau_i^{(2)} = \tau_i^{(22)}$. Duration of the customer service by the device 1 , 2 and 3 are independent random variables with densities functions:

$$f_1(t) = \frac{1}{\mu_1} e^{-\frac{1}{\mu_1}t}, \;\; t > 0,$$

$$f_2(t) = \frac{1}{\mu_2} e^{-\frac{1}{\mu_2}t}, \;\; t > 0,$$

$$f_3(t) = \frac{1}{\mu_3} e^{-\frac{1}{\mu_3}t}, \;\; t > 0.$$

Also define customer specific loss of waiting service r_i for player i and indicator

$$I\{t_i^{(j)}, T_j\} = \begin{cases} 1, & \text{if } t_i^{(j)} \leqslant T_j, \\ 0, & \text{if } t_i^{(j)} < T_j, \end{cases}$$

which define the time when customer begin to loose an amount R_j by waiting service, $i = 1, \dots, n$ $j = 1, 2, 3$. (For some period of time customer prefer to wait the service and after this it begin to loose.)

Q_{1i}, Q_{2i}, Q_{3i} - player i expected loss for 1'st, 2'nd and 3'rd service scheme respectively where $Q_{1i} = E(\tilde{Q}_{1i}), Q_{2i} = E(\tilde{Q}_{2i}), Q_{3i} = E(\tilde{Q}_{3i})$. So we can define

$$Q_{1i} = E(\tilde{Q}_{1i}) = E(r_i(\tau_i^{(11)} + \tau_i^{(12)}) + R_1 I\{t_i^{(1)}, T_1\} + c_1),$$

$$Q_{2i} = E(\tilde{Q}_{2i}) = E((r_i + c_2 2)\tau_i^{(22)} + R_2 I\{t_i^{(2)}, T_2\} + c_{21}),$$

$$Q_{3i} = E(\tilde{Q}_{3i}) = E((r_i + c_{32})\tau_i^{(32)} + R_3 I\{t_i^{(3)}, T_3\}), i = 1, \dots, n.$$

We consider the casualty functions below: $h_i = -H_i$, $i = 1, \dots, n$.

Customers choose order fulfillment schemes trying to minimize expected losses. So to find the optimal behavior of customers we should find mean value of $Q_{ji}, i = 1, \dots, n$ $j = 1, 2, 3$.

The equilibrium strategies for clients of company with different cases of ordering schemes is found. The existence of these equilibria is proved.

References

[1] V. Bure, "Game-theoretical model of a queueing system.", *Vestnik Sankt-Peterburgskogo Universiteta*, St. Petersburg, No. 9, 3-5 (2002).

[2] V. Feller *Introduction to Probability Theory and Its Applications.*Moscow: Nauka, (1984).

[3] A. Langevin, D. Riopel *Logistics systems: design and optimization.*, New York: Springer, (2005).

[4] L. Petrosyan, N. Zenkevich, E. Syomina, *Game Theory*, Moscow: Nauka, (1998).

Optimality Conditions and Duality for Nonsmooth Multiobjective Semi-Infinite Programming Problems with Support Functions under Generalized (C, α, ρ, d)-convexity

Singh V.
vinaysingh14aug@gmail.com
Department of Mathematics, Banaras Hindu University, Varanasi-221005, India

In this paper, we consider the following nonsmooth multiobjective semi-infinite programming problem:

$$Min \qquad f_i(x) + S(x|D_i)$$

$$subject \quad to \quad g_j(x) + S(x|E_j), \quad j \in J,$$

where J is an infinite index set, $f_i, i \in \{1, 2, ...9\}$ and $g_j, j \in J$ are locally Lipschitz functions from a nonempty open set $X \subseteq R^n$ to R. D_i and E_j are compact convex sets in R^n, and $S(x|D_i), S(x|E_j)$ are designate the support functions of compact sets.

We establish sufficient optimality conditions and duality results for nonsmooth multiobjective semi-infinite programming problem with support function under generalized (C, α, ρ, d)-convexity.

References

[1] N. Kanzi and S. Nobakhtian, "Optimality conditions for non-smooth semi-infinite programming *Optimization* 59, 717-727 (2010).

[2] D. H. Yuan, X. L. Liu, A. Chinchuluun and P. M. Pardalos, "Nondifferentiable minimax fractional programming problems *J. Optim. Theory Appl.*, 129 (1), pp. 185–199 (2006).

Differential Games with Random Duration and a Composite Cumulative Distribution Function

Shevkoplyas E., Gromov D.
ekaterina.shevkoplyas@gmail.com, gromov@cim.mcgill.ca
St. Petersburg State University, St.Petersburg, Russia, McGill University, Montreal, Canada

Game theory as a branch of mathematics investigates conflict processes controlled by many participants (players). In this paper we consider games with the fixed duration $[0, T]$, where T is a random variable with cumulative distribution function $F(t)$.

In many cases the cumulative distribution function (CDF) of the terminal time may change depending on some conditions, which can be expressed as a function of time and/or state. One particular example is the development of a mineral deposit. The probability of a breakdown may depend on the development stage. Namely, at the initial stage this probability is higher than during the routine mining operation. Thus, given a set of CDFs related to the single phases, we define a uniform composite distribution function for the terminal time. It is shown that under some week conditions on the individual CDFs, the composite distribution function is well defined.

Furthermore, it is shown that the Pareto-optimal strategy in this class of differential games can be found as a sequence of optimal control problems expressed in terms of the Pontryagin Maximum Principle with additional constraints imposed on the state and on the adjoint variables.

Finally, an application of the obtained theoretical results is presented. We investigate one simple model of non-renewable resource extraction, where the termination time is a random variable with a composite distribution function. Two different switching rules are studied and a qualitative analysis of the obtained results is presented.

References

[1] L. A. Petrosyan and N. V. Murzov, "Game-theoretic problems of mechanics", *Litovsk. Math. Sb.*, vol. 6, pp. 423–433, 1966, in Russian.

[2] L. A. Petrosyan and E. V. Shevkoplyas, "Cooperative solutions for games with random duration", *Game Theory and Applications*, vol. IX, pp. 125–139, 2003.

[3] J. Marin-Solano and E. V. Shevkoplyas, "Non-constant discounting in differential games with random time horizon", *Automatica*, Vol. 47, 12, 2011, pp. 2626–2638.

[4] M. E. Yaari, "Uncertain lifetime, life insurance, and the theory of the consumer", *The Review of Econimic Studies*, vol. 32, no. 2, pp. 137–150, 1965.

On one maximum robustness control problem

Slesar N.

nickslesar@gmail.com

Санкт-Петербургский государственный университет, Университетская наб., 7/9, Санкт-Петербург, 199034, Россия

In the report, we discuss the problem of transferring a material point along an interval by a bounded force at a given time with the minimal speed under the condition that the length of the interval and the initial speed are not known in advance. To construct a control function, a criterion of maximal robustness under uncertainty conditions is used.

Об одной задаче максимально робастного управления

В данном сообщении рассматривается задача перемещения материальной точки вдоль отрезка прямой ограниченной по величине силой за заданное время с минимальной скоростью при условии, что величина отрезка и начальная скорость движения заранее не известны. Для построения управления использован критерий максимальной робастности управления в условиях неопределенности [1, 2]. В современной терминологии робастным называется управление, обеспечивающее выполнение условий задачи управления для каждого значения параметра неопределенности из множества его возможных значений. Критерий максимальной робастности состоит в том, чтобы каждому управлению заданного класса сопоставить множество всех значений параметра неопределенности, для которых данное управление эффективно (т.е. удовлетворяются все условия на траекторию, задающие цель управления). Это множество, называемое множеством робастности, максимизируется в указанном ниже смысле на множестве всех управлений рассматриваемого класса. В докладе задача максимально робастного управления рассмотрена для случая, когда заранее не известным параметром (параметром неопределенности) является длина отрезка, и для случая, когда не известны как длина отрезка, так и начальная скорость. Для каждого случая построено максимальное множество робастности и реализующее это множество максимально робастное управление. В первом случае максимально робастное управление есть функция фазовых координат системы, во втором случае - функция фазовых координат и начальной скорости (без информации о длине отрезка).

Постановка задачи. Уравнение движения материальной точки вдоль прямой и начальный вектор движения

$$\ddot{x} = u, \quad x(0) = x_0, \;\; \dot{x}(0) = \dot{x}_0$$

определяют траекторию системы в фазовом пространстве $x(t; u)$. Здесь $u-$ управляющая функция заданного класса (напр., $u = u(\cdot|t), u(\cdot|x, \dot{x})$). Ограничения заданы в виде

$$|u| \leqslant 1, \quad x(T) = 0, \quad \dot{x}(T) = 0, \quad T - \text{задано.}$$

Рассмотрены два варианта неопределенности в математическом описании системы.

1. Неопределенности подвержено начальное положение x_0, которое может принимать любое значение. Введение неопределенности означает, что траектория системы определяется не только управлением u, но и начальным положением x_0, $\;x(t; u, x_0)$.

