An introduction to Donaldson-Witten theory

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Abstract

In these lecture notes a pedagogical introduction to Donaldson-Witten theory is given. After a survey of four-manifold topology, some basic aspects of Donaldson theory are presented in detail. The physical approach to Donaldson theory is based on topological quantum field theory (TQFT), and some general properties of TQFT's are explained. Finally, the TQFT underlying Donaldson theory (which is usually called Donaldson-Witten theory) is constructed in detail by twisting $\mathcal{N} = 2$ super Yang-Mills theory.

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Contents

1 Introduction

Donaldson-Witten theory has played an important role both in mathematics and in physics. In mathematics, Donaldson theory has been a fundamental tool in understanding the differential topology of four-manifolds. In physics, topological Yang-Mills theory, also known as Donaldson-Witten theory, is the canonical example of a topological quantum field theory. The development of the subject has seen a remarkable interaction between these two different approaches -one of them based on geometry, and the other one based on quantum field theory.

In these lectures, we give a self-contained introduction to Donaldson-Witten theory. Unfortunately, we are not going to be able to cover the whole development of the subject. A more complete treatment can be found in [30]. The organization of the lectures is as follows: in section 2, we review some elementary properties of four-manifolds. In section 3, we present Donaldson theory from a rather elementary point of view. A more detailed and rigorous point of view can be found in [13]. In section 4, we give a quick introduction to $\mathcal{N}=1$ supersymmetry. In section 5, we present the physical theory behind Donaldson-Witten theory, *i.e.* $\mathcal{N}=2$ supersymmetric Yang-Mills theory. In section 6, we give a general overview of topological field theories and we explain the twisting procedure. In section 7, we construct Donaldson-Witten theory in detail and show that its correlation functions are in fact the Donaldson invariants introduced in section 3. Finally, in section 8, we give a very brief overview of recent developments.

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2 Basics of four-manifolds

The purpose of this section is to collect a series of more or less elementary facts about the topology of four-manifolds that will be used in the rest of these lectures. We haven't made any attempt to be self-contained, and the reader should consult for example the excellent book [20] for a more complete survey. The first chapter of [13] gives also a very good summary. A general warning: in these lectures we will assume that the four-manifolds under consideration are closed, compact and orientable. We will also assume that they are endowed with a Riemannian metric.

2.1 Homology and cohomology

The most basic classical topological invariants of a four-manifold are the homology and cohomology groups $H_i(X, \mathbf{Z})$, $H^i(X, \mathbf{Z})$. These homology groups are abelian groups, and the rank of $H_i(X, \mathbf{Z})$ is called the i-th *Betti number* of X, and denoted by b_i. Remember that by Poincaré duality one has

$$H^i(X, \mathbf{Z}) \simeq H_{n-i}(X, \mathbf{Z}). \tag{2.1}$$

and hence $b_i = b_{n-i}$. We will also need the (co)homology groups with coefficients in other groups like $\mathbf{Z}_2$. To obtain these groups one uses the universal coefficient theorem, which states that

$$H_i(X, G) \simeq H_i(X, \mathbf{Z}) \otimes_{\mathbf{Z}} G \oplus \mathrm{Tor}(H_{i-1}(X, \mathbf{Z}), G). \tag{2.2}$$

Let's focus on the case $G = \mathbf{Z}_p$. Given an element x in $H_i(X, \mathbf{Z})$, one can always find an element in $H_i(X, \mathbf{Z}_p)$ by sending $x \to x \otimes 1$. This in fact gives a map:

$$H_i(X, \mathbf{Z}) \to H_i(X, \mathbf{Z}_p) \tag{2.3}$$

which is called the *reduction mod p* of the class x. Notice that, by construction, the image of (2.3) is in $H_i(X, \mathbf{Z}) \otimes \mathbf{Z}_p$. Therefore, if the torsion part in (2.2) is not zero, the map (2.3) is clearly not surjective. When the torsion product is zero, any element in $H_i(X, \mathbf{Z}_p)$ comes from the reduction mod p of an element in $H_i(X, \mathbf{Z})$. For the cohomology groups we have a similar result. Physicists are more familiar with the *de Rham cohomology groups*, $H^*_{DR}(X)$ which are defined in terms of differential forms. These groups are defined over $\mathbf{R}$, and therefore they are insensitive to the torsion part of the singular cohomology. Formally, one has $H^i_{DR}(X) \simeq (H^i(X, \mathbf{Z})/\mathrm{Tor}(H^i(X, \mathbf{Z}))) \otimes_{\mathbf{Z}} \mathbf{R}$.

Remember also that there is a nondegenerate pairing in cohomology, which in the de Rham case is the usual wedge product followed by integration. We will denote the pairing of the cohomology classes (or differential form representatives) α, β by (α, β).

Let's now focus on dimension four. Poincaré duality gives then an isomorphism between $H_2(X, \mathbf{Z})$ and $H^2(X, \mathbf{Z})$. It also follows that $b_1(X) = b_3(X)$. Recall that the Euler characteristic $\chi(X)$ of an n-dimensional manifold is defined as

$$\chi(X) = \sum_{i=0}^{n} (-1)^i b_i(X). \tag{2.4}$$

For a connected four-manifold X, we have then, using Poincaré duality, that

$$\chi(X) = 2 - 2b_1(X) + b_2(X). \tag{2.5}$$

2.2 The intersection form

An important object in the geometry and topology of four-manifolds is the *intersection form*,

$$Q : H^2(X, \mathbf{Z}) \times H^2(X, \mathbf{Z}) \to \mathbf{Z} \tag{2.6}$$

which is just the pairing restricted to the two-classes. By Poincaré duality, it can be defined on $H_2(X,\mathbf{Z}) \times H_2(X,\mathbf{Z})$ as well. Notice that Q is zero if any of the arguments is a torsion element, therefore one can define Q on the torsion free parts of homology and cohomology.

Another useful way to look at the intersection form is precisely in terms of intersection of submanifolds in X. One fundamental fact in this respect is that we can represent any two-homology class in a four-manifold by a closed, oriented surface S: given an embedding

$$i : S \hookrightarrow X, \tag{2.7}$$

we have a two-homology class $i_*([S]) \in H_2(X,\mathbf{Z})$, where $[S]$ is the fundamental class of S. Conversely, any $a \in H_2(X,\mathbf{Z})$ can be represented in this way, and $a = [S_a]$ [20]. One can also prove that

$$Q(a,b) = S_a \cup S_b,$$

where the right hand side is the number of points in the intersection of the two surfaces, couted with signs which depend on the relative orientation of the surfaces. If, moreover, η_{S_a}, η_{S_b} denote the Poincaré duals of the submanifolds S_a, S_b (see [8]), one has

$$Q(a,b) = \int_X \eta_{S_a} \wedge \eta_{S_b} = Q([\eta_{S_a}], [\eta_{S_b}]).$$

If we choose a basis $\{a_i\}_{i=1,\ldots,b_2(X)}$ for the torsion-free part of $H_2(X,\mathbf{Z})$, we can represent Q by a matrix with integer entries that we will also denote by Q. Under a change of basis, we obtain another matrix $Q \to C^T Q C$, where C is the transformation matrix. This matrix is obviously symmetric, and it follows by Poincaré duality that it is unimodular, *i.e.* it has $\det(Q) = \pm 1$. If we consider the intersection form on the real vector space $H_2(X,\mathbf{R})$, we see that it is a symmetric, bilinear, nondegenerate form, and therefore it is classified by its rank and its signature. The rank of Q, $\mathrm{rk}(Q)$, is clearly given by $b_2(X)$, the second Betti number. The number of positive and negative eigenvalues of Q will be denoted by $b_2^+(X)$, $b_2^-(X)$, respectively, and the *signature* of the manifold X is then defined as

$$\sigma(X) = b_2^+(X) - b_2^-(X). \tag{2.8}$$

We will say that the intersection form is *even* if $Q(a,a) \equiv 0 \mod 2$. Otherwise, it is *odd*. An element x of $H_2(X,\mathbf{Z})/\mathrm{Tor}(H_2(X,\mathbf{Z}))$ is called *characteristic* if

$$Q(x,a) \equiv Q(a,a) \mod 2 \tag{2.9}$$

for any $a \in H_2(X,\mathbf{Z})/\mathrm{Tor}(H_2(X,\mathbf{Z}))$. An important property of characteristic elements is that

$$Q(x,x) \equiv \sigma(X) \mod 8. \tag{2.10}$$

In particular, if Q is even, then the signature of the manifold is divisible by 8.

Examples.

(1) The simplest intersection form is:

$$n(1) \oplus m(-1) = \mathrm{diag}(1,\ldots,1,-1,\ldots,-1), \tag{2.11}$$

which is odd and has $b_2^+ = n$, $b_2^- = m$.

(2) Another important form is the hyperbolic lattice,

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.12}$$

which is even and has $b_2^+ = b_2^- = 1$.

(3) Finally, one has the even, positive definite form of rank 8

$$E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \tag{2.13}$$

which is the Dynkin diagram of the exceptional Lie algebra E_8.

Fortunately, unimodular lattices have been classified. The result depends on whether the intersection form is even or odd and whether it is definite (positive or negative) or not. Odd, indefinite lattices are equivalent to $p(1) \oplus q(-1)$, while even indefinite lattices are equivalent to $pH \oplus qE_8$. Definite lattices are more complicated, since they involve "exotic" cases. Clearly the intersection form is a homotopy invariant. It turns out that simply connected smooth four-manifolds are completely characterized topologically by the intersection form, *i.e.* two simply-connected, smooth four-manifolds are homeomorphic if their intersection forms are equivalent. This is a result due to Freedman. The classification of smooth four-manifolds up to diffeomorphism is another story, and this is the main reason to introduce new invariants which are sensitive to the differentiable structure. But before going into that, we have to give some more details about classical topology.

2.3 Self-dual and anti-self-dual forms

The Riemannian structure of the manifold X allows us to define the Hodge star operator $*$, which can be used to define an induced metric on the forms by:

$$\psi \wedge *\theta = (\psi, \theta) d\mu,$$

where $d\mu$ is the Riemannian volume element. Since $*^2 = 1$, the Hodge operator has eigenvalues ± 1, hence it gives a splitting of the two-forms $\Omega^2(M)$ in self-dual (SD) and anti-self-dual (ASD) forms, defined as the ± 1 eigenspaces of $*$ and denoted by $\Omega^{2,+}(X)$ and $\Omega^{2,-}(X)$, respectively. Given a differential form $\psi \in \Omega^2(M)$, its self-dual (SD) and anti-self-dual (ASD) parts will be denoted by $\psi^{\pm}$. Explicitly,

$$\psi^{\pm} = \frac{1}{2}(\psi \pm *\psi). \tag{2.14}$$

The Hodge operator lifts to cohomology and in four dimensions it maps

$$* : H^2(X) \to H^2(X). \tag{2.15}$$

The number of $+1$ eigenvalues of $*$ in $H^2(X)$ is precisely b_2^+, and the number of -1 eigenvalues is b_2^-. This means that we can interpret b_2^+ as the number of self-dual harmonic forms on X. This interpretation will be useful in the context of gauge invariants.

2.4 Characteristic classes

An important set of topological invariants of X is given by the characteristic classes of its real tangent bundle. The most elementary ones are the Pontriagin class $p(X)$ and the Euler class $e(X)$, both in $H^4(X, \mathbf{Z}) \simeq \mathbf{Z}$. These classes are then completely determined by two integers, once a generator of $H^4(X, \mathbf{Z})$ is chosen. These integers will be also denoted by $p(X)$, $e(X)$, and they give the Prontriagin number and the Euler characteristic of the four-manifold X, so $e(X) = \chi$. The Pontriagin number is related to the signature of the manifold through the Hirzebruch theorem, which states that:

$$p(X) = 3\sigma(X). \tag{2.16}$$

If a manifold admits an almost-complex structure, one can define a holomorphic tangent bundle $T^{(1,0)}(X)$. This is a complex bundle of rank $r = \dim(X)$, therefore we can associate to it the Chern character $c(T^{(1,0)}(X))$ which is denoted by $c(X)$. For a four-dimensional manifold, one has $c(X) = 1 + c_1(X) + c_2(X)$. Since $c_1(X)$ is a two-form, its square can be paired with the fundamental class of the four manifold. The resulting number can be expressed in terms of the Euler characteristic and the signature as follows:

$$c_1^2(X) = 2\chi(X) + 3\sigma(X). \tag{2.17}$$

Finally, the second Chern class of X is just its Euler class: $c_2(X) = e(X)$. If the almost complex structure is integrable, then the manifold X is complex, and it is called a complex surface. Complex surfaces provide many examples in the theory of four-manifolds. Moreover, there is a very beautiful classification of complex surfaces due to Kodaira, using techniques of algebraic geometry. The interested reader can consult [4, 6].