Множество допустимых пар (u, x_0) запишется в виде

$$M = \{(u, x_0) \mid |u| \leqslant 1,\ x(T; u, x_0) = 0,\ \dot{x}(T; u, x_0) = 0\}.$$

Заметим, множество допустимых управлений оказывается зависимым от значений x_0

$$U(x_0) = \{u \ \mid \ (u, x_0) \in M\}.$$

На траекторию $x(t; u, x_0)$ накладывается следующее условие оптимальности. Рассматривается функционал

$$J(u, x_0) = \max_{t \in [0,T]} \mid \dot{x}(t; u, x_0) \mid,$$

его сечение по переменной x_0 – $J_{x_0}(u)$ – и задача

$$J_{x_0}(u) \to \min_{u \in U(x_0)} . \tag{1}$$

Теперь, с учетом условия (1), множество допустимых пар задачи имеет вид

$$M^+ = \{(u, x_0) \mid u = \arg \min_{u \in U(x_0)} J_{x_0}(u)\},$$

а множество робастности каждого управления из заданного класса – вид

$$X_0(u) = \{x_0 \mid (u, x_0) \in M^+\}.$$

Ставится задача максимально робастного управления: найти управление u^*, такое что

$$X_0(u^*) \supseteq X_0(u) \quad \forall u. \tag{2}$$

2. Для случая, когда неопределенности подвержено начальное положение x_0 и начальная скорость $\dot{x}_0$, задача максимально робастного управления формулируется аналогично.

Решение задачи. Для случая неопределенности начального положения материальной точки решение задачи (2) получено на классе управлений как функций фазовых координат $u = u(x, \dot{x})$. Максимально робастное управление имеет вид поверхности над фазовой плоскостью со значениями $-1, 0, +1$ и линиями переключений в виде соответствующих парабол. Максимальное множество робастности зависит от значений начальной скорости. Для нулевой скорости это отрезок $[-0.25T^2, 0.25T^2]$.

Для случая неопределенности начального положения и начальной скорости решение задачи (2) на множестве управляющих функций $u = u(x, \dot{x})$ не существует, но было получено на множестве управляющих функций $u = u(x, \dot{x}, \dot{x}_0)$. Максимально робастное управление имеет вид поверхности над фазовой плоскостью со значениями $-1, 0, +1$ и линиями переключений в виде соответствующих парабол, одна из которых имеет коэффициент, зависящий от начальной скорости $\dot{x}_0$.

Заключение. Рассмотрен пример максимально робастного управления механической системой с одной степенью свободы с неопределенностью двух типов. В задаче с неизвестной длиной отрезка было показано, что для управлений как функций времени решение задачи не существует, а для функций фазовых координат решение существует, и оно было построено вместе с максимальным множеством значений длины отрезка. В задаче с неизвестной длиной отрезка и с неизвестной начальной скоростью максимально робастное управление среди функций фазовых координат не существует, оно было построено только как функция фазовых координат и начальной скорости. Рассмотренные примеры позволяют сделать вывод, что по мере роста объема информации, которое использует управление, растет “богатство” множеств робастности. Подход дает возможность исследовать различные классы управляющих функций и для решения задачи максимально робастного управления выбирать нужный класс. Следует отметить, что максимальное множество робастности, соответствующее максимально робастному управлению, описывает предельные возможности системы быть управляемой (соблюдать условия, накладываемые на траектории системы в пространстве состояний) при неопределенности заданного типа.

ЛИТЕРАТУРА

[1] Кулагин В.В., "Максимальная робастность в задаче оптимального управления"// *Тезисы доклада. X международный семинар "Устойчивость и колебания нелинейных систем управления"*, Москва: Институт проблем управления РАН, (2008).

[2] Кулагин В.В., Слесарь Н.О. Максимально робастное перемещение материальной точки вдоль отрезка прямой за заданное время с минимальной скоростью. *Вестник Нижегородского университета им. Н.И. Лобачевского*, № 4, часть 2, СС 189-190, (2011).

Nonsmooth computational mechanics. Applications on masonry and steel structures

Stavroulakis G.E., Drosopoulos G.A., Stavroulaki M.E.
gestavr@dpem.tuc.gr, gdrosopoulos@isc.tuc.gr,
mstavr@mred.tuc.gr
Technical University of Crete, Institute of Computational Mechanics and Optimization, GR-73100 Chania, Greece

Nonsmooth mechanics uses theoretical results and algorithms from nonsmooth analysis and optimization for the study of structures with unilateral behaviour. Unilateral contact, stick-slip in friction, multi-block and cracked continua and other similar effects induce nonsmoothness in structures. Usually a finite element discretization is used and the resulting system of equations, inequalities, variational inequalities, nonsmooth equations etc has many degrees of freedom. Therefore specialized techniques for the efficient numerical treatment of the arising problems are required. Previous publications of the first author include [1], [2], [3].

A review of recent results in the area of computational nonsmooth mechanics and practical applications will be given in the talk. Two applications in which the authors are actively involved in the last years, namely modelling of masonry monuments and bolted connections of steel structures, will be presented in more details.

Unilateral modelling of discontinuous and multiblock masonry structures can be used for both structural analysis and collapse modelling and prediction. Detailed investigation of unilateral models for two-dimensional masonry bridges has been compared with analytical and experimental results during the last years. The developed models can be used for further strength and reinforcement studies as well as for damage prediction or explanation in existing monuments. Detailed results in this direction have been published in [4], [5], [6].

The mechanical behaviour of bolted steel structure connections is dominated from the unilateral contact and friction effects between the involved parts of the connection. Extended end-plate steel connections and top and seat angle bolted steel connections with double web angles have been studied. Detailed finite element models including unilateral contact and friction nonlinearities together with classical elastoplasticity and large displacements are used. The results are comparable with experimental measurements taken at the Jordan University of Science and Technology [7]. Furthermore, in order to access combined thermomechanical behaviour, relevant to combined earthquake and fire loadings, the finite element models have been extended accordingly. Different fire and loading scenarios have been considered. Strength reduction, which is a well-known fact for steel structures, is clearly demonstrated. In addition unilateral interface opening due to thermal loads appear as well [8].

References

[1] V.F. Dem'yanov, G.E. Stavroulakis, L.N. Polyakova, P.D. Panagiotopoulos, *Quasidifferentiability and nonsmooth modelling in mechanics, engineering and economics*, Kluwer Academic Publishers, Springer, (1996).

[2] D.Y. Gao, R. Ogden, G.E. Stavroulakis (Eds.), *Nonsmooth/Nonconvex Mechanics: Modeling, Analysis and Numerical Methods*, Kluwer Academic Publishers, Springer, (2001).

[3] J. Haslinger, G.E. Stavroulakis (Eds.), *Nonsmooth Mechanics of Solids*, CISM Lecture Notes Vol. 485, Springer, (2006).

[4] B. Leftheris, A. Sapounaki, M.E. Stavroulaki, G.E. Stavroulakis, *Computational Mechanics for Heritage Structures*, Wit - Computational Mechanics Publications, (2006).

[5] G.A. Drosopoulos, G.E. Stavroulakis, C.V. Massalas, "Limit analysis of a single span masonry bridge with unilateral frictional contact interfaces", *Engineering Structures*, 28, 1864—1873 (2006).

[6] M.E. Stavroulaki, G.D. Drosopoulos, G.E. Stavroulakis, 'Unilateral analysis and damage identification in masonry structures', *CC2011 The Thirteenth International Conference on Civil, Structural and Environmental Engineering Computing ,Chania, Crete, Greece 6-9 September 2011*, Civil Comp Press, (2011).

[7] G.A. Drosopoulos, G.E. Stavroulakis, K.M. Abdalla, "3D finite element analysis of end-plate steel joints", *Steel and Composite Structures. An international journal*, 11, No. 2, 93–116, (2012).

[8] A. Kalogeropoulos, G.A. Drosopoulos, G.E. Stavroulakis, "Thermal-stress analysis of a three-dimensional end-plate steel joint", *Construction and Building Materials*, 29, 619–626, (2012).

Shor's acceleration for Polyak's subgradient method (in Russian)

Stetsyuk P.
stetsyukp@gmail.com
Институт кибернетики НАН Украины, проспект Глушкова, 40, г. Киев, 03187, Украина

Ускоренный по Шору субградиентный метод Поляка

Пусть $f(x)$ – выпуклая функция векторного аргумента $x \in R^n$ и ее субградиент $\partial f(x)$ удовлетворяет условию:

$$(x - x^*, \partial f(x)) \geqslant m(f(x) - f^*), \quad \text{где } m \geqslant 1, \tag{1}$$

для $\forall x \in R^n$ и $\forall x^* \in X^*$. Здесь R^n – евклидово пространство размерности n со скалярным произведением (x, y); X^* – множество точек минимума функции $f(x)$; f^* – минимальное значение функции $f(x)$: $f^* = f(x^*)$, $x^* \in X^*$. Параметр m задает величину максимального сдвига по выпуклости функции $f(x)$ и введен для учета специальных классов выпуклых функций. Для кусочно-линейной негладкой функции $m = 1$, для квадратичной гладкой функции $m = 2$.

Пусть известно f^*. Для нахождения точки $x^* \in X^*$ можно использовать субградиентный метод Поляка [1]:

$$x_{k+1} = x_k - h_k \frac{\partial f(x_k)}{\|\partial f(x_k)\|}, \quad h_k = \frac{m\,(f(x_k) - f^*)}{\|\partial f(x_k)\|}, \quad k = 0, 1, 2, \ldots . \tag{2}$$

Здесь шаг h_k задает величину максимального сдвига в направлении нормированного антисубградиента, при котором для выпуклой функции $f(x)$ условие (1) гарантирует, что угол между антисубградиентом и направлением из точки x_{k+1} на точку минимума будет нетупым.

Обоснование сходимости метода (2) очень простое, так как для всех точек итерационного процесса справедливы неравенства

$$\|x_{k+1} - x^*\|^2 \leqslant \|x_k - x^*\|^2 - \left(\frac{m(f(x_k) - f^*)}{\|\partial f(x_k)\|}\right)^2, \quad k = 0, 1, \ldots .$$

Отсюда для каждой итерации $K > 1$ имеем неравенство

$$\|x_K - x^*\|^2 \leqslant \|x_0 - x^*\|^2 - \sum_{k=0}^{K-1} \left(\frac{m(f(x_k) - f^*)}{\|\partial f(x_k)\|}\right)^2,$$

из которого, используя расходимость ряда, легко придти к противоречию, что $\|x_{\overline{K}} - x^*\|^2 < 0$ для некоторого $\overline{K}$.

Медленную сходимость метода Поляка для овражных функций определяет тупой угол между двумя последовательными субградиентами. Чем ближе угол к 180 градусам, тем более медленной будет сходимость. Так, например, для негладкой функции двух переменных $f(x_1, x_2) = |x_1| + k\,|x_2|$ скорость сходимости метода (2) определяется геометрической прогрессией со знаменателем $\sqrt{1 - 1/k^2}$, который близок к 1 даже при сравнительно небольших значениях k.

Ускорить метод (2) можно, если пространство переменных преобразовать так, чтобы тупой угол между двумя последовательными субградиентами уменьшался. Тупой угол между двумя векторами можно преобразовать в прямой с помощью „одноранговогo эллипсоидального оператора“ [2]. Для двух нормированных векторов ξ и η из R^n это реализует линейный оператор из R^n в R^n, который в матричной форме имеет вид:

$$T_1(\xi, \eta) = I - \frac{1}{1 - (\xi, \eta)^2}\left(\left(1 - \sqrt{1 - (\xi, \eta)^2}\right)\eta - (\xi, \eta)\xi\right)\eta^T. \qquad (3)$$

Здесь $\xi, \eta \in R^n$ – векторы, такие, что $\|\xi\| = 1$, $\|\eta\| = 1$ и их скалярное произведение удовлетворяет условию $(\xi, \eta)^2 \neq 1$, I – единичная матрица размера $n \times n$. Для оператора $T_1(\xi, \eta)$ существует обратный $T_1^{-1}(\xi, \eta)$:

$$T_1^{-1}(\xi, \eta) = I + \frac{1}{\sqrt{1 - (\xi, \eta)^2}}\left(\left(1 - \sqrt{1 - (\xi, \eta)^2}\right)\eta - (\xi, \eta)\xi\right)\eta^T. \qquad (4)$$

Оператор $T_1(\xi, \eta)$ связан с преобразованием в шар специального эллипсоида, описанного вокруг тела, которое получено в результате пересечения шара и двух полупространств, проходящих через центр шара.

Субградиентный метод метод Поляка с преобразованием пространства имеет следующий вид:

$$x_{k+1} = x_k - h_k B_k \frac{B_k^T \partial f(x_k)}{\left\|B_k^T \partial f(x_k)\right\|}, \quad h_k = \frac{m(f(x_k) - f^*)}{\left\|B_k^T \partial f(x_k)\right\|}, \quad k = 0, 1, 2, \ldots, \qquad (5)$$

где матрица $B_0 = I$, а матрица B_{k+1} пересчитывается по следующему правилу:

$$B_{k+1} = \begin{cases} B_k, & \text{если } (\xi, \eta) \geqslant 0 \\ B_k T^{-1}(\xi, \eta) & \text{иначе} \end{cases},$$

где $\xi = \dfrac{B_k^T \partial f(x_k)}{\left\|B_k^T \partial f(x_k)\right\|}$ и $\eta = \dfrac{B_k^T \partial f(x_{k+1})}{\left\|B_k^T \partial f(x_{k+1})\right\|}$.

Пусть $A_k = B_k^{-1}$, $A_{k+1} = B_{k+1}^{-1}$. Для $\forall x^* \in X^*$ и всех точек итерационного процесса (5) справедливы неравенства

$$\|A_{k+1}(x_{k+1} - x^*)\|^2 \leqslant \|A_k(x_k - x^*)\|^2 - \left(\frac{m(f(x_k){-}f^*)}{\|B_k^T \partial f(x_k)\|}\right)^2, \quad k = 0, 1, \ldots. \tag{6}$$

Неравенства (6) означают, что в методе (5) преобразование пространства таково, что в каждом очередном преобразованном пространстве переменных гарантируется уменьшение расстояния до множества точек минимума. Благодаря этому для каждой итерации $k > 1$ имеем неравенство

$$\|A_k(x_k - x^*)\|^2 \leqslant \|x_0 - x^*\|^2 - \sum_{i=0}^{k-1} \left(\frac{m(f(x_i) - f^*)}{\|B_i^T \partial f(x_i)\|}\right)^2.$$

Антиовражная техника в методе (5) направлена на уменьшение степени овражности поверхностей уровня выпуклых функций подобно тому, как это сделано в r-алгоритмах [3]. Детерминант матрицы B_k стремится к нулю, так как, если на k-м шаге реализуется преобразование пространства, то

$$\det(B_{k+1}) = \det(B_k)\det(T^{-1}(\xi, \eta)) = \det(B_k)\sqrt{1 - \cos^2\varphi},$$

где φ – угол между двумя последовательными субградиентами. Для овражных функций это обеспечивает ускоренную сходимость метода (5) по отношению к методу (2) при произвольной начальной стартовой точке x_0. Так, например, для овражной функции двух переменных $f(x_1, x_2) = |x_1| + k\,|x_2|$ метод (5) будет находить точку x^* не более чем за две итерации независимо от значения k.

Отметим, что метод (5) можно назвать ускоренным методом Поляка за счет преобразования пространства переменных которое свойственно r-алгоритмам Шора [3]. Действительно, он обладает двумя характерными чертами r-алгоритмов. Во-первых, движение из точки осуществляется в направлении антисубградиента и, во-вторых, растяжение пространства производится в направлении разности нормированных последовательных субградиентов, если угол между ними тупой. Если нормы субградиентов одинаковы, то это направление будет совпадать с разностью двух последовательных субградиентов, но в отличие от r-алгоритмов второй субградиент получен не согласно шагу наискорейшего спуска, а согласно шагу Поляка в преобразованном пространстве переменных.

ЛИТЕРАТУРА

[1] Поляк Б.Т., *Введение в оптимизацию*, М.: Наука (1983).

[2] Стецюк П.И., “Ортогонализующие линейные операторы в выпуклом программировании”, *Кибернетика и системный анализ*, No. 3, 97–119 (1997).

[3] Шор Н.З., *Методы минимизации недифференцируемых функций и их приложения*, Киев: Наукова думка, (1979).

Non-differentiable Optimization Problems with Hidden Non-convex Structures

Strekalovsky A.

strekal@icc.ru

Institute for System Dynamics and Control Theory SB RAS, Lermontov St. 134, Irkutsk, 664033, Russia

We consider various non-convex optimization problems with the functions of A.D.Alexandrov [3, 4, 5] (D.C. functions) that might be non-smooth. For example, the problems of financial and medical diagnostics turn out often to be of the kind.

The problems of hierarchical structure [7, 8] provide for another type of optimization problems with hidden non-convex structures, as complementarily problems [9] as well.

As well known, the direct application of classical optimization schemes (methods) may have unpredictable consequences while the computational results are often interpreted only in the content aspect. On the other hand the crucial question of nowadays optimization is how to escape a stationary or local solution (provided by local search method) with improving of the cost function. We present new mathematical tools (global search theory) which allow to do it (see [5]).

The global search theory consist of two principal stages, as follows:

a) local search (several new special local search methods have been developed at least two decades);
b) procedures of escaping out of the stationary and local solutions (based on global optimality conditions).

This new apparatus is oriented to deal with non-differentiable objects and to use in interior of the general scheme the classical and non-smooth optimization methods [1, 2]. The results of numerous computational simulations show the competitive effectiveness of the developed methods [6].

References

[1] F.H. Clarke, *Optimization and Nonsmooth Analysis*, 2nd ed., Classics in Applied Mathematics, Vol. 5, SIAM, Philadelphia, (1990).

[2] V.F. Demyanov, L.V. Vasiliev, *Nondifferentiable Optimization*, Springer-Optimization Software, New York, (1985).