There is another set of characteristic classes which is perhaps less known in physics. These are the Stiefel-Whitney classes of real bundles F over X, denoted by $w_i(F)$. They take values in $H^i(X, \mathbf{Z}_2)$, and a precise definition can be found in [20, 31], for example. The Stiefel-Whitney classes of a four-manifold X are defined as $w_i(X) = w_i(TX)$. The first Stiefel-Whitney class of a manifold measures its orientability, so we will always have $w_1(X) = 0$. The second Stiefel-Whitney class plays an important role in what follows. This is a two-cohomology class with coefficients in $\mathbf{Z}_2$, and it has three important properties. If the manifold admits an almost complex structure, then

$$c_1(X) \equiv w_2(X) \mod 2, \tag{2.18}$$

i.e. $w_2(X)$ is the reduction mod 2 of the first Chern class of the manifold. This is a general property of $w_2(X)$ for any almost-complex manifold. In four dimensions, $w_2(X)$ satisfies in addition two other properties: first, it always has a integer lift to an integer class [21] (for example, it the manifold is almost complex, then $c_1(X)$ is such a lift). The second property is the Wu formula, which states that

$$(w_2(X), \alpha) = (\alpha, \alpha) \mod 2, \tag{2.19}$$

for any $\alpha \in H^2(X, \mathbf{Z})$. The l.h.s can be interpreted as the pairing of α with the integer lift of $w_2(X)$. A corollary of the Wu formula is that an integer two-cohomology class is characteristic if and only if it is an integer lift of $w_2(X)$.

2.5 Examples of four-manifolds

(1) A simple example is the four-sphere, $\mathbf{S}^4$. It has $b_1 = b_2 = 0$, and therefore $\chi = 2$, $\sigma = 0$, $Q = 0$.

(2) Next we have the complex projective space $\mathbf{CP}^2$. Recall that this is the complex manifold obtained from $\mathbf{C}^3 - \{(0,0)\}$ by indentifying $z_i \simeq \lambda z_i$, $i = 1, 2, 3$, with $\lambda \neq 0$. $\mathbf{CP}^2$ has $b_1 = 0$ and $b_2 = 1$. In fact, the basic two-homology class is the so-called class of the hyperplane h, which is given in projective coordinates by $z_1 = 0$. It is not difficult to prove that $h^2 = 1$, so $Q_{\mathbf{CP}^2} = (1)$. Notice that h is in fact a $\mathbf{CP}^1$, therefore it is an embedded sphere in $\mathbf{CP}^2$. The projective plane with the opposite orientation will be denoted by $\overline{\mathbf{CP}^2}$, and it has $Q = (-1)$.

(3) An easy way to obtain four-manifolds is by taking products of two Riemann surfaces. A simple example are the so-called product ruled surfaces $\mathbf{S}^2 \times \Sigma_g$, where Σ_g is a Riemann surface of genus g. This manifold has $b_1 = 2g$, $b_2 = 2$. The homology classes have the submanifold representatives $\mathbf{S}^2$ and Σ_g. They have self intersection zero and they intersect in one point, therefore $Q = H$, the hyperbolic lattice, with $b_2^+ = b_2^- = 1$. One then has $\chi = 4(1-g)$.

(4) Our last example is a hypersurface of degree d in $\mathbf{CP}^3$, described by a homogeneous polynomial $\sum_{i=1}^4 z_i^d = 0$. We will denote this surface by S_d. For $d = 4$, one obtains the so-called $K3$ surface.

Exercise 2.1. 1) Compute $c_1(S_d)$ and $c_2(S_d)$. Deduce the values of χ and σ.

2) Use the classification of unimodular symmetric, bilinear forms to deduce Q_{K3} (for help, see [20]).

3 Basics of Donaldson invariants

Donaldson invariants can be mathematically motivated as follows: as we have mentioned, Freedman's results imply that two simply-connected smooth manifolds are homeomorphic if and only if they have the same intersection form. However, the classification of four-manifolds up to diffeomorphism turns out to be much more subtle: most of the techniques that one uses in dimension ≥ 5 to approach this problem (like cobordism theory) fail in four dimensions. For example, four dimensions is the only dimension in which a fixed homeomorphism type of closed four-manifolds is represented by infinitely many diffeomorphism types, and $n = 4$ is the only dimension where there are "exotic" $\mathbf{R}^n$'s, *i.e.* manifolds which are homeomorphic to $\mathbf{R}^n$ but not diffeomorphic to it. One has to look then for a new class of invariants of differentiable manifolds in order to solve the classification problem, and this was the great achievement of Donaldson. Remarkably, the new invariants introduced by Donaldson are defined by looking at instanton configurations of nonabelian gauge theories on the four-manifold. We will give here a sketch of the mathematical procedure to define Donaldson invariants, in a rather formal way and without entering into the difficult parts of the theory. The interested reader can consult the excellent book by Donaldson and Kronheimer [13]. Other useful resources include [16–18], on the mathematical side, and [9, 34] on the physical side. The reference [15] gives a very nice review of the mathematical background.

3.1 Yang-Mills theory on a four-manifold

Donaldson theory defines differentiable invariants of smooth four-manifolds starting from Yang-Mills fields on a vector bundle over the manifold. The basic framework is then gauge theory on a four-manifold, and the moduli space of ASD connections. Here we review very quickly some basic notions of gauge connections on manifolds. A more detailed account can be found for example in [10, 15].

Let G be a Lie group (usually we will take $G = SO(3)$ or $SU(2)$). Let $P \to M$ be a principal G-bundle over a manifold M with a connection A, taking values in the Lie algebra of G, $\mathbf{g}$. Given a vector space V and a representation ρ of G in $GL(V)$, we can form an associated vector bundle $E = P \times_G V$ in the standard way. G acts on V through the representation ρ. The connection A on P induces a connection on the vector bundle E (which we will also denote by A) and a covariant derivative ∇_A. Notice that, while the connection A on the principal bundle is an element in $\Omega^1(P, \mathbf{g})$, the induced connection on the vector bundle E is better understood in terms of a local trivialization U_α. On each U_α, the connection 1-form A_α is a $\mathrm{gl}(V)$ valued one-form (where $\mathrm{gl}(V)$ denotes the Lie algebra of $\mathrm{GL}(V)$) and the transformation rule which glues together the different descriptions is given by:

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + i g_{\alpha\beta}^{-1} d g_{\alpha\beta}, \tag{3.1}$$

where $g_{\alpha\beta}$ are the transition functions of E.

Recall that the representation ρ induces a representation of Lie algebras $\rho_* : \mathbf{g} \to \mathrm{gl}(V)$. We will identify $\rho_*(\mathbf{g}) = \mathbf{g}$, and define the adjoint action of G on $\rho_*(\mathbf{g})$ through the representation ρ. On M one can consider the *adjoint bundle* $\mathbf{g}_E$, defined by:

$$\mathbf{g}_E = P \times_G \mathbf{g}, \tag{3.2}$$

which is a subbundle of $\mathrm{End}(E)$. For example, for $G = SU(2)$ and V corresponding to the fundamental representation, $\mathbf{g}_E$ consists of Hermitian, trace-free endomorphisms of E. If we look at (3.1), we see that the difference of two connections is an element in $\Omega^1(\mathbf{g}_E)$ (the one-forms on X with values in the bundle $\mathbf{g}_E$). Therefore, we can think about the space of all connections $\mathcal{A}$ as an affine space with tangent space at A given by $T_A\mathcal{A} = \Omega^1(\mathbf{g}_E)$.

The curvature F_A of the vector bundle E associated to the connection A can be also defined in terms of the local trivialization of E. On U_α, the curvature F_α is a $\mathrm{gl}(V)$-valued two-form that behaves under a change of trivialization as:

$$F_\beta = g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta}, \tag{3.3}$$

and this shows that the curvature can be considered as an element in $\Omega^2(\mathbf{g}_E)$.

The next geometrical objects we must introduce are gauge transformations, which are automorphisms of the vector bundle E, $u : E \rightarrow E$ preserving the fibre structure (*i.e.*, they map one fibre onto another) and descend to the identity on X. They can be described as sections of the bundle $\mathrm{Aut}(E)$. Gauge transformations form an infinite-dimensional Lie group $\mathcal{G}$, where the group structure is given by pointwise multiplication. The Lie algebra of $\mathcal{G} = \Gamma(\mathrm{Aut}(E))$ is given by $\mathrm{Lie}(\mathcal{G}) = \Omega^0(\mathbf{g}_E)$. This can be seen by looking at the local description, since on an open set U_α the gauge transformation is given by a map $u_\alpha : U_\alpha \rightarrow G$, where G acts through the representation ρ. As it is well-known, the gauge transformations act on the connections as

$$u^*(A_\alpha) = u_\alpha A_\alpha u_\alpha^{-1} + i du_\alpha u_\alpha^{-1} = A_\alpha + i(\nabla_A u_\alpha) u_\alpha^{-1}, \tag{3.4}$$

where $\nabla_A u_\alpha = du_\alpha + i[A_\alpha, u_\alpha]$, and they act on the curvature as:

$$u^*(F_\alpha) = u_\alpha F_\alpha u_\alpha^{-1}. \tag{3.5}$$

3.2 $SU(2)$ and $SO(3)$ bundles

In these lectures we will restrict ourselves to the gauge groups $SU(2)$ and $SO(3)$, and E corresponding to the fundamental representation. Therefore, E will be a two-dimensional complex vector bundle or a three-dimensional real vector bundle, respectively. $SU(2)$ bundles over a compact four-manifold are completely classified by the second Chern class $c_2(E)$ (for a proof, see for example [16]).

In the case of a $SO(3)$ bundle V, the isomophism class is completely classified by the first Pontriagin class

$$p_1(V) = -c_2(V \otimes \mathbf{C}), \tag{3.6}$$

and the Stiefel-Whitney class $w_2(V) \in H^2(X, \mathbf{Z}_2)$. These characteristic classes are related by

$$w_2(V)^2 = p_1(V) \bmod 4. \tag{3.7}$$

$SU(2)$ bundles and $SO(3)$ bundles are of course related: given an $SU(2)$ bundle, we can form an $SO(3)$ bundle by taking the bundle $\mathbf{g}_E$ in (3.2). However, although an $SO(3)$ bundle can be always regarded locally as an $SU(2)$ bundle, there are global obstructions to lift the $SO(3)$ group to an $SU(2)$ group. The obstruction is measured precisely by the second Stiefel-Whitney class $w_2(V)$. Therefore, we can view $SU(2)$ bundles as a special case of $SO(3)$ bundle with zero Stiefel-Whitney class, and this is what we are going to do in these lectures. When the $SO(3)$ bundle can be lifted to an $SU(2)$ bundle, one has the relation:

$$p_1(V) = -4c_2(E). \tag{3.8}$$

Chern-Weil theory gives a representative of the characteristic class $p_1(V)/4$ in terms of the curvature of the connection:

$$\frac{1}{4} p_1(V) = \frac{1}{8\pi^2} \mathrm{Tr} F_A^2, \tag{3.9}$$

where F_A is a Hermitian, trace-free matrix valued two-form. Notice that Hermitian, trace-free matrices have the form:

$$\xi = \begin{pmatrix} a & -ib+c \\ ib+c & -a \end{pmatrix}, \quad a, b, c \in \mathbf{R}, \tag{3.10}$$

so the trace is a positive definite form:

$$\mathrm{Tr}\xi^2 = 2(a^2+b^2+c^2) = 2|\xi|^2, \qquad \xi \in \mathbf{su}(2). \tag{3.11}$$

We define the *instanton number* k as:

$$k = -\frac{1}{8\pi^2}\int_X \mathrm{Tr}F_A^2. \tag{3.12}$$

Notice that, if V has not a lifting to an $SU(2)$ bundle, the instanton number is not an integer. If V lifts to E, then $k = c_2(E)$.