[3] J.-B. Hiriart-Urruty, "Generalized Differentiability, Duality and Optimization for Problem dealing with Difference of Convex Functions", *Lecture Notes in Economics and Mathematical Systems*, Springer-Verlag, Berlin, 256, 37–69 (1985).

[4] H. Tuy, "D.c. Optimization: Theory, Methods and Algorithms", *Handbook of Global Optimization*, Ed. by R. Horst, P.M. Pardalos, Kluwer, Dordrecht, 149–216 (1995).

[5] A.S. Strekalovsky, *Elements of Nonconvex Optimization*, Nauka, Novosibirsk, (2003) [in Russian].

[6] T.V. Gruzdeva, A.S. Strekalovsky, A.V. Orlov, O.V. Druzinina, "Nonsmooth Minimization Problems for the Difference of Two Convex Functions", *Numerical Methods and Programming*, 12, No. 2, 139–151 (2011) [in Russian].

[7] A.S. Strekalovsky, A.V. Orlov, A.V. Malyshev, "On Computational Search for Optimistic Solutions in Bilevel Problems", *Journal of Global Optimization*, 48, No. 1, 159–172 (2010).

[8] A.S. Strekalovsky, A.V. Orlov, and A.V. Malyshev, "Numerical Solution of a Class of Bilevel Programming Problems", *Numerical Analysis and Applications*, 3, No. 2, 165–173 (2010).

[9] E. O. Mazurkevich, E. G. Petrova and A. S. Strekalovsky, "On the Numerical Solution of the Linear Complementarity Problem", *Computational Mathematics and Mathematical Physics*, 49, No. 8, 1318–1331 (2009).

On a method of solving a bilevel optimization problem

Sukach M.P.

nskmike@yandex.ru

St. Petersburg State University, Universitetskii prospekt 35, Petergof, 198504, St. Petersburg, Russia

The problem of finding optimal values of one functional on the set of optimal solutions for another functional is discussed in the report. It is proposed to solve the stated bilevel optimization problem by reducing it to an unconstrained optimization problem. The reduction is done by employing the Exact Penalization Technique.

О методе решения задачи двухуровневой оптимизации

В данной работе рассматривается задача нахождения оптимального значения функционала на решениях другой оптимизационной задачи. Предлагается решать эту задачу сведением к задаче безусловной минимизации некоторого функционала, который (даже в случае гладкости исходных функционалов) является существенно негладким. Указанное сведение проводится с помощью теории точных штрафных функций. Данный подход был предложен в работе[1].

Постановка задачи. Пусть $f, f_1 : \mathbb{R}^n \to \mathbb{R}$ – заданные непрерывные функции, $G \subset R^n$ – заданное множество. Положим

$$\Omega = \{x \in G \mid f_1(x) \leqslant f_1(y)\ \forall y \in G\},$$

т.е. $\Omega \subset G$ - множество точек минимума функции f_1 на множестве G. Предположим, что $\Omega \neq \varnothing$. Задача

$$f(x) \longrightarrow \inf_{x \in \Omega} \tag{1}$$

называется задачей двухуровневой оптимизации.

Штрафные функции в двухуровневой задаче. Для того чтобы применить теорию точных штрафных функций(более подробно с ней можно познакомиться в [2]), представим множество Ω в виде $\Omega = \{x \in G \mid \varphi(x) = 0\}$, где

$$\varphi(x) = \sup_{y \in G}(f_1(x) - f_1(y)) = f_1(x) - f_{1G}^*, \quad f_{1G}^* = \inf_{y \in G} F_1(y).$$

Рассмотрим случай, когда множество G представимо в виде

$$G = \{x \in R^n \mid \varphi_1(x) = 0\},$$

где $\varphi_1(x) \geqslant 0\ \forall x \in R^n$. Пусть $\lambda_1 \geqslant 0, \lambda_2 \geqslant 0$ фиксированы. Определим штрафную функцию

$$F_\lambda(x) = f(x) + \lambda_1[\varphi(x) + \lambda_2\varphi_1(x)]. \tag{2}$$

Определение. Функция $F_\lambda(x)$ называется точной штрафной функцией для функции f на множестве Ω, если существует такой штрафной параметр λ_1^*, что для любых $\lambda_1 \geqslant \lambda_1^*$ множество точек минимума функции $F_\lambda(x)$ на всем пространстве R^n совпадает с множеством точек минимума функции f на множестве Ω.

Таким образом, если $F_\lambda(x)$ точная штрафная функция, то исходная задача (1) эквивалентна задаче безусловной минимизации штрафной функции (2). Для того чтобы воспользоваться методом точных штрафных функций на практике, необходимо знать значение точного штрафного параметра. Его оценку можно построить на основании Теоремы 3.4.1 из [2] по формулам

$$\lambda_1^* \geqslant \frac{2L}{a}, \quad \exists\, \delta > 0,\ a > 0 :\ \varphi^{\downarrow}(x) + \lambda_2\varphi_1^{\downarrow}(x) < -a < 0\ \ \forall x \in \Omega_\delta \backslash \Omega,$$

где L — константа Липшица для функции $f(x)$ на множестве $\Omega_\delta \backslash \Omega$, $\varphi^{\downarrow}(x) = \liminf_{y \to x} \frac{f(x)-f(y)}{\|x-y\|}$ — скорость наискорейшего спуска, $\Omega_\delta = \{x \in \mathbb{R} \mid \varphi(x) < \delta\}$.

Если $F_\lambda(x)$ является точной штрафной функцией и кодифференцируема, тогда используя метод усечённого кодифференциального спуска [3] получаем решение. Приведены численные результаты решения задачи, полученные с помощью пакета MATLAB [4]. Данная задача также рассматривается в работе [5] и других работах автора.

Работа выполнена при финансовой поддержке РФФИ, грант № 12-01-00752.

ЛИТЕРАТУРА

[1] *Демьянов В.Ф., Факкиней Ф.* Задачи двухуровневой оптимизации и штрафные функции // Известия высших учебных заведений, 2003. № 12. с. 49–61.

[2] *Демьянов В.Ф.* Условия экстремума и вариационное исчисление // М.: Высшая школа, 2005. 336 с.

[3] *Demyanov V.F., Bagirov A.M., Rubinov A.M.* A method of truncated codifferential with application to some problems of cluster analysis // Journal of Global Optimization. 2002. Vol. 23. № 1. P. 63–80.

[4] *Андрамонов М.Ю., Тамасян Г.Ш.* Реализация аналитического кодифференцирования в пакете MATLAB // Вычислительные методы и программирование. 2007. т. 8. с. 1-5.

[5] *Stephan Dempe* Foundations of Bilevel Programming // NONCONVEX OPTIMIZATION AND ITS APPLICATIONS Volume 61, 2002.

Characterization theorem for best polynomial spline approximation with free knots

Sukhorukova N., Ugon J.

n.sukhorukova@ballarat.edu.au, j.ugon@ballart.edu.au

Centre for Informatics and Applied Optimization University of Ballarat, P.O. Box 663, Ballarat, Vic. 3350, Australia

We derive a necessary condition for a best approximation by piecewise polynomial functions. We apply nonsmooth nonconvex analysis to obtain this result, which is also a necessary and sufficient condition for the inf-stationarity in the sense of Demyanov-Rubinov. We start from identifying a special property of the knots. Then, basing on this property, we construct a characterization theorem for best free knots polynomial spline approximation, which is stronger than the existing necessary optimality conditions.

References

[1] V. F. Demyanov and A. M. Rubinov (eds.), Quasidifferentiability and related topics, Kluwer Academic Publisher, 2000, 400 p.

[2] G. Nürnberger, Approximation by spline functions, Springer-Velgar, 1989.

Direct methods in the simplest variational problem in the parametric form

Tamasyan G.Sh.
grigoriytamasjan@mail.ru
St. Petersburg State University, Universitetskii prospekt 35, Petergof, 198504, St. Petersburg, Russia

In the talk, the simplest problem of Calculus of Variations in the parametric form is considered. Necessary optimality conditions are derived by employing the Exact Penalization Theory. New effective direct numerical methods are proposed.

Прямые методы в простейшей вариационной задаче в параметрической форме

Многие прикладные задачи вариационного исчисления более удобно изучать, если искомую кривую представить в параметрическом виде. При этом следует заметить, что каждая линия допускает не единственное параметрическое представление [1]–[4]. В связи с этим, для того чтобы исследуемый на экстремум функционал зависел от линии, а не от ее параметрического представления накладываются дополнительные требования «условия однородности» [1, 2].

Пусть $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$. Через $P^1[t_0, t_1]$ обозначим класс непрерывно-дифференцируемых на $[t_0, t_1]$ функций с кусочно-непрерывной и ограниченной на $[t_0, t_1]$ производной. Исследуем на экстремум функционал

$$I(x, y) = \int_{t_0}^{t_1} F(x, y, x', y')\, dt,$$

где функцию F будем считать непрерывно-дифференцируемой по всем своим аргументам на $\mathbb{R}^4$, а граничные условия имеют вид

$$x(t_0) = x_0, \quad x(t_1) = x_1, \quad y(t_0) = y_0, \quad y(t_1) = y_1.$$

Требуется найти x_*, $y_* \in \Omega$, такие, что

$$I(x_*, y_*) = \min_{[x,y]\in\Omega} I(x, y), \tag{1}$$

где

$$\Omega = \left\{\, x, y \in P^1[t_0, t_1] \;\middle|\; x(t_0) = x_0,\ x(t_1) = x_1,\ y(t_0) = y_0,\ y(t_1) = y_1 \right\}. \tag{2}$$

Не умаляя общности будем считать, что $t_0 = 0$, $t_1 = 1$, а подинтегральная функция F четырех переменных x, y, x', y' является положительно однородной первой степени относительно x', y', т.е.

$$F(x, y, kx', ky') = kF(x, y, x', y') \quad \forall\, k > 0.$$

Переформулируем поставленную выше задачу (1), (2). Обозначим через $z_1(t) = x'(t)$, $z_2(t) = y'(t)$, тогда

$$x(t) = x_0 + \int_0^t z_1(\gamma)\, d\gamma, \quad y(t) = y_0 + \int_0^t z_2(\tau)\, d\tau.$$

Положим

$$Z = \Big\{ z = [z_1, z_2] \in P[0,1] \times P[0,1] \;\Big|\; x_0 + \int_0^1 z_1(\gamma)\, d\gamma = x_1, \\ y_0 + \int_0^1 z_2(\tau)\, d\tau = y_1 \Big\},$$

где $P[0,1]$ — множество кусочно-непрерывных, ограниченных на отрезке $[0,1]$ функций. Введем функционал

$$f(z_1, z_2) = \int_0^1 F\Big(x_0 + \int_0^t z_1(\gamma)\, d\gamma, y_0 + \int_0^t z_2(\gamma)\, d\gamma, z_1(t), z_2\Big) dt.$$

Несложно показать, что задача (1), (2) эквивалентна задаче $f(z) \longrightarrow \min\limits_{z \in Z}$.

Множество Z можно представить в эквивалентном виде

$$Z = \big\{ [z_1, z_2] \in P[0,1] \times P[0,1] \;\big|\; \varphi(z) = 0 \big\},$$

где

$$\varphi(z) = |\varphi_1(z_1)| + |\varphi_2(z_2)| = \left| \int_0^1 z_1(\gamma)\, d\gamma + x_0 - x_1 \right| + \left| \int_0^1 z_2(\gamma)\, d\gamma + y_0 - y_1 \right|.$$

Пусть $\lambda \geqslant 0$ фиксировано. Введем функцию $\Phi_\lambda(z) = f(z) + \lambda\varphi(z)$. Функция $\Phi_\lambda(z)$ называется *штрафной функцией*, а число λ — *штрафным параметром.* В [5] представлены ряд теорем, при выполнении которых $\Phi_\lambda(z)$ является функцией точного штрафа.

Пусть z_1, $z_2 \in P[0,1]$ фиксированы, $\varepsilon > 0$. Выберем произвольное v_1, $v_2 \in P[0,1]$. Положим

$$z_{1\varepsilon}(t) = z_1(t) + \varepsilon v_1(t), \quad z_{2\varepsilon}(t) = z_2(t) + \varepsilon v_2(t).$$

Применяя классические вариации функций z_1 и z_2, имеем

$$\Phi_\lambda(z_\varepsilon) = \Phi_\lambda(z) + \varepsilon\Bigg[\int_0^1 \left(F'_{z_1} + \int_t^1 F'_x d\tau\right) v_1(t)\,dt + \\ + \int_0^1 \left(F'_{z_2} + \int_t^1 F'_y d\tau\right) v_2(t)\,dt\Bigg] +$$

$$+ \lambda \max\left\{\varphi_1(z_1) - |\varphi_1(z_1)| + \varepsilon\int_0^1 v_1(t)\,dt, \right. \\ \left. - \varphi_1(z_1) - |\varphi_1(z_1)| - \varepsilon\int_0^1 v_1(t)\,dt\right\} +$$

$$+ \lambda \max\left\{\varphi_2(z_2) - |\varphi_2(z_2)| + \varepsilon\int_0^1 v_2(t)\,dt, \right. \\ \left. - \varphi_2(z_2) - |\varphi_2(z_2)| - \varepsilon\int_0^1 v_2(t)\,dt\right\} + o(\varepsilon). \quad (3)$$

Из (3) следует, что функционал $\Phi_\lambda(z)$ гиподифференцируем в точке z. Отметим, что гиподифференциальное отображение $d\Phi_\lambda(z)$ является непрерывным в метрике Хаусдорфа [6, 5]. Известно, что необходимое условие минимума функции $\Phi_\lambda(z)$ является

$$[0, 0_{P[0,T]}, 0_{P[0,T]}] \in d\Phi_\lambda(z). \quad (4)$$

Метод гиподифференциального спуска. Выберем произвольное z^0. Пусть уже найдено z^k. Если $\varphi(z^k) = 0$ и выполнено условие (4), то точка z^k является стационарной, и процесс прекращается.

Если же $\varphi(z^k) \neq 0$ или $\varphi(\widetilde{z}^k) = 0$, но условие (4) не выполнено, то возьмем функцию $G_\lambda(z^k)$ — наименьший по норме гипоградиент функционала Φ_λ в точке z^k.

Далее решается задача одномерной минимизации

$$\min_{\beta \geqslant 0} \Phi_\lambda(z^k - \beta G_\lambda(z^k)) = \Phi_\lambda(z^k - \beta_k G_\lambda(z^k)).$$

Теперь положим $z^{k+1} = z^k - \beta_k G_\lambda(z^k)$. Имеем $\Phi_\lambda(z^{k+1}) < \Phi_\lambda(z^k)$. Пользуясь непрерывностью в метрике Хаусдорфа гиподифференциального отображения, как функции z, можно показать, что описанный метод сходится в следующем смысле: $\|G\| \to 0$. Вопрос о существовании предельных точек последовательности $\{z^k\}$ остается открытым.

Работа выполнена при финансовой поддержке Российского фонда фундаментальных исследований (грант № 12-01-00752).

ЛИТЕРАТУРА

[1] *Смирнов В.И., Крылов В.И., Канторович Л.В.* Вариационное исчисление. Л.: КУБУЧ, 1933. 204 с.

[2] *Лаврентьев М.А., Люстерник Л.А.* Основы вариационного исчисления. Том 1. Часть 2. М.-Л.: ОНТИ, 19195. 400 с.

[3] *Гюнтер Н.М.* Курс вариационного исчисления. М.: Гостехиздат, 1941. 308 с.

[4] *Гельфанд И.М., Фомин С.В.* Вариационное исчисление. М.: Физматгиз, 1961. 228 с.

[5] *Демьянов В.Ф.* Условия экстремума и вариационные задачи. М.: Высшая школа, 2005. 335 с.

[6] *Демьянов В.Ф., Рубинов А.М.* Основы негладкого анализа и квазидифференциальное исчисление. М.: Наука, 1990. 432 с.

Implicit Multifunctions Theorems: recent developments

Théra M.
michel.thera@unilim.fr
XLIM and University of Limoges

Setting up equations or equations systems, and solving them is an important problem in mathematics. A central question is whether the solutions to an equation involving parameters, that is, the equation generally speaking form $F(x,p)=0$, may be viewed as a function of those parameters. Such a function is called an **implicit function** defined by the equation

$$F(x,p)=0, \tag{1}$$

Finding an implicit function is one of the most important, and one of the oldest paradigm in modern mathematics, and has many applications to algebra, differential geometry, differential topology, functional analysis, partial differential equations, and many other areas of mathematics. The first idea for implicit function theorems goes back to Newton and later to Leibniz, Lagrange, Cauchy, Dini, Nash. The classical implicit function theorem is a device to solve equations and systems of equations.

When the target space is m-dimensional, solving Equation (1) is equivalent to finding the solution of the system of equations

$$F_i(x,p)=0, \quad i=1,\ldots,m. \tag{2}$$

and it is well known that if X is of dimension n and $m = n$, then, if $(\bar{x}, \bar{p})$ satisfies (2) and the partial gradients of the $F_i, i = 1, \ldots, m$ with respect to x are continuous and linearly independent at $(\bar{x}, \bar{p})$, then the classical implicit function theorem tells us that for any p near $\bar{p}$, there is a unique solution $x = s(p)$ of (2); furthermore the function s is continuous at p. However, the classical implicit function theorem **cannot handle** a system of inequalities, of the form

$$F_i(x, p) \leqslant 0, \quad i = 1, \ldots, m. \tag{3}$$

These systems are extremely important in optimization problems with inequalities constraints. So, it is necessary to consider equations in the form (1), where $F : X \times P \rightrightarrows Y$ is a set-valued mapping, which defines (3). The study of the behavior of such parameterized generalized equations is related to implicit multifunction theorems and plays a central role in variational analysis, especially, in investing problems of sensitivity analysis with respect to parameters. A recent book by Dontchev & Rockafellar highlights many variant forms of implicit multifunction theorems as well as their their applications. The study of implicit multifunction theorems arises in problems related to metric regularity, open covering properties, variational inequalities and many other areas. Similarly to the classical implicit function theorem, we also want to setting up a sufficient condition for the existence of a solutionto a generalized equation and give the formular for calculating of the derivative (coderivative) of the implicit multifunction (if may be).
In this presentation, we use an approach based on the error bound property of the lower semicontinuous envelope of distance functions to the images of set-valued mappings to derive implicit multifunction results. This approach was introduced by Ngai & Théra and allows to avoid the completeness of the image space.