The topological invariant $w_2(V)$ for $SO(3)$ bundles may be less familiar to physicists, but it has been used by 't Hooft [39] when $X = \mathbf{T}^4$, the four-torus, to construct gauge configurations called *torons*. To construct torons, one considers $SU(N)$ gauge fields on a four-torus of lengths a_μ, $\mu = 1, \cdots, 4$. To find configurations which are topologically nontrivial, we require of the gauge fields to be periodic up to a gauge transformation in two directions:

$$\begin{aligned} A_\mu(a_1, x_2) &= \Omega_1(x_2) A_\mu(0, x_2), \\ A_\mu(x_1, a_2) &= \Omega_2(x_1) A_\mu(x_1, 0), \end{aligned} \tag{3.13}$$

where we have denoted by $\Omega\, A$ the action of the gauge transformation Ω on the connection A. Looking at the corners, we find the compatibility condition

$$\Omega_1(a_2)\Omega_2(0) = \Omega_2(a_1)\Omega_1(0)Z, \tag{3.14}$$

where $Z \in C(SU(N)) = \mathbf{Z}_N$ is an element in the center of the gauge group. We can allow a nontrivial Z since a gauge transformation which is in the center of $SU(N)$ does not act on the $SU(N)$ gauge fields. This means that when we allow torons we are effectively dealing with an $SU(N)/\mathbf{Z}_N$ gauge theory. For $SU(2)$, this means that we are dealing with an $SO(3)$ theory, and the toron configurations are in fact topologically nontrivial $SO(3)$ gauge fields with nonzero Stiefel-Whitney class.

3.3 ASD connections

The splitting (2.14) between SD and ASD forms extends in a natural way to bundle-valued forms, in particular to the curvature associated to the connection A, $F_A \in \Omega^2(\mathbf{g}_E)$. We call a connection ASD if

$$F_A^+ = 0. \tag{3.15}$$

It is instructive to consider this condition in the case of $X = \mathbf{R}^4$ with the Euclidean metric. If $\{dx_1, dx_2, dx_3, dx_4\}$ is an oriented orthonormal frame, a basis for SD (ASD) forms is given by:

$$\{dx_1 \wedge dx_2 \pm dx_3 \wedge dx_4, dx_1 \wedge dx_4 \pm dx_2 \wedge dx_3, dx_1 \wedge dx_3 \pm dx_4 \wedge dx_1\}, \tag{3.16}$$

with $\pm$ for SD and ASD, respectively. If we write $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$, then the ASD condition reads:

$$\begin{aligned} F_{12} + F_{34} &= 0 \\ F_{14} + F_{23} &= 0 \\ F_{13} + F_{42} &= 0. \end{aligned} \tag{3.17}$$

Notice that the second Chern class density can be written as

$$\mathrm{Tr}(F_A^2) = \{|F_A^+|^2 - |F_A^-|^2\}d\mu, \tag{3.18}$$

where $d\mu$ is the volume element and the norm is defined as:

$$|\psi|^2 = \frac{1}{2}\mathrm{Tr}(\psi \wedge *\psi). \tag{3.19}$$

We then see that, with our conventions, if A is an ASD connection the instanton number k is positive. This gives a topological constraint on the existence of ASD connections.

One of the most important properties of ASD connections is that they minimize the Yang-Mills action

$$S_{\mathrm{YM}} = \frac{1}{2}\int_X F \wedge *F = \frac{1}{4}\int_X d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu} \tag{3.20}$$

in a given topological sector. This is so because the integrand of (3.20) can be written as $|F_A^+|^2 + |F_A^-|^2$, therefore

$$S_{\mathrm{YM}} = \frac{1}{2}\int_X |F_A^+|^2 d\mu + 8\pi^2 k, \tag{3.21}$$

which is bounded from below by $8\pi^2 k$. The minima are attained precisely when (3.15) holds.

The ASD condition is a nonlinear differential equation for non-abelian gauge connections, and it defines a subspace of the (infinite dimensional) configuration space of connections $\mathcal{A}$ [1]. This subspace can be regarded as the zero locus of the section

$$s : \mathcal{A} \longrightarrow \Omega^{2,+}(\mathbf{g}_E) \tag{3.22}$$

given by

$$s(A) = F_A^+. \tag{3.23}$$

Our main goal is to define a finite-dimensional moduli space starting from $s^{-1}(0)$. The key fact to take into account is that the section (3.22) is equivariant with respect to the action of the gauge group: $s(u^*(A)) = u^*(s(A))$. Therefore, if a gauge connection A satisfies the ASD condition, then any gauge-transformed connection $u^*(A)$ will also be ASD. To get rid of the gauge redundancy in order to obtain a finite dimensional moduli space, one must "divide by $\mathcal{G}$" *i.e.* one has to quotient $s^{-1}(0)$ by the action of the gauge group. We are thus led to define the moduli space of ASD connections, $\mathcal{M}_{\mathrm{ASD}}$, as follows:

$$\mathcal{M}_{\mathrm{ASD}} = \big\{[A] \in \mathcal{A}/\mathcal{G} \,|\, s(A) = 0\big\}, \tag{3.24}$$

[1] In the following, when we refer to the space of smooth connections we will proceed on a purely formal level, and we will avoid the hard functional analysis which is needed in order to give a rigorous treatment. We refer to [13, 16] for details concerning this point.

where $[A]$ denotes the gauge-equivalence class of the connection A. Notice that, since s is gauge-equivariant, the above space is well-defined. The fact that the ASD connections form a moduli space is well-known in field theory. For example, on $\mathbf{R}^4$ $SU(2)$ instantons are parameterized by a finite number of data (which include, for example, the position of the instanton), giving $8k-3$ parameters for instanton number k [22]. The moduli space $\mathcal{M}_{\mathrm{ASD}}$ is in general a complicated object, and in the next subsections we will analyze some of its aspects in order to provide a local model for it.

3.4 Reducible connections

In order to analyze $\mathcal{M}_{\mathrm{ASD}}$, we will first look at the map

$$\mathcal{G} \times \mathcal{A} \to \mathcal{A} \tag{3.25}$$

and the associated quotient space $\mathcal{A}/\mathcal{G}$. The first problem we find when we quotient by $\mathcal{G}$ is that, if the action of the group is not free, one has singularities in the resulting quotient space. If we want a smooth moduli space of ASD connections, we have to exclude the points of $\mathcal{A}$ which are fixed under the action of $\mathcal{G}$. To characterize these points, we define the isotropy group of a connection A, Γ_A, as

$$\Gamma_A = \{u \in \mathcal{G} | u(A) = A\}, \tag{3.26}$$

which measures the extent at which the action of $\mathcal{G}$ on a connection A is not free. If the isotropy group is the center of the group $C(G)$, then the action is free and we say that the connection A is *irreducible*. Otherwise, we say that the connection A is *reducible*. Reducible connections are well-known in field theory, since they correspond to gauge configurations where the gauge symmetry is broken to a smaller subgroup. For example, the $SU(2)$ connection

$$A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \tag{3.27}$$

should be regarded in fact as a $U(1)$ connection in disguise. It is clear that a constant gauge transformation of the form $u\sigma_3$ leaves (3.27) invariant, therefore the isotropy group of A is bigger than the center of $SU(2)$. We will denote the space of irreducible connections by $\mathcal{A}^*$. It follows from the definition that the reduced group of gauge transformations $\hat{\mathcal{G}} = \mathcal{G}/C(G)$ acts freely on $\mathcal{A}^*$.

By using the description of u as a section of $\mathrm{Aut}(E)$ and the action on A given in (3.4), we see that

$$\Gamma_A = \{u \in \Gamma(\mathrm{Aut}(E)) | \nabla_A u = 0\}, \tag{3.28}$$

i.e. the isotropy group at A is given by the covariantly constant sections of the bundle $\mathrm{Aut}(E)$. It follows that Γ_A is a Lie group, and its Lie algebra is given by

$$\mathrm{Lie}(\Gamma_A) = \{f \in \Omega^0(\mathbf{g}_E) | \nabla_A f = 0\}. \tag{3.29}$$

Therefore, a useful way to detect if Γ_A is bigger than $C(G)$ (and has positive dimension) is to study the kernel of ∇_A in $\Omega^0(\mathbf{g}_E)$. Reducible connections correspond then to a non-zero kernel of

$$-\nabla_A : \Omega^0(\mathbf{g}_E) \to \Omega^1(\mathbf{g}_E). \tag{3.30}$$

In the case of $SU(2)$ and $SO(3)$, a reducible connection has precisely the form (3.27), with isotropy group $\Gamma_A/C(G) = U(1)$. This means, topologically, that the $SU(2)$ bundle E splits as:

$$E = L \oplus L^{-1}, \tag{3.31}$$

with L a complex line bundle, while a reducible $SO(3)$ bundle splits as

$$V = \mathbf{R} \oplus T, \tag{3.32}$$

where $\mathbf{R}$ denotes the trivial rank-one real bundle over X. The above structure for V is easily derived by considering the real part of $\mathrm{Sym}^2(E)$. Notice that, if V admits a $SU(2)$ lifting E, then $T = L^2$. There are topological constraints to have these splittings, because (3.31) implies that $c_2(E) = -c_1(L)^2$, and (3.32) that

$$p_1(V) = c_1(T)^2. \tag{3.33}$$

When E exists, the first Chern class $\lambda = c_1(L)$ is an integral cohomology class. However, when $w_2(E) \neq 0$, then it follows from (3.7) that L does not exist as a line bundle, since its first Chern class is not an integral class but lives in the lattice

$$H^2(X, \mathbf{Z}) + \frac{1}{2} w_2(V). \tag{3.34}$$

In particular, one has that

$$c_1(T) \equiv w_2(V) \bmod 2. \tag{3.35}$$

Therefore, reductions of V are in one-to-one correspondence with cohomology classes $\alpha \in H^2(M; \mathbf{Z})$ such that $\alpha^2 = p_1(V)$. In the following, when we study the local model of $\mathcal{M}_{\mathrm{ASD}}$, we will restrict ourselves to irreducible connections.

3.5 A local model for the moduli space

To construct a local model for the moduli space means essentially to give a characterization of its tangent space at a given point. The way to do that is to consider the tangent space at an ASD connection A in $\mathcal{A}$, which is isomorphic to $\Omega^1(\mathbf{g}_E)$, and look for the directions in this vector space which preserve the ASD condition and which are not gauge orbits (since we are quotienting by $\mathcal{G}$). The local model for $\mathcal{M}_{\mathrm{ASD}}$ was first obtained by Atiyah, Hitchin and Singer in [2].

Let us first address the second condition. We want to find out which directions in the tangent space at a connection A are pure gauge, *i.e.* we want to find slices of the action of the gauge group $\hat{\mathcal{G}}$. The procedure is simply to consider the derivative of the map (3.25) in the $\mathcal{G}$ variable at a point $A \in \mathcal{A}^*$ to obtain

$$C : \mathrm{Lie}(\mathcal{G}) \longrightarrow T_A\mathcal{A}, \tag{3.36}$$

which is nothing but (3.30) (notice the minus sign in ∇_A, which comes from the definition of the action in (3.4)). Since there is a natural metric in the space $\Omega^*(\mathbf{g}_E)$, we can define a formal adjoint operator:

$$C^\dagger : \Omega^1(\mathbf{g}_E) \longrightarrow \Omega^0(\mathbf{g}_E) \tag{3.37}$$

given by $C^\dagger = \nabla_A^\dagger$. We can then orthogonally decompose the tangent space at A into the gauge orbit Im C and its complement:

$$\Omega^1(\mathbf{g}_E) = \text{Im } C \oplus \text{Ker } C^\dagger. \tag{3.38}$$

Here we used the fact that IM C is closed as a consequence of its Fredholm property, which follows from the injectivity of the leading symbol of C. (3.38) is precisely the slice of the action we were looking for. Locally, this means that the neighbourhood of $[A]$ in $\mathcal{A}^*/\mathcal{G}$ can be modelled by the subspace of $T_A\mathcal{A}$ given by Ker $\nabla_A^\dagger$. Furthermore, the isotropy group Γ_A has a natural action on $\Omega^1(\mathbf{g}_E)$ given by the adjoint multiplication, as in (3.5). If the connection is reducible, the moduli space is locally modelled on (Ker $\nabla_A^\dagger)/\Gamma_A$ (see [13,16]).