Through these new characterizations, it is possible to investigate implicit multifunction theorems based on coderivatives and on contigent derivatives, as well as the perturbation stability of implicit multifunctions.

Nonlinear extremal problems and approximation theory

Tikhomirov V.
vmtikh@googlemail.com
Moscow State University, Russia

In this report the intercommunication between general principles of optimization and the theory of approximation and recovery will be illustrated by solution of some concrete extremal problems of this theory (such as

Landau-Kolmogorov inequalities, polynomial inequalities, extremal properties of splines, optimal recovery of functions and operators etc).

Quantitative Semicontinuity Properties of Variational Systems with Application to Parametric Optimization

Uderzo A.
amos.uderzo@unimib.it
University of Milano-Bicocca, Via Bicocca degli Arcimboldi, 8, Milan 20126, Italy

Given a set-valued mapping $F : P \times X \longrightarrow 2^Y$ and a function $f : P \times X \longrightarrow Y$, since the seminal paper [3] by *parametric generalized equation* the problem is meant of finding $x \in X$ such that

$$(\mathcal{GE}_p) \qquad f(p,x) \in F(p,x),$$

with the variable p playing the role of a parameter subject to perturbation. The related solution mapping $G : P \longrightarrow 2^X$ implicitly defined by $(\mathcal{GE}_p)$, namely

$$G(p) = \{x \in X : \ f(p,x) \in F(p,x)\},$$

is often referred to as a *variational system*. Variational systems are an abstract formalism that appears in a variety of particular forms within several topics of nonsmooth optimization and variational analysis: for instance, as constraining systems, as nonlinear programming problems, as variational conditions (including variational inequalities, complementarity problems, fixed points and equilibria).

The present talk is mainly devoted to the analysis of some stability properties of variational systems associated with a parametric generalized equation $(\mathcal{GE}_p)$. Whenever P, X and Y are endowed with a metric space structure, it becomes possible to investigate “quantitative"[1] semicontinuity properties of G, starting from adequate assumptions on the problem data f and F. In particular, the analysis here proposed focusses its attention on the following two properties of G (see [1, 2, 4]):

— *Lipschitz lower semicontinuity* at a point $(\bar{p}, \bar{x}) \in P \times X$, with $\bar{x} \in G(\bar{p})$, which postulates the existence of positive constants ζ and l such that

$$G(p) \cap B(\bar{x}, ld(p,\bar{p})) \neq \varnothing, \quad \forall p \in B(\bar{p}, \zeta);$$

— *calmness* at a point $(\bar{p}, \bar{x}) \in P \times X$, with $\bar{x} \in G(\bar{p})$, which postulates the existence of positive constants ζ, δ and ℓ such that

$$G(p) \cap B(\bar{x}, \delta) \subseteq B(G(\bar{p}), \ell d(p,\bar{p})), \quad \forall p \in B(\bar{p}, \zeta).$$

[1]Note that the adjective “quantitative"is to be read here in contrast to “merely topological".

Here $B(x,r)$ denotes the open ball centered at x with radius $r > 0$, whereas $B(S,r)$ denotes the r-enlargement of set S in a metric space. Notice that both such properties are of different nature with respect to the Aubin property (see [2, 4]), which has been the subject of many investigations for specific as well as for abstract variational systems (see [1, 2]). In fact, it turns out that they may occur in circumstances where the latter fails. The technique employed in the analysis relies on the use of derivative-like objects, which are well known in nonsmooth analysis.

As an illustration of the achieved results, some applications are presented to parametric constrained optimization, establishing local solvability and solution stability of extremum problems subject to perturbation near a reference value of the parameter.

References

[1] D. Klatte and B. Kummer, *Nonsmooth equations in optimization. Regularity, calculus, methods and applications*, Kluwer, Dordrecht, (2002).

[2] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation, I: Basic Theory*, Springer, Berlin, (2006).

[3] S.M. Robinson, “Generalized equations and their solutions. I. Basic theory. Point-to-set maps and mathematical programming”, *Math. Program. Stud.*, 10, 128–141 (1979).

[4] R.T. Rockafellar and J.-B. Wets, *Variational Analysis*, Springer, Berlin, (1998).

Solving Piecewise Linear Programming problems using codifferentials

Ugon J., Bagirov A.

j.ugon@ballarat.edu.au, a.bagirov@ballarat.edu.au

SITE, University of Ballarat, Ballarat, Australia

In this talk, necessary and sufficient local and global optimality conditions for continuous piecewise linear functions are investigated. These conditions are formulated using the concept of codifferential. Since any continuous piecewise linear function can be represented as a maxima of minima of linear functions we use this representation to demonstrate that in many practical situations these conditions can be efficiently checked.

Algorithms for local and global minimization of functions represented as a difference of two polyhedral functions are proposed. These algorithms are based on the concept of codifferential. We prove that the proposed algorithms are finite convergent. Examples are presented to demonstrate the performance of the global search algorithm.

Reduction of upper exhausters

Urbański R., Grzybowski J., Pallaschke D.
rich@amu.edu.pl, jgrz@amu.edu.pl, lh09@rz.uni-karlsruhe.de
Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland, University of Karlsruhe, Kaiserstr. 12, 76128 Karlsruhe, Germany

We present equivalent conditions for reducing upper exhauster to one set or to an element of Minkowski–Rådström–Hörmander space. We connect reducing upper exhauster with the property of translation of the intersection of a family of convex sets. We also discuss the minimality of upper exhausters and answer Demyanov's question on the uniqueness of minimal exhauster.

References

[1] V. F. Demyanov and V. A. Roshchina, "Exhausters and subdifferentials in nonsmooth analysis", *Optimization*, 57, 41-–56 (2008).

[2] J. Grzybowski, D. Pallaschke and R. Urbański, "Reduction of finite exhausters", *J. Glob. Optim.*, 46, 589-601, (2010).

Nonsmooth Analyze Methods for Constructions of Differential Games and Control Problems Solutions Approximations

Ushakov V.N., Uspenskii A.A., Lebedev P.D.
ushak@imm.uran.ru, uspen@imm.uran.ru, pleb@yandex.ru
Institute of Mathematics and Mechanics of the Ural Branch of RAS , 620990, Ekaterinburg, S. Kovalevskaya str., 16, Russia

Both numerical and analytical and difference algorithms for approximate solutions of differential games and control problems are proposed using convex and nonsmooth analyze methods.

Difference operators and connected mesh algorithms of differential game coast function approximate calculation for dynamic conflict controlled system considered on a finite time interval [1]. These algorithms are based on unification constructions [2, 3]. Regularization of nonsmooth functions defined in points of a mesh with local convex down (or up) hulls [4, 5]. Sub differentials (or super differentials) of convex hulls and quasi differentials (by V.F. Demyanov) are used instead of differences of functions, which are traditional in mesh schemes [6].

The problem of appearance of optimal result function singularities in control problem with simple movements is studied. The singularities research technique uses local diffeomorphism properties [7]. Elements of numerical and analytical procedures for control sets construction with quite common properties of the target set boarder are given [8, 9]. Finite-difference operator for optimal result function approximation based on generalized derivative (which is sometimes similar to Schwartz derivative) is suggested.

Application of the researched results for numerical construction of Hamilton type PDE Cauchy and Dirichlet problems generalized (minimax) solutions is discussed [10].

The study is illustrated with computing modeling of dynamical problems solutions results.

This work was supported by the Presidium of RAS Program "Mathematical Control Theory" (with finance support of the Ural Branch of RAS, project no. 09-P-1-1015) and by the Russian Foundation for Basic Research (projects no. 11-01-12088 ofi-m-2011 and no. 11-01-00427-a)

References

[1] N.N. Krasovskii and A.I. Subbotin. *Positional Differential Games* Nauka, Moscow, 1974. [in Russian].

[2] N.N. Krasovskii "About Differential Game Unification Problem", *Dokl AN SSSR*, Vol. 226, no 6. Pp. 1260–1263. (1976). [in Russian].

[3] V.N. Ushakov "On Stable Bridge Construction Problem in Approaching and Deviation Differential Game" *Izv AN SSSR. Techn Cybern.* No 4. Pp.29–36. (1980). [in Russian].

[4] A.M. Tarasyev, A.A. Uspenskii, and V.N. Ushakov. "Approximation Operators and Finite-Derivative Schemes for Generalized Solutions of Hamilton-Jacobi Equations Solutions" *Izv AN SSSR. Techn Cybern.* No 3. Pp. 173-185. (1994). [In Russian].