We have obtained a local model for the orbit space $\mathcal{A}^*/\mathcal{G}$, and now we need to enforce the ASD condition. Let A be an irreducible ASD connection, verifying $F_A^+ = 0$, and let $A+a$ be another ASD connection, where $a \in \Omega^1(\mathbf{g}_E)$. The condition we get on a starting from $F_{A+a}^+ = 0$ is $p^+(\nabla_A a + a \wedge a) = 0$, where p^+ is the projector on the SD part of a two-form. At linear order we find:

$$p^+\nabla_A a = 0. \tag{3.39}$$

Notice that the map $p^+\nabla_A$ is nothing but the linearization of the section s, ds:

$$ds : T_A\mathcal{A} \longrightarrow \Omega^{2,+}(\mathbf{g}_E) \tag{3.40}$$

The kernel of ds corresponds to tangent vectors that satisfy the ASD condition at linear order (3.39). We can now give a precise description of the tangent space of $\mathcal{M}_{\text{ASD}}$ at $[A]$: we want directions which are in Ker ds but which are not in Im ∇_A. First notice that, since s is gauge-equivariant, Im $\nabla_A \subset$ Ker ds. This can be checked by direct computation:

$$p^+\nabla_A\nabla_A\phi = [F_A^+, \phi] = 0, \qquad \phi \in \Omega^0(\mathbf{g}_E), \tag{3.41}$$

since A is ASD. Taking now into account (3.38), we finally find:

$$T_{[A]}\mathcal{M}_{\text{ASD}} \simeq (\text{Ker}\, ds) \cap (\text{Ker}\, \nabla_A^\dagger). \tag{3.42}$$

This space can be regarded as the kernel of the operator $D = p^+\nabla_A \oplus \nabla_A^\dagger$:

$$D : \Omega^1(\mathbf{g}_E) \longrightarrow \Omega^0(\mathbf{g}_E) \oplus \Omega^{2,+}(\mathbf{g}_E). \tag{3.43}$$

Since Im $\nabla_A \subset$ Ker ds there is a short exact sequence:

$$0 \longrightarrow \Omega^0(\mathbf{g}_E) \xrightarrow{\nabla_A} \Omega^1(\mathbf{g}_E) \xrightarrow{p^+\nabla_A} \Omega^{2,+}(\mathbf{g}_E) \longrightarrow 0. \tag{3.44}$$

This complez is called the *instanton deformation complex* or *Atiyah-Hitchin-Singer (AHS) complex* [2], and gives a very elegant local model for the moduli space of ASD connections. In particular, one has that

$$T_{[A]}\mathcal{M}_{\text{ASD}} = H_A^1, \tag{3.45}$$

where H_A^1 is the middle cohomology group of the complex (3.44):

$$H_A^1 = \frac{\text{Ker } p^+\nabla_A}{\text{Im } \nabla_A}. \tag{3.46}$$

The index of the AHS complex (3.44) is given by

$$\text{ind} = \dim H_A^1 - \dim H_A^0 - \dim H_A^2, \tag{3.47}$$

where $H_A^0 = \operatorname{Ker} \nabla_A$ and $H_A^2 = \operatorname{Coker}\ p^+ \nabla_A$. This index is usually called the *virtual* dimension of the moduli space. When A is an irreducible connection (in particular, $\operatorname{Ker} \nabla_A = 0$) and in addition it satisfies $H_A^2 = 0$, it is called a *regular* connection [13]. For these connections, the dimension of $T_{[A]}\mathcal{M}_{\rm ASD}$ is given by the virtual dimension. This index can be computed for any gauge group G using the Atiyah-Singer index theorem. The computation is done in [2], and the result for $SO(3)$ is:

$$\dim\ \mathcal{M}_{\rm ASD} = -2p_1(V) - \frac{3}{2}(\chi + \sigma), \tag{3.48}$$

where $p_1(V)$ denotes the first Pontriagin number (*i.e.* the Pontriagin class (3.6) integrated over X) and χ, σ are the Euler characteristic and signature of X,respectively.

Exercise 3.1. *Dimension of instanton moduli space.* Compute $\mathcal{M}_{\rm ASD}$ using the index theorem for the twisted Dirac operator. *Hint*: use that $\Omega^1(X) \simeq S^+ \otimes S^-$, and $\Omega^{2,+} \simeq S^+ \otimes S^+$.

The conclusion of this analysis is that, if A is an irreducible ASD connection, the moduli space in a neighbourhood of this point is smooth and can be modelled by the cohomology (3.46). If the connection is also regular, the index of the instanton deformation complex gives minus the dimension of moduli space. Of course, the most difficult part of Donaldson theory is to find the global structure of $\mathcal{M}_{\rm ASD}$. In particular, in order to define the invariants one has to compactify the moduli space. We are not going to deal with these subtle issues here, and refer the reader to the references mentioned at the beginning of this section.

3.6 Donaldson invariants

Donaldson invariants are roughly defined in terms of integrals of differential forms in the moduli space of irreducible ASD connections. These differential forms come from the rational cohomology ring of $\mathcal{A}^*/\mathcal{G} = \mathcal{B}^*$, and it is necessary to have an explicit description of this ring. The construction involves the *universal bundle* or *universal instanton* associated to this moduli problem, and goes as follows: if the gauge group is $SU(2)$, we consider the $SO(3)$ bundle $\mathbf{g}_E$ associated to E, and if the gauge group is $SO(3)$ we consider the vector bundle V. We will denote both of them by $\mathbf{g}_E$, since the construction is the same in both cases. We then consider the space $\mathcal{A}^* \times \mathbf{g}_E$. This can be regarded as a bundle:

$$\mathcal{A}^* \times \mathbf{g}_E \to \mathcal{A}^* \times X \tag{3.49}$$

which is the pullback from the bundle $\pi : \mathbf{g}_E \to X$. The space $\mathcal{A}^* \times \mathbf{g}_E$ is called a *family of tautological connections*, since the natural connection on $\mathcal{A}^* \times \mathbf{g}_E$ is tautological in the $\mathbf{g}_E$ direction and trivial in the $\mathcal{A}^*$ direction: at the point (A, p), the connection is given by $A_\alpha(\pi(p))$ (where we have chosen a trivialization of $\mathbf{g}_E$ as in section 3.1, and $\pi(p) \in U_\alpha$). Since the group of reduced gauge transformations $\hat{\mathcal{G}}$ acts on both factors, $\mathcal{A}^*$ and $\mathbf{g}_E$, the quotient

$$\mathbf{P} = \mathcal{A}^* \times_{\hat{\mathcal{G}}} \mathbf{g}_E \tag{3.50}$$

is a $G/C(G)$-bundle over $\mathcal{B}^* \times X$. This is the *universal bundle* associated to E (or V). In the case of $G = SU(2)$ or $SO(3)$, the universal bundle is an $SO(3)$ bundle (since $SU(2)/\mathbf{Z}_2 = SO(3)$ and $SO(3)$ has no center). Its Pontriagin class $p_1(\mathbf{P})$ can be computed using Chern-Weil theory in terms of the curvature of a connection on $\mathbf{P}$. One can construct a natural connection on $\mathbf{P}$, called the *universal connection*, by considering the quotient of the tautological connection (see [9,13] for details). The curvature of the universal connection will be denoted by $K_{\mathbf{P}}$. It is a form in $\Omega^2(\mathcal{B}^* \times X, \mathbf{g}_E)$, and splits according to the bigrading of $\Omega^*(\mathcal{B}^* \times X)$ into three pieces: a two-form with respect to $\mathcal{B}^*$, a two-form with respect to X, and a mixed form (one-form on $\mathcal{B}^*$ and one-form on X), all with values in $\mathbf{g}_P$. The Pontriagin class is:

$$\frac{p_1(\mathbf{P})}{4} = \frac{1}{8\pi^2}\mathrm{Tr}(K_{\mathbf{P}} \wedge K_{\mathbf{P}}) \tag{3.51}$$

and defines a cohomology class in $H^4(\mathcal{B}^* \times X)$. By decomposing according to the bigrading, we obtain an element in $H^*(\mathcal{B}^*) \otimes H^*(X)$. To get differential forms on $\mathcal{B}^*$, we just take the slant product with *homology* classes in X (*i.e.* we simply pair the forms on X with cyles on X). In this way we obtain the *Donaldson map*:

$$\mu : H_i(X) \longrightarrow H^{4-i}(\mathcal{B}^*). \tag{3.52}$$

One can prove [13] that the differential forms obtained in this way actually generate the cohomology ring of $\mathcal{B}^*$. Finally, after restriction to $\mathcal{M}_{\mathrm{ASD}}$ we obtain the following differential forms on the moduli space of ASD connections:

$$\begin{array}{rcl} x \in H_0(X) & \rightarrow & \mathcal{O}(x) \in H^4(\mathcal{M}_{\mathrm{ASD}}), \\ \delta \in H_1(X) & \rightarrow & I_1(\delta) \in H^3(\mathcal{M}_{\mathrm{ASD}}), \\ S \in H_2(X) & \rightarrow & I_2(S) \in H^2(\mathcal{M}_{\mathrm{ASD}}). \end{array} \tag{3.53}$$

There are also cohomology classes associated to three-cycles in X, but we will not consider them in these lectures. In the next lecture we will see that the Donaldson map arises very naturally in the context of topological field theory in what is called the *descent procedure.* In any case, we can now formally define the Donaldson invariants as follows. Consider the space

$$\mathbf{A}(X) = \mathrm{Sym}(H_0(X) \oplus H_2(X)) \otimes \wedge^* H_1(X), \tag{3.54}$$

with a typical element written as $x^\ell S_{i_1} \cdots S_{i_p} \delta_{j_1} \cdots \delta_{j_q}$. The *Donaldson invariant* corresponding to this element of $\mathbf{A}(X)$ is the following intersection number:

$$\begin{array}{c} \mathcal{D}_X^{w_2(V),k}(x^\ell S_{i_1} \cdots S_{i_p} \delta_{j_1} \cdots \delta_{j_q}) = \\ \int_{\mathcal{M}_{\mathrm{ASD}}(w_2(V),k)} \mathcal{O}^\ell \wedge I_2(S_{i_1}) \wedge \cdots \wedge I_2(S_{i_p}) \wedge I_1(\delta_{j_1}) \wedge \cdots \wedge \ I_1(\delta_{j_q}), \end{array} \tag{3.55}$$

where we denoted by $\mathcal{M}_{\mathrm{ASD}}(w_2(V), k)$ the moduli space of ASD connections specified by the second Stiefel-Whitney class $w_2(V)$ and the instanton number k. Notice that, since the integrals of differential forms are different from zero only when the dimension of the space equals the total degree of the form, it is clear that the integral in (3.55) will be different from zero only if the degrees of the forms add up to $\dim(\mathcal{M}_{\rm ASD}(w_2(V),k))$. It follows from (3.55) that Donaldson invariants can be understood as functionals:

$$\mathcal{D}_X^{w_2(V),k} : \mathbf{A}(X) \to \mathbf{Q}. \tag{3.56}$$

The reason that the values of the invariants are rational rather than integer is subtle and has to do with the fact that they are rigorously defined as intersection numbers only in certain situations (the so-called *stable range*). Outside this range, there is a natural way to extend the definition which involves dividing by 2 (for more details, see [17]).

It is very convenient to pack all Donaldson invariants in a generating function. Let $\{\delta_i\}_{i=1,\dots,b_1}$ be a basis of one-cycles, and $\{S_i\}_{i=1,\dots,b_2}$ a basis of two-cycles. We introduce the formal sums

$$\delta = \sum_{i=1}^{b_1} \zeta_i \, \delta_i, \qquad\qquad S = \sum_{i=1}^{b_2} v_i \, S_i, \tag{3.57}$$

where v_i are complex numbers, and ζ_i are Grassmann variables. We then define the *Donaldson-Witten generating function* as:

$$Z_{DW}^{w_2(V)}(p, \zeta_i, v_i) = \sum_{k=0}^{\infty} \mathcal{D}_X^{w_2(V),k}(\mathrm{e}^{px+\delta+S}), \tag{3.58}$$

where in the right hand side we are summing over all instanton numbers, *i.e.* we are summing over all topological configurations of the $SO(3)$ gauge field with a *fixed* $w_2(V)$. This gives a formal power series in p, ζ_i and v_i. The Donaldson invariants are the coefficients of this formal series. If we assign degree 4 to p, 2 to v_i and 3 to ζ_i, and we fix the total degree (*i.e.* we fix k), we get a finite polynomial which encodes all the Donaldson invariants for a fixed instanton number. Therefore, Donaldson invariants at fixed instanton number can be also regarded as polynomials in the (dual of the) cohomology of the manifold. Sometimes we will also write (3.58) as a functional $Z_{DW}^{w_2(V)}(p, S, \delta)$. It should be mentioned that in the math literature the most common object is the so-called *Donaldson series* [24], which is defined when $\delta = 0$ as follows:

$$\mathbf{D}^{w_2(V)}(S) = Z_{DW}^{w_2(V)}(p, S)\big|_{p=0} + \frac{1}{2}\frac{\partial}{\partial p} Z_{DW}^{w_2(V)}(p, S)\big|_{p=0}. \tag{3.59}$$