[5] T.B. Tokmantsev, A.A. Uspenskii, and V.N. Ushakov. "Computing Approximations Of Stable Bridges in Differential Games on a Finite Time Interval" *Control Theory and Hamilton–Jacobi Equations Generalized Solution Theory. Papers of the Seminar Dedicated to the 60th Anniversary of A.I. Subbotin.* , Ekaterinburg. Vol 1, Pp. 294–302. (2006). [In Russian].

[6] V.F.Demyanov and A.M.Rubinov *Quasidifferential Calculus.* New York. Optimization Software. (1986).

[7] V.I. Arnol'd. *Singularities of Caustics and Wave Fronts.* Springer-Verlag, Berlin, 2001.

[8] A.A. Uspenskii and P.D. Lebedev. "Optimal Result Function Construction in Velocity Problem Based on Symmetry Set" *Automation and Remote Control.* No 7. Pp. 50–57. (2009). [In Russian].

[9] A.A. Uspenskii and P.D. Lebedev. "Singularity Set Construction Algorithms for One Type of Velocity Ptoblem" *Vestnik Udm Univer. Math, Maech and Comp Science.* Izhevsk. No 3. Pp.30–41. (2010). [In Russian].

[10] A.I. Subbotin. *Generalized Solutions of First-Order PDEs. The Dynamical Optimization Perspective.* Boston, Birkhauser. (1995).

Numerical solution for minimax problems

Vinogradova T., Pupysheva J.
vinogradova@inbox.ru, j_poupycheva@mail.ru
Санкт-Петербургский государственный электротехнический университет, ул. Профессора Попова, д.5, Санкт-Петербург, 197376, Россия,
Санкт-Петербургский государственный университет, факультет прикладной математики — процессов управления, Университетский пр., д.35, Санкт-Петербург, 198504, Россия

The study of a wide class of extremal problems with nonsmooth criterion functions can be done by making use of different approaches. In the talk, for solving minimax problems, the theorem of Grachev-Evtushenko [3] is employed. By means of this theorem a minimax problem is reduced to a Cauchy problem for ODEs. Then, the problem thus obtained is reduced to a polynomial Cauchy problem, which, in turn, is solved by means of the Tailor series method. The method proposed is used to solve one of well-known minimax problems - namely, the Mandelstam one.

О решении минимаксных задач

Исследование обширного класса экстремальных задач с негладкой целевой функцией, в частности минимаксных задач, можно проводить в разных направлениях. Некоторые из возможных подходов описаны в [1]. Построение приближенных методов на основе необходимых условий различной силы предлагается в [2]. Мы же собираемся воспользоваться другими соображениями. Будем опираться на теорему Грачева-Евтушенко [3], которая сводит минимаксную задачу к задаче Коши для обыкновенных дифференциальных уравнений. Далее полученная задача сводится к полиномиальной задаче Коши, к которой применяется метод рядов Тейлора [4]. Предлагаемый метод применяется к одной из минимаксных задач, а именно к задаче Мандельштама: пусть $X = (x_1, x_2, \ldots, x_n)$ и $F(X,t) = |\sum_{k=1}^{n} \cos(kt + x_k)|$. Среди всех $X \in E_n$ требуется найти вектор X^* такой, что $\max\limits_{t\in[0,2\pi]} F(X^*,t) = \min\limits_{\{X\}} \max\limits_{t\in[0,2\pi]} F(X,t)$.

ЛИТЕРАТУРА

[1] Демьянов В.Ф., Малоземов В.Н., *Введение в минимакс*, М.: Наука, (1972).

[2] Виноградова Т.К., “О необходимых условиях в минимаксных задачах управления”, *Известия Санкт-Петербургского электротехнического института*, No. 449, 39–44 (1992).

[3] Евтушенко Ю.Г., *Методы решения экстремальных задач и их применение в системах оптимизации*, М.: Наука, (1983).

[4] Babadzhanjanz L.K., Sarkissian D.R., “Taylor series method for dynamic systems with control: convergence and error estimates”, *Journal of Mathematical Sciences. Springer New York*, Vol. 139, No. 6, 7025-7046 (2006).

Prediction of Network Dynamics under Various Time and Uncertainty Assumptions. Investigations in Education, Finance, Economics, Biology and Medicine - Recent Contributions Supported by Optimization

Weber G., Kuter S., Kerimov A., Temoçin B., Kürüm E.

gweber@metu.edu.tr, semihkuter@yahoo.com,
azer.kerimov@gmail.com, busras@gmail.com, efsun.kurum@gmail.com

Institute of Applied Mathematics, Middle East Technical University, Ankara, 06800, Turkey, Department of Forest Engineering, Faculty of Forestry, Cankiri Karatekin University, Cankiri, 18200, Turkey

This presentation introduces recent research efforts and results in identifying and predicting time dependent processes, with various degrees of model discontinuity and uncertainty. This generalization is gradually unfolded, while always motivated by real-world challenges and applications of them.

We aim at displaying joy and interest in state-of-the art optimization and to invite to education, research and developing our countries by its help, in our scientific community. In fact, we try to "make appetite" for this.

References

[1] S.Z. Alparslan Gök, G.-W. Weber, "Cooperative games under elipsoidal uncertainty in the proceedings of PCO 2010, 3rd Global Conference on Power Control and Optimization, February 2-4, 2010, Gold Coast, Queensland, Australia (ISBN: 978-983-44483-1-8).

[2] E. Kropat, G.-W.Weber, and P.C. Sekhar, "Regulatory networks under ellipsoidal uncertainty - optimization theory and dynamical systems preprint no. 138, Institute of Applied Mathematics, METU, 2009.

[3] S. Pickl, and G.-W. Weber, "Optimization of a time-discrete nonlinear dynamical system from a problem of ecology - an analytical and numerical approach Journal of Computational Technologies 6, 1, 2001, pp. 43-52.

[4] G.-W. Weber, S.Z. Alparslan-Gök, and B. Söyler, "A new mathematical approach in environmental and life sciences: gene-environment Networks and their dynamics Environmental Modeling & Assessment 14, 2 (April 2009), 2009a, pp. 267-288.

[5] G.-W. Weber, R. Branzei, and S.Z. Alparslan-Gök, "On the elipsoidal core for cooperative games under ellipsoidal uncertainty in the Proceedings of 2nd International Conference on Engineering Optimization, September 6-9 2010, Lisbon, Portugal.

Smoothness Versus Nonsmoothness in the Modeling of Economic Equilibrium: the Need for a New Paradigm

Wets R.
rjbwets@ucdavis.edu
Department of Mathematics, University of California, Davis

In the theory of economic equilibrium, rigorous mathematical models with proofs of existence did not arrive until the 1950's. At first, full attention was paid to the possibility that some agents might start with, or end up with, zero quantities of some goods at their survival boundaries. Soon after, because ways of working with inequality constraints had not yet been developed well, boundary behavior was shunned. Assumptions precluded it, and the subject turned to tools of smooth differential geometry and topology aimed chiefly at generic results. Now this all could, and should, change.

Some Results on the Equivalence of Demyanov Difference and Minkowski Difference

Xia Z., Lin S.
zqxiazhh@dlut.edu.cn, lsdsl@163.com
School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China

Demyanov difference is an important concept in non-smooth analysis and optimization, especially in quasidifferentiable analysis in the sense of Demyanov and Rubinov. It is important and necessary, especially for the kernelled quasidifferential of a quasidifferentiable function, to study the problem of the equivalence of the Demyanov difference and the Minkowski difference of two nonempty convex compact sets. Gao gave a sufficient condition for the case in R^n $(n \geqslant 2)$. Song *et al* gave a necessary and sufficient condition for the case in R^2.

However a necessary and sufficient condition for the case of R^n $(n > 2)$ has not been given. In this paper, this problem is transformed into an equivalent problem of solving a *generalized equation*

$$P_{A \dot{-} B}(v) = P_{A-B}(v), \quad \forall\, v \in R^n,$$

which is presented by using the support function $P_{A \dot{-} B}(\cdot)$ of the Demyanov difference of two nonempty convex compact sets $A \dot{-} B$, where $A, B \in Y_n$ and Y_n is the set of all nonempty convex compact subsets in R^n. The equivalence of the Demyanov difference and the Minkowski difference of two nonempty convex compact sets in R^n is equivalent to the existence of the solution of the generalized equation, and some properties of the generalized equation are given. In order to conveniently determine whether the sets A and B satisfy the equality

$$A \dot{-} B = A - B.$$

The existence problem of the solution of the generalized equation can be transformed to the following problem: finding A, $B \in Y_n$, such that for any $v \in R^n$, there exists a $u \in R^n$, the following inequalities

$$\partial P_A(u) \cap \partial P_A(v) \neq \varnothing \quad \text{and} \quad \partial P_B(u+v) \cap \partial P_B(-v) \neq \varnothing$$

hold, where ∂"denotes the symbol of subdifferential in the sense of convex analysis. Some properties of the sets A and B which can satisfy the above problem are given.

References

[1] V. F. Demyanov and A. M. Rubinov, *Constructive Nonsmooth Analysis*, Verlag Peter Lango, (1995).