The Donaldson series can then be regarded as a map:

$$\mathbf{D}^{w_2(V)} : \mathrm{Sym}(H_2(X)) \to \mathbf{Q}. \tag{3.60}$$

The basic goal of Donalson theory is the computation of the generating functional (3.58) (or, in the simply-connected case, of the Donaldson series (3.59)). Many results have been obtained along the years for different four-manifolds (a good review is [36]). The major breakthrough in this sense was the structure theorem of Kronheimer and Mrowka [24] (see also [19]) for the Donaldson series of simply-connected four-manifolds with $b_2^+ > 1$ and of the so-called *Donaldson simple type*. A four-manifold is said to be of *Donaldson simple type* if

$$\left(\frac{\partial^2}{\partial p^2} - 4\right) Z_{DW}^{w_2(V)}(p, S) = 0, \tag{3.61}$$

for all choices of $w_2(V)$. When this holds, then, according to the results of Kronheimer and Mrowka, the Donaldson series has the following structure:

$$\mathbf{D}^{w_2(V)}(S) = \exp(S^2/2) \sum_{s=1}^{p} a_s \mathrm{e}^{(\kappa_s, S)}, \tag{3.62}$$

for finitely many homology classes $\kappa_1, \cdots, \kappa_p \in H_2(X, \mathbf{Z})$ and nonzero rational numbers $a_1, \cdots, a_p$. Furthermore, each of the classes κ_i is characteristic. The classes κ_i are called *Donaldson basic classes.*

A simple example of this situation is the $K3$ surface. In this case, the Donaldson-Witten generating functional is given by

$$Z_{DW}^{w_2(V)} = \frac{1}{2} \mathrm{e}^{2\pi i \lambda_0^2} \Big(\mathrm{e}^{\frac{S^2}{2} + 2p} - i^{-w_2(V)^2} \mathrm{e}^{-\frac{S^2}{2} - 2p} \Big). \tag{3.63}$$

In this expression, $2\lambda_0$ is a choice of an integer lifting of $w_2(E)$. The overall factor $\mathrm{e}^{2\pi i \lambda_0^2}$ gives a dependence on the choice a such a lifting, and this is due to the fact that the orientation of instanton moduli space depends on such a choice [13]. From the above expression one can deduce that for example for $w_2(V) = 0$, one has

$$\int I_2(S)^2 = (S, S), \quad \int I_2(S)^6 = \frac{1}{8}(S, S)^3, \tag{3.64}$$

and so on. Notice that in the first integral in (3.64) we integrate over the moduli space $\mathcal{M}_{\mathrm{ASD}}$ with instanton number $k = 2$, and in the second one we have $k = 6$. According to (3.63), $K3$ is of simple type, and the Donalson series is simply given by:

$$\mathbf{D}^{w_2(V)} = \mathrm{e}^{S^2/2}, \tag{3.65}$$

which satisfies indeed the structure theorem of Kronheimer and Mrowka and shows that $K3$ has only one Donaldson basic class, namely $\kappa = 0$.

4 $\mathcal{N} = 1$ supersymmetry

In this section, we give some useful background on supersymmetry. Since our motivation is the construction of topological field theories, our presentation will be rather sketchy. The standard reference is [40]. A very useful and compact presentation can be found in the excellent review by Álvarez-Gaumé and Hassan [1], which is the main source for this very quick review. We follow strictly the conventions of [1], which are essentially those in [40], although there are some important differences. Some of these conventions can be found in Appendix A. Another useful reference, intended for mathematicians, is [11].

4.1 The supersymmetry algebra

Supersymmetry is the only nontrivial extension of Poincaré symmetry which is compatible with the general principles of relativistic quantum field theory. In $\mathbf{R}^{1,3}$ one introduces $\mathcal{N}$ fermionic generators

$$Q_u = \begin{pmatrix} Q_{\alpha u} \\ \overline{Q}^{\dot{\alpha} u} \end{pmatrix} \tag{4.1}$$

where $u = 1, \cdots, \mathcal{N}$. The superPoincaré algebra extends the usual Poincaré algebra, and the (anti)commutators of the fermionic generators are:

$$
\begin{aligned}
\{Q_{\alpha u}, \overline{Q}_{\dot{\beta} v}\} &= 2\epsilon_{uv}\sigma^{\mu}_{\alpha\dot{\beta}}P_{\mu} \\
[P_{\mu}, Q_{\alpha v}] &= 0 \\
[M_{\mu\nu}, Q_{\alpha u}] &= -(\sigma_{\mu\nu})_{\alpha}{}^{\beta}Q_{\beta u} \\
\{Q_{\alpha u}, Q_{\beta v}\} &= 2\sqrt{2}\epsilon_{\alpha\beta}Z_{uv} \\
\left[P_{\mu}, \overline{Q}_{\dot{\alpha} u}\right] &= 0 \\
\left[M_{\mu\nu}, \overline{Q}^{\dot{\alpha} u}\right] &= -(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\overline{Q}^{\dot{\beta} u}
\end{aligned}
\tag{4.2}
$$

where $u, v = 1, \cdots, \mathcal{N}$, and $M_{\mu\nu}$ are the generators of the Lorentz group $SO(4) \simeq SU(2)_+ \times SU(2)_-$. The terms Z_{uv} are the so-called central charges. They satisfy

$$Z_{uv} = -Z_{vu} \tag{4.3}$$

and they commute with all the generators of the algebra.

When the central charges vanish, the theory has an internal $U(\mathcal{N})$ symmetry:

$$Q_{\alpha v} \to U_v{}^w Q_{\alpha w} \qquad \bar{Q}^{v}_{\dot{\alpha}} \to U^{*w}_{v}\bar{Q}_{\dot{\alpha}}{}^{w}, \tag{4.4}$$

where $U \in U(\mathcal{N})$ is a unitary matrix. This symmetry is called in physics an R-symmetry, and it is denoted by $U(\mathcal{N})_{\mathcal{R}}$. The generators of this symmetry will be denoted by B_a, and their commutation relations with the fermionic supercharges are:

$$[Q_{\alpha v}, B_a] = (b_a)_v{}^w Q_{\alpha w} \qquad [\overline{Q}_{\dot{\alpha}}{}^{w}, B_a] = -\overline{Q}_{\dot{\alpha}}{}^{v}(b_a)_v{}^w \tag{4.5}$$

where $b_a = b_a^+$. The central charges are linear combinations of the $U(N)$ generators

$$Z_{uv} = d_{uv}{}^a B_a. \tag{4.6}$$

If the central charges are not zero, the internal symmetry gets reduced to USp($\mathcal{N}$), formed by the unitary transformations that leave invariant the 2-form (4.6) in $\mathcal{N}$ dimensions. The $U(1)_R$ of the internal symmetry (4.4), with generator R, gives a chiral symmetry of the theory,

$$[Q_{\alpha v}, R] = Q_{\alpha v}, \qquad [\overline{Q}^{w}_{\dot{\alpha}}, R] = -\overline{Q}^{v}_{\dot{\alpha}}. \tag{4.7}$$

This symmetry is typically anomalous, quantum-mechanically, and the quantum effects break it down to a discrete subgroup.

4.2 $\mathcal{N} = 1$ superspace and superfields

In order to find a local realization of supersymmetry, one has to extend the usual Minkowski space to the so-called *superspace*. In this section we are going to develop the basics of $\mathcal{N} = 1$ superspace, which is extremely useful to formulate supersymmetric field multiplets and supersymmetric Lagrangians. Therefore, we are going to construct a local realization of the supersymmetry algebra (4.2) when we have two supercharges Q_α, $\bar{Q}^{\dot{\alpha}}$.

The superspace is obtained by adding four spinor coordinates $\theta^\alpha, \bar{\theta}_{\dot{\alpha}}$ to the four space-time coordinates x^μ. The generator of supersymmetric transformations in superspace is

$$-i\xi^\alpha Q_\alpha - i\bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} \tag{4.8}$$

where ξ^α, $\bar{\xi}_{\dot{\alpha}}$ are (fermionic) transformation parameters[2]. Under this generator, the superspace coordinates transform as

$$\begin{aligned} x^\mu &\rightarrow x^\mu + i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta} \\ \theta &\rightarrow \theta + \xi\,, \\ \bar{\theta} &\rightarrow \bar{\theta} + \bar{\xi}\,. \end{aligned} \tag{4.9}$$

The representation of the supercharges acting on the superspace is then given by

$$Q_\alpha = i\left(\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\,\partial_\mu\right), \qquad \overline{Q}_{\dot{\alpha}} = -i\left(\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu\right) \tag{4.10}$$

and they satisfy $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu$. Since $P_\mu = -i\partial_\mu$, this gives a representation of the supersymmetry algebra. It is also convenient to introduce the super-covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu\,, \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\sigma^\mu_{\alpha\dot{\alpha}}\theta^\alpha\,\partial_\mu\,, \tag{4.11}$$

which satisfy $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu$ and commute with Q and $\bar{Q}$.

A *superfield* is just a function on the superspace $F(x, \theta, \bar{\theta})$. Since the θ-coordinates are anti-commuting, the Taylor expansion in the fermionic coordinates truncates after a finite number of terms. Therefore, the most general $\mathcal{N} = 1$ superfield can always be expanded as

$$\begin{aligned} F(x,\theta,\bar{\theta}) &= f(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ &+ \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\psi(x) + \theta\theta\bar{\theta}\bar{\theta}d(x)\,. \end{aligned} \tag{4.12}$$

Under a supersymmetry transformation (4.8), the superfield transforms as $\delta F = (\xi Q + \bar{\xi}\bar{Q})F$, and from this expression one can obtain the transformation of the components.

The generic superfield gives a reducible representation of the supersymmetry algebra. Therefore, in order to obtain irreducible representations one must impose constraints. There are two different $\mathcal{N} = 1$ irreducible supermultiplets:

a) Chiral multiplet: The $\mathcal{N} = 1$ scalar multiplet is a superfield which satisfies the following constraint:

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{4.13}$$

and it is called the chiral superfield. The constraint can be easily solved by noting that, if $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, then

$$\bar{D}_{\dot{\alpha}}y^\mu = 0, \quad \bar{D}_{\dot{\alpha}}\theta^\beta = 0\,. \tag{4.14}$$

Therefore, any function of (y, θ) is a chiral superfield. We can then write

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta^2 F(y)\,, \tag{4.15}$$

[2]This is the only case in which we do not follow the conventions of [1]: their susy charges are $-i$ times ours.

and we see that a chiral superfield contains two complex scalar fields, ϕ and F, and a Weyl spinor ψ_α. In a similar way we can define an anti-chiral superfield by $D_\alpha \Phi^\dagger = 0$, which can be expanded as

$$\Phi^\dagger(y^\dagger, \bar\theta) = \phi^\dagger(y^\dagger) + \sqrt{2}\bar\theta\bar\psi(y^\dagger) + \bar\theta^2 F^\dagger(y^\dagger)\,, \tag{4.16}$$

where, $y^{\mu\dagger} = x^\mu - i\theta\sigma^\mu\bar\theta$.

Exercise 4.1. Show that, in terms of the original variables, Φ and $\Phi^\dagger$ take the form

$$\begin{aligned}\Phi(x,\theta,\bar\theta) &= A(x) + i\theta\sigma^\mu\bar\theta\partial_\mu A - \frac{1}{4}\theta^2\bar\theta^2\nabla^2 A + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\bar\theta + \theta\theta F(x)\,,\\ \Phi^\dagger(x,\theta,\bar\theta) &= A^\dagger(x) - i\theta\sigma^\mu\bar\theta\partial_\mu A^\dagger - \frac{1}{4}\theta^2\bar\theta^2\nabla^2 A^\dagger + \sqrt{2}\bar\theta\bar\psi(x)\\ &+ \frac{i}{\sqrt{2}}\bar\theta\bar\theta\,\theta\sigma^\mu\,\partial_\mu\bar\psi + \bar\theta\bar\theta F^\dagger(x)\,.\end{aligned} \tag{4.17}$$

Here, $\nabla^2 = \partial_\mu\partial^\mu$.

b) Vector Multiplet: this is a real superfield satifying $V = V^\dagger$. In components, it takes the form

$$\begin{aligned}V(x,\theta,\bar\theta) &= C + i\theta\chi - i\bar\theta\bar\chi + \frac{i}{2}\theta^2(M+iN) - \frac{i}{2}\bar\theta^2(M-iN) - \theta\sigma^\mu\bar\theta A_\mu\\ &+ i\theta^2\bar\theta(\bar\lambda + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi) - i\bar\theta^2\theta(\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi) + \frac{1}{2}\theta^2\bar\theta^2(D - \frac{1}{2}\nabla^2 C)\,.\end{aligned} \tag{4.18}$$

By performing an abelian gauge transformation $V \to V + \Lambda + \Lambda^\dagger$, where Λ ($\Lambda^\dagger$) are chiral (antichiral) superfields, one can set $C = M = N = \chi = 0$. This is the so called Wess-Zumino gauge, where

$$V = -\theta\sigma^\mu\bar\theta A_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2 D\,. \tag{4.19}$$

In this gauge, $V^2 = \frac{1}{2}A_\mu A^\mu\theta^2\bar\theta^2$ and $V^3 = 0$. The Wess-Zumino gauge breaks supersymmetry, but not the gauge symmetry of the abelian gauge field A_μ. The Abelian field strength is defined by

$$W_\alpha = -\frac{1}{4}\bar D^2 D_\alpha V\,, \qquad \bar W_{\dot\alpha} = -\frac{1}{4}D^2\bar D_{\dot\alpha}V\,,$$

and W_α is a chiral superfield. In the Wess-Zumino gauge it takes the form

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D - \frac{i}{2}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha F_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar\lambda)_\alpha\,. \tag{4.20}$$

The non-Abelian case is similar: V is in the adjoint representation of the gauge group, $V = V_A T^A$, and the gauge transformations are

$$e^{-2V} \to e^{-i\Lambda^\dagger}e^{-2V}e^{i\Lambda}$$

where $\Lambda = \Lambda_A T^A$. The non-Abelian gauge field strength is defined by

$$W_\alpha = \frac{1}{8}\bar D^2 e^{2V} D_\alpha e^{-2V}$$

and transforms as

$$W_\alpha \to W'_\alpha = e^{-i\Lambda} W_\alpha e^{i\Lambda} .$$

In components, it takes the form

$$W_\alpha = T^a \left(-i\lambda^a_\alpha + \theta_\alpha D^a - \frac{i}{2}(\sigma^\mu \bar{\sigma}^\nu \theta)_\alpha F^a_{\mu\nu} + \theta^2 \sigma^\mu \nabla_\mu \bar{\lambda}^a \right) \tag{4.21}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu , \qquad \nabla_\mu \bar{\lambda}^a = \partial_\mu \bar{\lambda}^a + f^{abc} A^b_\mu \bar{\lambda}^c .$$

4.3 Construction of $\mathcal{N} = 1$ Lagrangians

In the previous subsection we have constructed supermultiplets of $\mathcal{N} = 1$ supersymmetry. The next step is to construct manifestly supersymmetric Lagrangians. Again, this is easily done in superspace.

The most general $\mathcal{N} = 1$ supersymmetric Lagrangian for the scalar multiplet (including the interaction terms) is given by

$$\mathcal{L} = \int d^4\theta \, K(\Phi, \Phi^\dagger) + \int d^2\theta \mathcal{W}(\Phi) + \int d^2\bar{\theta} \bar{\mathcal{W}}(\Phi^\dagger) . \tag{4.22}$$

We are following here the usual rules of Grasmmannian integration, and the θ-integrals pick up the highest component of the superfield. In our conventions, $\int d^2\theta \theta^2 = 1$ and $\int d^2\bar{\theta}\bar{\theta}^2 = 1$. The kinetic term for the scalar fields A_i has the form

$$g^{ij} \partial_\mu A_i \partial^\mu A^\dagger_j \tag{4.23}$$

where

$$g^{ij} = \frac{\partial^2 K}{\partial A_i \partial A^\dagger_j} \tag{4.24}$$

is in general a nontrivial metric for the space of fields Φ. This has the form of a Kähler metric derived from a Kähler potential $K(A_i, A^\dagger_j)$. For this reason, the function $K(\Phi, \Phi^\dagger)$ is referred to as the *Kähler potential*. The simplest Kähler potential, corresponding to the flat metric, is

$$K(\Phi, \Phi^\dagger) = \sum_{i=1} \Phi^\dagger_i \Phi_i$$

which gives the free Lagrangian for a massless scalar and a massless fermion with an auxiliary field which can be eliminated by its equation of motion:

$$\mathcal{L} = \sum_i \Phi^\dagger_i \Phi_i \mid_{\theta^2\bar{\theta}^2} = \partial_\mu A^\dagger_i \partial^\mu A_i + F^\dagger_i F_i - i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i . \tag{4.25}$$

Exercise 4.2. Show that

$$\begin{aligned} \Phi^\dagger_i \Phi_j \mid_{\theta^2\bar{\theta}^2} &= -\frac{1}{4} A^\dagger_i \nabla^2 A_j - \frac{1}{4} \nabla^2 A^\dagger_i A_j + F^\dagger_i F_j + \frac{1}{2} \partial_\mu A^\dagger_i \partial^\mu A_j \\ &- \frac{i}{2} \psi_j \sigma^\mu \partial_\mu \bar{\psi}_i + \frac{i}{2} \partial_\mu \psi_j \sigma^\mu \bar{\psi}_i . \end{aligned} \tag{4.26}$$

and from this derive (4.25).

The function $W(\Phi)$ in (4.22) is an arbitrary holomorphic function of chiral superfields, and it is called the *superpotential.* It can be expanded as,

$$
\begin{aligned}
\mathcal{W}(\Phi_i) &= \mathcal{W}(A_i + \sqrt{2}\theta\psi_i + \theta\theta F_i) \\
&= \mathcal{W}(A_i) + \frac{\partial \mathcal{W}}{\partial A_i}\sqrt{2}\theta\psi_i + \theta\theta\left(\frac{\partial \mathcal{W}}{\partial A_i}F_i - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial A_i A_j}\psi_i\psi_j\right) .
\end{aligned} \tag{4.27}
$$

Supersymmetric interaction terms can be constructed in terms of the superpotential and its conjugate. Finally, we have to mention that there is $U(1)_R$ symmetry that acts as follows:

$$
\begin{aligned}
R\,\Phi(x,\theta) &= e^{2in\alpha}\Phi(x, e^{-i\alpha}\theta)\,, \\
R\,\Phi^\dagger(x,\bar\theta) &= e^{-2in\alpha}\Phi^\dagger(x, e^{i\alpha}\bar\theta)\,.
\end{aligned} \tag{4.28}
$$

Under this, the component fields transform as

$$
\begin{aligned}
A &\longrightarrow e^{2in\alpha}A\,, \\
\psi &\longrightarrow e^{2i(n-1/2)\alpha}\psi\,, \\
F &\longrightarrow e^{2i(n-1)\alpha}F\,.
\end{aligned} \tag{4.29}
$$

Let us now present the Lagrangian for vector superfields. The super Yang-Mills Lagrangian with a θ-term can be written as

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{8\pi}\mathrm{Im}\left(\tau\,\mathrm{Tr}\int d^2\theta\, W^\alpha W_\alpha\right) \\
&= -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\tilde F^{a\mu\nu} + \frac{1}{g^2}\left(\frac{1}{2}D^aD^a - i\lambda^a\sigma^\mu\nabla_\mu\bar\lambda^a\right),
\end{aligned} \tag{4.30}
$$

where $\tau = \theta/2\pi + 4\pi i/g^2$, and $\tilde F^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$.

Exercise 4.3. Using the normalization $\mathrm{Tr}T^aT^b = \delta^{ab}$, show that

$$
\mathrm{Tr}(W^\alpha W_\alpha\mid_{\theta\theta}) = -2i\lambda^a\sigma^\mu\nabla_\mu\bar\lambda^a + D^aD^a - \frac{1}{2}F^{a\mu\nu}F^a_{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\,, \tag{4.31}
$$

and from here derive (4.30).

Now we can present the general Lagrangian that describes chiral multiplets coupled to a gauge field. Let the chiral superfields Φ_i belong to a given representation of the gauge group in which the generators are the matrices T^a_{ij}. The kinetic energy term $\Phi^\dagger_i\Phi_i$ is invariant under global gauge transformations $\Phi' = e^{-i\Lambda}\Phi$. In the local case, to insure that Φ' remains a chiral superfield, Λ has to be a chiral superfield. The supersymmetric gauge invariant kinetic energy term is then given by $\Phi^\dagger e^{-2V}\Phi$. The full $\mathcal{N}=1$ supersymmetric Lagrangian is

$$
\mathcal{L} = \frac{1}{8\pi}\,\mathrm{Im}\left(\tau\mathrm{Tr}\int d^\theta\, W^\alpha W_\alpha\right) + \int d^2\theta d^2\bar\theta\,\Phi^\dagger e^{-2V}\Phi + \int d^2\theta\,\mathcal{W} + \int d^2\bar\theta\,\bar{\mathcal{W}}\,. \tag{4.32}
$$

Exercise 4.4. Expand (4.32) in components to obtain

$$\begin{aligned}
\mathcal{L} \;=\;& -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\widetilde{F}^{a\mu\nu} - \frac{i}{g^2}\,\lambda^a\sigma^\mu\nabla_\mu\bar{\lambda}^a + \frac{1}{2g^2}D^aD^a \\
+\;& (\partial_\mu A - iA^a_\mu T^a A)^\dagger(\partial^\mu A - iA^{a\mu}T^a A) - i\,\bar{\psi}\bar{\sigma}^\mu(\partial_\mu\psi - iA^a_\mu T^a\psi) \\
-\;& D^a A^\dagger T^a A - i\sqrt{2}\,A^\dagger T^a\lambda^a\psi + i\sqrt{2}\,\bar{\psi}T^a A\bar{\lambda}^a + F_i^\dagger F_i \\
+\;& \frac{\partial\mathcal{W}}{\partial A_i}F_i + \frac{\partial\bar{\mathcal{W}}}{\partial A_i^\dagger}F_i^\dagger - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial A_i\partial A_j}\,\psi_i\psi_j - \frac{1}{2}\frac{\partial^2\bar{\mathcal{W}}}{\partial A_i^\dagger\partial A_j^\dagger}\,\bar{\psi}_i\bar{\psi}_j\,.
\end{aligned} \tag{4.33}$$

In (4.33), the auxiliary fields F and D^a can be eliminated by using their equations of motion. The terms involving these fields, thus, give rise to the scalar potential

$$V = \sum_i \left|\frac{\partial\mathcal{W}}{\partial A_i}\right|^2 - \frac{1}{2}g^2(A^\dagger T^a A)^2\,. \tag{4.34}$$

5 $\mathcal{N}=2$ super Yang-Mills theory

To construct topological field theories in four dimensions, we are actually interested in models with two supersymmetries (*i.e.* with eight supercharges). In this section we will present $\mathcal{N}=2$ supersymmetric Yang-Mills theory in some detail, following the conventions in [27,32]. We will use the $\mathcal{N}=1$ superspace formalism of the previous section as our starting point, and then we will write the supersymmetric transformations in a manifest $\mathcal{N}=2$ supersymmetric way.