[2] V. F. Demyanov and A. M. Rubinov, *Quasidifferentiability and Related Topics*, Kluwer Academic Publishers, Netherlands, (2000).

[3] Yan Gao, *Representative of quasidifferentials and its formula for a quasidifferentiable function*, Set-Valued Analysis, 13, 323-336(2005).

[4] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, (1970).

Conically equivalent convex sets and applications

Zaffaroni A.

alberto.zaffaroni@unimore.it

Universitá di Modena, viale Berengario 51, 41121, Modena, Italy

Given a l.s.c. sublinear function h on a normed space X and a cone $K \subseteq X$, two closed, convex sets A and B in X^* are said to be K-equivalent if the support functions of A and B coincide on K. We characterize the greatest set in an equivalence class, analyze the equivalence between two sets, find conditions for the existence and the uniqueness of the least element with respect to inclusion, extending previous results. The main assumption is that the cone K is open. A relevant role is played by the set of weak* support points of A determined by elements of K.

We give some applications to the study of sublinear gauges of convex radiant sets and of superlinear cogauges of convex coradiant sets. Moreover we study the minimality of a second order hypodifferential.

Space Transformation in Quasi-Newton Methods and Methods of Conjugate Directions (in Russian)

Zhurbenko N.

zhurbnick@yandex.ru

Институт кибернетики НАН Украины, проспект Глушкова, 40, г. Киев, 03187, Украина

Преобразование пространства в квазиньютоновких методах и методах сопряженных градиентов

Общая схема квазиньютоновских методов безусловной минимизации функции $f(x)$ в R^n :

$$x_k = x_{k-1} - h_k H_k g_{k-1}, k = 1, 2, \dots, \tag{1}$$

где h_k шаговый множитель, $g(x_{k-1})-$ градиент функции $f(x)$ в точке x_{k-1}, H_k – симметричная положительно определенная матрица. Матрица H_k рекуррентно пересчитывается с выполнением так называемого квазиньютоновского условия:

$$H_{k+1}(g_k - g_{k-1}) = h_k H_k g_{k-1}. \tag{2}$$

Шаг квазиньтоновского метода соответствует шагу градиентного метода в пространстве с определяемой матрицей H_k^{-1} метрике. Поэтому квазиньтоновские методы обычно называют методами переменной метрики. Квазиньтоновские методы обеспечивают решение задачи минимизации квадратичной функции $f(x) = (Cx, x) + (b, x)$ за n шагов. При этом $H_n = C^{-1}$.

Общеизвестна следующая интерпретация квазиньтоновского метода с позиции преобразования пространства. Представим H_k в виде $H_k = B_k B_k^T$, где B_k невырожденная матрица $n \times n$. Положим $A_k = B_k^{-1}$. Пусть Y "преобразованное" соответствующим матрице A_k линейным оператором пространство: $Y = A_k X$ Градиент $\tilde{g}(y)$ функции $\tilde{f}(y) = f(B_k x)$ соответствующей функции $f(x)$ в пространстве Y равен $B_k^T g(x)$. Поэтому шаг квазиньютоновского метода соответствует шагу градиентного алгоритма в "преобразованном" пространстве.

Для квадратичной функции $\tilde{f}(y) = (\tilde{C}_{k+1} y, y) + (B_{k+1}^T b, y)$, где $\tilde{C}_{k+1} = B_{k+1}^T C B_{k+1}$ – матрица квадратичной части функции в преобразованном пространстве. Положим: $p_k = H_k g_{k-1}$, $\tilde{p}_k = A_{k+1} p_k$, (p_k - направление движения алгоритма (1), $\tilde{p}_k$ - направление в преобразованном пространстве, соответствующее направлению p_k).

Пусть матрица преобразования B_{k+1} удовлетворяет следующему условию

$$\tilde{C}_k \tilde{p}_k = \tilde{p}_k. \tag{3}$$

Справедливо следующее утверждение.

Квазиньтоновское условие (2) и условие (3) (для квадратичной функции) эквивалентны.

Естественная форма корректировки обратной матрицы преобразования пространства B_k имеет вид $B_{k+1} = B_k T_k$ (такая форма соответствует последовательному преобразованию пространства). Нетрудно показать, что для квадратичной функции условие (3) эквивалентно следующему (не содержащую явно матрицу) условию

$$T_k T_k^T (\tilde{g}_k - \tilde{g}_{k-1}) = h_k \tilde{g}_{k-1} \quad (\tilde{g}_k = B_k^T g_k, \quad \tilde{g}_{k-1} = B_k^T g_{k-1}) \tag{4}$$

Таким образом, условия (4) и (2) эквивалентны. Этот факт легко устанавливается непосредственно из условия (2) без использования условия (3). Отметим, что представление известных вариантов квазиньютоновских методов (Давидона–Флетчера–Пауэлла, Бройдена–Флетчера–Шенно) в форме преобразования пространства приведено, например, в работе [1].

Пусть матрица преобразования B_{k+1} удовлетворяет следующему условию (вектор $\tilde{p}_k$ является собственным вектором матрицы квадратичной функции в преобразованном пространстве; собственное значение вектора равно λ)

$$\tilde{C}_k \tilde{p}_k = \lambda \tilde{p}_k, \tag{5}$$

где $\lambda > 0$ (условие (3) соответствует (5) при $\lambda = 1$).

Оказывается, что градиентный метод с преобразованием пространства при выполнении условия (5) является методом сопряженных градиентов. Само условие (5) соответствует известному условию C-ортогональности [2], [3]:

$$H_k C p_j = \lambda p_k, j = 0, 1, \ldots, k-1. \tag{6}$$

Условие C-ортогональности (6) является основой построения различных вариантов алгоритма сопряженных направлений [3] в форме (1).

Корректировка матрицы H_k в классических вариантах методов (1) определяется в аддитивной форме: $H_{k+1} = H_k + \Delta H_k$. При этом рекуррентные формулы пересчета матриц H_k не используют в явном виде представление этих матриц в форме произведения $B_k B_k^T$. Для алгоритмов (1) в форме преобразования пространства рекурентные соотношения определяются непосредственно для матриц B_k. При этом положительная определенность матриц $H_k = B_k B_k^T$ обеспечивается, в определенном смысле, независимо от погрешностей вычисления матриц B_k. Это обеспечивает численную устойчивость алгоритмов.

Для квадратичной функции при выполнении условия (5) получаем следующую геометрическую интерпретацию алгоритмов (1) в форме преобразования пространства. Образ траектории метода (1) в текущем преобразованном пространстве соответствует движению по (взаимно ортогональным) собственным векторам матрицы $\tilde{C}_k$, соответствующей матрице квадратичной функции в этом пространстве.

Приведенная интерпретация квазиньютоновского условия (2) и условия C-ортогональности (6) позволяет строить новые модификации методов рассматриваемого класса. Так в работах [4], [5] разработано семейство алгоритмов на основе использования условия (5) и операторов растяжения пространства [6]. Как частный случай в это семейство входит предельный вариант r-алгоритма [6] с бесконечным коэффициентом растяжения.

ЛИТЕРАТУРА

[1] Стецюк П.И., “Линейные операторы в квазиньютоновских методах. ” *Теория и приложения методов оптимизации.*, Киев, Ин-т кибернетики им. В.М.Глушкова НАН Украины, с.3–8 (1998).

[2] H. Y. Huang, “Unified approach to quadratically convergent algorithms for function minimization”, *J. Optim. Theory and Appl.*, v. 5, 405–423 (1970).

[3] Пшеничный Б.Н., Данилин Ю.М., *Численные методы в экстремальных задачах*, -М.: Наука, (1975).

[4] Журбенко Н.Г., “Построение семейства методов сопряженных направлений на основе использования оператора растяжения пространства ” *Теория и приложения методов оптимизации.*, Киев, Ин-т кибернетики им. В.М.Глушкова НАН Украины, с.12–18 (1998).

[5] Журбенко Н.Г., “Квазиньтоновские алгоритмы минимизации на основе использования оператора растяжения пространства ” *Теория и приложения методов оптимизации.*, Киев, Ин-т кибернетики им. В.М.Глушкова НАН Украины, с.45–50 (1999).

[6] Шор Н.З., Журбенко Н.Г., “Метод минимизации, использующий операцию растяжения пространства в направлении разности двух последовательных градиентов” *Кибернетика*, No. 3, 51–59 (1971).

AUTHOR INDEX — АВТОРСКИЙ УКАЗАТЕЛЬ

Научное издание

Конструктивный негладкий анализ и смежные вопросы

Тезисы докладов международной конференции

Компьютерная верстка: *Г. Ш. Тамасян*

Подписано к печати 14.05.12. Формат 60×84 1/16.
Бумага офсетная. Печать цифровая. Печ. л. 10,58.
Тираж 160 экз. Заказ 5466.
Отпечатано в Отделе оперативной полиграфии
химического факультета СПбГУ

198504, Санкт-Петербург, Старый Петергоф, Университетский пр., 26
Тел.: (812) 428-40-43, 428-69-19