$\mathcal{N}=2$ Yang-Mills theory contains a real superfields V, and a massless chiral superfield Φ, both in the adjoint representation of the gauge group G. The action, written in $\mathcal{N}=1$ superspace, is a particular case of the general action (4.32) with $W=0$, and it reads:

$$\int d^4x d^2\theta \mathrm{Tr}(W^2) + \int d^4x d^2\bar{\theta}\mathrm{Tr}(W^{\dagger 2}) + \int d^4x d^2\theta d^2\bar{\theta}\mathrm{Tr}(\Phi^{\dagger k}\mathrm{e}^{2V_{kl}}\Phi^l). \tag{5.1}$$

In this equation, $V_{kl} = T^a{}_{kl}V^a$, where T^a is a Hermitian basis for the Lie algebra in the adjoint representation, and the real superfield is in the WZ gauge, with components A_μ, $\lambda^1{}_\alpha = \lambda_{2\alpha}$ and D (all in the adjoint representation of the gauge group):

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu - i\bar{\theta}^2\theta\lambda^1 - i\theta^2\bar{\theta}\bar{\lambda}^2 + \frac{1}{2}\theta^2\bar{\theta}^2 D. \tag{5.2}$$

Notice that the conjugate of $\lambda^1{}_\alpha$ is $\bar{\lambda}_{1\dot{\alpha}} = -\bar{\lambda}^2{}_{\dot{\alpha}}$. The chiral superfield Φ, also in the adjoint representation, has components ϕ, $\lambda^2{}_\alpha = -\lambda_{1\alpha}$ and F:

$$\begin{aligned}
\Phi \;&= \phi^a + \sqrt{2}\theta\lambda^2 + \theta^2 F, \\
\Phi^\dagger \;&= \phi^\dagger + \sqrt{2}\bar{\theta}\bar{\lambda}^1 + \bar{\theta}^2\overline{F},
\end{aligned} \tag{5.3}$$

and the conjugate of $\lambda^2{}_\alpha$ is $\bar{\lambda}_{2\dot{\alpha}} = \bar{\lambda}^1{}_{\dot{\alpha}}$. We can now write the action in components. First, we redefine the auxiliary field D as $D \to D + [\phi,\phi^\dagger]$. The action then reads:

$$S \;= \int d^4x \mathrm{Tr}\big\{\nabla_\mu\phi^\dagger\nabla^\mu\phi - i\lambda^1\sigma^\mu\nabla_\mu\bar{\lambda}_1 - i\lambda^2\sigma^\mu\nabla_\mu\bar{\lambda}_2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} +$$

$$+\tfrac{1}{2}D^2+|F|^2-\tfrac{1}{2}[\phi,\phi^\dagger]^2-i\sqrt{2}\lambda^{1\alpha}[\phi^\dagger,\lambda^2_\alpha]+i\sqrt{2}\bar{\lambda}_{1\dot{\alpha}}[\bar{\lambda}^{\dot{\alpha}}_2,\phi]\big\}. \tag{5.4}$$

This action is not manifestly $\mathcal{N}=2$ supersymmetric, since the $\mathcal{N}=2$ supersymmetric algebra has an internal $SU(2)_R$ symmetry, which is not manifest in (5.4). $SU(2)_R$ invariance is easily achieved: the scalars ϕ and the gluons A_μ are singlets, while the gluinos λ_v, $v=1,2$ form a doublet. The auxiliary fields form a real triplet:

$$D^{vw}=\left(\begin{array}{cc}\sqrt{2}F & iD\\ iD & \sqrt{2F}\end{array}\right), \tag{5.5}$$

and the $SU(2)_R$ indices are raised and lowered with the matrices ϵ_{vw}, ϵ^{vw}. Notice that $D^{vw}=D^*_{vw}$. Finally, by covariantizing the $\mathcal{N}=1$ transformations, one finds the $\mathcal{N}=2$ transformations:

$$\begin{array}{rcl}
[Q_{v\alpha},\phi] &=& i\sqrt{2}\lambda_{v\alpha},\\
\left[Q_{v\alpha},\phi^\dagger\right] &=& 0,\\
[Q_{v\alpha},A_\mu] &=& (\sigma_\mu)_{\alpha\dot{\beta}}\bar{\lambda}_v{}^{\dot{\beta}}\\
[Q_{v\alpha},\lambda_{w\beta}] &=& -i\epsilon_{\alpha\beta}D_{vw}-\left[\phi,\phi^\dagger\right]\epsilon_{\alpha\beta}\epsilon_{vw}-i(\sigma^{\mu\nu})_{\alpha\beta}\epsilon_{vw}F_{\mu\nu},\\
\left[Q_{v\alpha},\bar{\lambda}_{w\dot{\beta}}\right] &=& \sqrt{2}(\bar{\sigma}^\mu)_{\dot{\beta}\alpha}\nabla_\mu\phi^\dagger\epsilon_{vw},\\
[Q_{u\alpha},D_{vw}] &=& (\sigma^\mu)_{\alpha\dot{\beta}}\big(\epsilon_{uv}(\nabla_\mu\bar{\lambda})_w{}^{\dot{\beta}}+\epsilon_{uw}(\nabla_\mu\bar{\lambda})_v{}^{\dot{\beta}}\big)+\sqrt{2}\big(\epsilon_{uv}\left[\lambda_{w\alpha},\phi^\dagger\right]+\epsilon_{uw}\left[\lambda_{v\alpha},\phi^\dagger\right]\big),\\
\left[\overline{Q}_{v\dot{\alpha}},\phi\right] &=& 0,\\
\left[\overline{Q}_{v\dot{\alpha}},\phi^\dagger\right] &=& i\sqrt{2}\bar{\lambda}_{v\dot{\alpha}},\\
\left[\overline{Q}_{v\dot{\alpha}},A_\mu\right] &=& (\bar{\sigma}_\mu)_{\dot{\alpha}\beta}\lambda_v{}^\beta,\\
[Q_{v\alpha},\lambda_{w\beta}] &=& -\sqrt{2}(\sigma^\mu)_{\beta\dot{\alpha}}\nabla_\mu\phi\epsilon_{vw},\\
\left[\overline{Q}_{v\dot{\alpha}},\bar{\lambda}_{w\dot{\beta}}\right] &=& i\epsilon_{\dot{\alpha}\dot{\beta}}D_{vw}+\left[\phi,\phi^\dagger\right]\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{vw}+i(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\epsilon_{vw}F_{\mu\nu},\\
\left[\overline{Q}_{u\dot{\alpha}},D_{vw}\right] &=& -(\bar{\sigma}^\mu)_{\dot{\alpha}\beta}\big(\epsilon_{uv}(\nabla_\mu\lambda)_w{}^\beta+\epsilon_{uw}(\nabla_\mu\lambda)_v{}^\beta\big)\\
&-& \sqrt{2}\big(\epsilon_{uv}\left[\bar{\lambda}_{w\dot{\alpha}},\phi\right]+\epsilon_{uw}\left[\bar{\lambda}_{v\dot{\alpha}},\phi\right]\big).
\end{array} \tag{5.6}$$

The action reads, once the $SU(2)_R$ symmetry is manifest,:

$$\begin{array}{rcl}
S &=& \displaystyle\int d^4x\mathrm{Tr}\big\{\nabla_\mu\phi^\dagger\nabla^\mu\phi-i\lambda^v\sigma^\mu\nabla_\mu\bar{\lambda}_v-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{4}D_{vw}D^{vw}\\
&-& \displaystyle\frac{1}{2}\left[\phi,\phi^\dagger\right]^2-\frac{i}{\sqrt{2}}\epsilon^{vw}\lambda_v{}^\alpha\left[\phi^\dagger,\lambda_{w\alpha}\right]-\frac{i}{\sqrt{2}}\epsilon_{vw}\bar{\lambda}^v{}_{\dot{\alpha}}\left[\bar{\lambda}^{w\dot{\alpha}},\phi\right]\big\}.
\end{array} \tag{5.7}$$

The above action also has a classical $U(1)_R$ symmetry:

$$\begin{array}{rclcrcl}
A_\mu &\to& A_\mu, && D_{vw} &\to& D_{vw},\\
\lambda_{v\alpha} &\to& \mathrm{e}^{i\varphi}\lambda_{v\alpha}, && \phi &\to& \mathrm{e}^{2i\varphi}\phi,\\
\bar{\lambda}^v{}_{\dot{\alpha}} &\to& \mathrm{e}^{-i\varphi}\bar{\lambda}^v{}_{\dot{\alpha}}, && \phi^\dagger &\to& \mathrm{e}^{-2i\varphi}\phi^\dagger.
\end{array} \tag{5.8}$$

6 Topological field theories from twisted supersymmetry

In this section, we introduce topological field theories and we give a brief general overview of their properties, focusing on the so-called theories of the Witten or cohomological type. We then explain the twisting procedure, which produces topological field theories from $\mathcal{N} = 2$ theories, and put it in practice with the examples of the previous section. General introductions to topological field theories can be found in [7, 9, 25], among other references.

6.1 Topological field theories: basic properties

Topological field theories (TFT's) were first introduced by Witten in [43]. A quantum field theory is topological if, when put on a manifold X with a Riemannian metric $g_{\mu\nu}$, the correlation functions of some set of operators do not depend (at least formally) on the metric. We then have

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle = 0, \tag{6.1}$$

where $\mathcal{O}_{i_1}, \cdots, \mathcal{O}_{i_n}$ are operators in the theory. There are two different types of TFT's: in the TFT's theories of the Schwarz type, one tries to define all the ingredients in the theory (the action, the operators, and so on) without using the metric of the manifold. The most important example is Chern-Simons theory, introduced by Witten in [45]. In the TFT's of the Witten type, one has an explicit metric dependence, but the theory has an underlying scalar symmetry δ acting on the fields in such a way that the correlation functions of the theory do not depend on the background metric. More precisely, if the energy-momentum tensor of the theory $T_{\mu\nu} = (\delta/\delta g^{\mu\nu}) S(\phi_i)$ can be written as

$$T_{\mu\nu} = -i\delta G_{\mu\nu} \tag{6.2}$$

where $G_{\mu\nu}$ is some tensor, then (6.1) holds for any operator $\mathcal{O}$ which is δ-invariant. This is because:

$$\begin{aligned} & \frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O}_{i_1} \mathcal{O}_{i_2} \cdots \mathcal{O}_{i_n} \rangle = \langle \mathcal{O}_{i_1} \mathcal{O}_{i_2} \cdots \mathcal{O}_{i_n} T_{\mu\nu} \rangle \\ = \quad & -i \langle \mathcal{O}_{i_1} \mathcal{O}_{i_2} \cdots \mathcal{O}_{i_n} \delta G_{\mu\nu} \rangle = \pm i \langle \delta (\mathcal{O}_{i_1} \mathcal{O}_{i_2} \cdots \mathcal{O}_{i_n} G_{\mu\nu}) \rangle = 0. \end{aligned} \tag{6.3}$$

In this derivation we have used the fact that δ is a symmetry of the classical action $S(\phi_i)$ and of the quantum theory. Such a symmetry is called a *topological symmetry*. In some situations this symmetry is anomalous, and the theory is not strictly topological. However, in most of the interesting cases, this dependence is mild and under control. We will see a very explicit example of this in Donaldson theory on manifolds of $b_2^+ = 1$.

If the theory is topological, as we have described it, the natural operators are then those which are δ-invariant. On the other hand, operators which are δ-exact decouple from the theory, since their correlation functions vanish. The operators that are in the cohomology of δ are called *topological observables*:

$$\mathcal{O} \in \frac{\mathrm{Ker}\, \delta}{\mathrm{Im}\, \delta}. \tag{6.4}$$

The topological symmetry δ is not nilpotent: in general one has

$$\delta^2 = Z \tag{6.5}$$

where Z is a certain transformation in the theory. It can be a local transformation (a gauge transformation) or a global transformation (for example, a global $U(1)$ symmetry). The appropriate framework to analyze the observables is then *equivariant cohomology*, and for consistency one has to consider only operators that are invariant under the transformation generated by Z (for example, gauge invariant operators). Equivariant cohomology turns out to be a very natural language to describe TFT's with local and global symmetries, but we are not going to explore it in these lectures. The interested reader should look at [9, 26].

The structure of topological field theories of the Witten type leads immediately to a general version of the Donaldson map [43]. Remember that, starting with the curvature of the universal bundle, this map associates cohomology classes in the instanton moduli space to homology classes in the four-manifold. One can easily see that in any theory where (6.2) is satisfied, one can define topological observables associated to homology cycles in spacetime. If (6.2) holds, then one has:

$$P_\mu = T_{0\mu} = -i\delta G_\mu, \tag{6.6}$$

where

$$G_\mu \equiv G_{0\mu}. \tag{6.7}$$

In the theories that we are goint to consider, δ is essentially given by a supersymmetric transformation, and therefore it is a Grassmannian symmetry. It follows that G_μ is an anticommuting operator. On the other hand, from the point of view of the Lorentz group, they are a scalar and a one-form, respectively. Then, topological field theories of Witten type violate the spin-statistics theorem. Consider now a δ-invariant operator $\phi^{(0)}(x)$. The *descent operators* are defined as

$$\phi^{(n)}_{\mu_1\mu_2\cdots\mu_n}(x) = G_{\mu_1}G_{\mu_2}\cdots G_{\mu_n}\phi^{(0)}(x), \quad n = 1, \cdots, d, \tag{6.8}$$

where d is the dimension of the spacetime manifold. Since the G_{μ_i} anticonmute, $\phi^{(n)}$ is antisymmetric in the indices $\mu_1, \cdots, \mu_n$, and therefore it gives an n-form:

$$\phi^{(n)} = \frac{1}{n!}\phi^{(n)}_{\mu_1\mu_2\cdots\mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}. \tag{6.9}$$

As an immediate consequence of (6.6) and the δ-invariance of $\phi^{(0)}$, one has the following *descent equations*:

$$d\phi^{(n)} = \delta\phi^{(n+1)}, \tag{6.10}$$

where d is the exterior derivative and we have taken into account that $P_\mu = -i\partial_\mu$. The descent equations can be also obtained by considering the cohomology of the operator $d + \delta$, see [5, 9] for more details. Using now (6.10) it is easy to see that the operator

$$W^{(\gamma_n)}_{\phi^{(0)}} = \int_{\gamma_n} \phi^{(n)}, \tag{6.11}$$

where $\gamma_n \in H_n(X)$, is a topological observable:

$$\delta W_{\phi^{(0)}}^{(\gamma_n)} = \int_{\gamma_n} \delta\phi^{(n)} = \int_{\gamma_n} d\phi^{(n-1)} = \int_{\partial\gamma_n} \phi^{(n-1)} = 0, \tag{6.12}$$

since $\partial\gamma_n = 0$.

Exercise 6.1. *Homology and observables.* Show that, if γ_n is trivial in homology (*i.e.*, if it is ∂-exact), then $W_{\phi^{(0)}}^{(\gamma_n)}$ is δ-exact.

Therefore, given a (scalar) topological observable, one can construct a family of topological observables

$$W_{\phi^{(0)}}^{(\gamma_{i_n})}, \quad i_n = 1, \cdots, b_n; \quad n = 1, \cdots, d, \tag{6.13}$$

in one-to-one correspondence with the homology classes of spacetime. This descent procedure is the analog of the Donaldson map in Donaldson-Witten and Seiberg-Witten theory. Notice that any family of operators $\phi^{(n)}$ that satisfies the descent equations (6.10) gives topological observables. The explicit realization (6.8) in terms of the G operator can then be regarded as a canonical solution to (6.10).

6.2 Twist of $\mathcal{N} = 2$ supersymmetry

In the early eighties, Witten noticed in two seminal papers [41, 42] that supersymmetry has a deep relation to topology. The simplest example of such a relation is supersymmetric quantum mechanics, which provides a physical reformulation (and in fact a refinement) of Morse theory [42]. Other examples are $\mathcal{N} = 2$ theories in two and four dimensions. In 1988 Witten discovered that, by changing the coupling to gravity of the fields in an $\mathcal{N} = 2$ theory in two or four dimensions, a theory satisfying the requirements of the previous subsection was obtained [43, 44]. This redefinition of the theory is called *twisting*. We are now going to explain in some detail how this works in the four-dimensional case.

The $\mathcal{N} = 2$ supersymmetry algebra (with no central charges) is:

$$\begin{array}{rclrcl}
\{Q_{\alpha v}, \overline{Q}_{\dot\beta w}\} & = & 2\epsilon_{vw}\sigma^{\mu}_{\alpha\dot\beta}P_\mu, & \{Q_{\alpha v}, Q_{\beta w}\} & = & 0, \\
{[P_\mu, Q_{\alpha v}]} & = & 0, & \left[P_\mu, \overline{Q}\dot\alpha v\right] & = & 0, \\
{[M_{\mu\nu}, Q_{\alpha v}]} & = & -(\sigma_{\mu\nu})_\alpha{}^\beta Q_{\beta v}, & \left[M_{\mu\nu}, \overline{Q}^{\dot\alpha v}\right] & = & -(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\overline{Q}^{\dot\beta v}, \\
{[Q_{\alpha v}, B^a]} & = & -\frac{1}{2}(\tau^a)_v{}^w Q_{\alpha w}, & \left[\overline{Q}_{\dot\alpha}{}^v, B^a\right] & = & \frac{1}{2}\overline{Q}_{\dot\alpha}{}^w(\tau^a)_w{}^v, \\
{[Q_{\alpha v}, R]} & = & Q_{\alpha v}, & \left[\overline{Q}_{\dot\alpha v}, R\right] & = & -\overline{Q}_{\dot\alpha v}.
\end{array} \tag{6.14}$$

Here, the indices $v, w \in \{1, 2\}$. The twisting procedure consists of redefining the coupling to gravity of the theory, *i.e.* in redefining the spins of the fields. To do this, we couple the fields to the $SU(2)_+$ spin connection according to their isospin. This means that we *add* to the Lagrangian the term

$$J^R_\mu \omega^\mu_+, \tag{6.15}$$

where J^R_μ is the $SU(2)_R$ current of the theory, and ω^μ_+ is the $SU(2)_+$ spin connection. We then have a new rotation group $\mathcal{K}' = SU'(2)_+ \otimes SU(2)_-$, where $SU'(2)_+$ is the diagonal of $SU(2)_+ \times SU(2)_R$. In practice, the twist means essentially that the $SU(2)_R$ indices v, w become spinorial indices $\dot\alpha, \dot\beta$, and we have the change $\overline{Q}_{\dot\alpha v} \to \overline{Q}_{\dot\alpha\dot\beta}$ and $Q_{\alpha v} \to Q_{\alpha\dot\beta}$. It is easy to check that the *topological supercharge*

$$\overline{\mathcal{Q}} \equiv \epsilon^{\dot\alpha\dot\beta}\overline{Q}_{\dot\alpha\dot\beta} = \overline{Q}_{\dot1\dot2} - \overline{Q}_{\dot2\dot1}. \tag{6.16}$$

is a scalar with respect to $\mathcal{K}'$. This topological supercharge will provide the topological symmetry δ that we need for a topological theory. The $\mathcal{N}=2$ algebra also gives a natural way to construct the operator G_μ defined in (6.7). In fact, define:

$$G_\mu = \frac{i}{4}(\bar\sigma_\mu)^{\dot\alpha\gamma}Q_{\gamma\dot\alpha}. \tag{6.17}$$

Using now the $\{Q, \overline{Q}\}$ anticommutator one can show that

$$\{\overline{\mathcal{Q}}, G_\mu\} = \partial_\mu. \tag{6.18}$$

This means that the supersymmetry algebra by itself almost guarantees (6.2), since it implies that the momentum operator P_μ is $\overline{\mathcal{Q}}$-exact. In the models that we will consider, (6.2) is true (at least on-shell). Finally, notice that from the anticommutator $\{\overline{Q}, \overline{Q}\}$ in (6.14) follows that the topological supercharge is nilpotent $\overline{\mathcal{Q}}^2 = 0$ (in the absence of central charge).

Our main conclusion is that *by twisting $\mathcal{N}=2$ supersymmetry one can construct a quantum field theory that satisfies (almost) all the requirements of a topological field theory of the Witten type.*

7 Donaldson-Witten theory

Donaldson-Witten theory (also known as topological Yang-Mills theory) is the topological field theory that results from twisting $\mathcal{N}=2$ Yang-Mills theory in four dimensions. Historically it was the first TFT of the Witten type, and as we will see it provides a physical realization of Donaldson theory.

7.1 The topological action

Remember that $\mathcal{N}=2$ super Yang-Mills theory contains a gauge field A_μ, two gluinos $\lambda_{v\alpha}$ and a complex scalar ϕ, all of them in the adjoint representation of the gauge group $\mathcal{G}$. In the off-shell formulation, we also have auxiliary fields D_{vw} in the **3** of the internal $SU(2)_R$. The total symmetry group of the theory is

$$\mathcal{H} = SU(2)_+ \times SU(2)_- \times SU(2)_R \times U(1)_R. \tag{7.1}$$

Under the twist, the fields in the $\mathcal{N}=2$ supermultiplet change their spin content as follows:

$$\begin{array}{ccc} A_\mu\ (1/2,1/2,0)^0 & \to & A_\mu\ (1/2,1/2)^0, \\ \lambda_{v\alpha}\ (1/2,0,1/2)^{-1} & \to & \psi_{\dot\beta\alpha}\ (1/2,1/2)^1, \\ \bar\lambda_{v\dot\alpha}\ (0,1/2,1/2)^1 & \to & \eta\ (0,0)^{-1}, \quad \chi_{\dot\alpha\dot\beta}\ (1,0)^{-1}, \\ \phi\ (0,0,0)^{-2} & \to & \phi\ (0,0)^{-2}, \\ \phi^\dagger\ (0,0,0)^2 & \to & \phi^\dagger\ (0,0)^2, \\ D_{vw}\ (0,0,1)^0 & \to & D_{\dot\alpha\dot\beta}\ (1,0)^0, \end{array} \tag{7.2}$$

where we have written the quantum numbers with respect to the group H before the twist, and with respect to the group $\mathcal{H}' = SU(2)'_+ \otimes SU(2)_- \otimes U(1)_R$ after the twist. In the topological theory, the $U(1)_R$ charge is usually called *ghost number*. The η and χ fields are given by the antisymmetric and symmetric pieces of $\overline{\lambda}_{\dot\alpha\dot\beta}$, respectively. More precisely:

$$\chi_{\dot\alpha\dot\beta} = \overline{\lambda}_{(\dot\alpha\dot\beta)}, \quad \eta = \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\overline{\lambda}_{\dot\alpha\dot\beta}. \tag{7.3}$$

From the $\mathcal{N}=2$ action it is straightforward to find:

$$\begin{aligned} S &= \int d^4x\,\sqrt{g}\mathrm{Tr}\,\{\nabla_\mu\phi\nabla^\mu\phi^\dagger - i\psi^{\dot\beta}{}_\alpha\nabla^{\dot\alpha\alpha}\chi_{\dot\beta\dot\alpha} - i\psi_{\dot\alpha\alpha}\nabla^{\dot\alpha\alpha}\eta - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ \frac{1}{4}D_{\dot\alpha\dot\beta}D^{\dot\alpha\dot\beta} - \frac{1}{2}[\phi,\phi^\dagger]^2 - \frac{i}{\sqrt{2}}\chi^{\dot\alpha\dot\beta}[\phi,\chi_{\dot\alpha\dot\beta}] + i\sqrt{2}\eta[\phi,\eta] - \frac{i}{\sqrt{2}}\psi_{\dot\alpha\alpha}[\psi^{\dot\alpha\alpha},\phi^\dagger]\} \end{aligned} \tag{7.4}$$

where $\nabla^{\dot\alpha\alpha} = \bar\sigma^{\mu\dot\alpha\alpha}\nabla_\mu$. The $\overline{\mathcal{Q}}$-transformations are easily obtained from the $\mathcal{N}=2$ transformations:

$$\begin{array}{rclrcl} [\overline{\mathcal{Q}},\phi] &=& 0, & [\overline{\mathcal{Q}},\phi^\dagger] &=& 2\sqrt{2}i\eta, \\ [\overline{\mathcal{Q}},A_\mu] &=& \psi_\mu, & \{\overline{\mathcal{Q}},\chi_{\dot\alpha\dot\beta}\} &=& i(F^+_{\dot\alpha\dot\beta} - D_{\dot\alpha\dot\beta}), \\ \{\overline{\mathcal{Q}},\eta\} &=& [\phi,\phi^\dagger], & [\overline{\mathcal{Q}},D] &=& (2\nabla\psi)_+ + 2\sqrt{2}[\phi,\chi]. \\ \{\overline{\mathcal{Q}},\psi_\mu\} &=& 2\sqrt{2}\nabla_\mu\phi, & & & \end{array} \tag{7.5}$$

In (7.5), $\psi_\mu = \sigma_{\mu\alpha\dot\beta}\psi^{\alpha\dot\beta}$ and $F^+_{\dot\alpha\dot\beta} = \bar\sigma^{\mu\nu}_{\dot\alpha\dot\beta}F_{\mu\nu}$ is the selfdual part of $F_{\mu\nu}$. It is not difficult to show that the action of Donaldson-Witten theory is $\overline{\mathcal{Q}}$-exact up to a topological term, *i.e.*

$$S = \{\overline{\mathcal{Q}}, V\} - \frac{1}{2}\int F\wedge F, \tag{7.6}$$

where

$$V = \int d^4x\,\mathrm{Tr}\,\{\,\frac{i}{4}\chi_{\dot\alpha\dot\beta}(F^{\dot\alpha\dot\beta} + D^{\dot\alpha\dot\beta}) - \frac{1}{2}\eta[\phi,\eta] + \frac{1}{2\sqrt{2}}\psi_{\alpha\dot\alpha}\nabla^{\dot\alpha\alpha}\phi^\dagger\,\}. \tag{7.7}$$

As we will see in a moment, this has important implications for the quantum behavior of the theory.

Exercise 7.1. *The Lagrangian of Donaldson-Witten theory.* Derive (7.4) and (7.6).

One of the most interesting aspects of the twisting procedure is the following: if we put the original $\mathcal{N}=2$ Yang-Mills theory on an arbitrary Riemannian four-manifold, using the well-known prescription of minimal coupling to gravity, we find global obstructions to have a well-defined theory. The reason is very simple: not every four-manifold is Spin, and therefore the fermionic fields $\lambda_{\alpha v}$ are not well-defined unless $w_2(X)=0$. However, after the twisting, all fields are differential forms on X, and therefore the twisted theory makes sense globally on an arbitrary Riemannian four-manifold.

7.2 The observables

The observables of Donaldson-Witten theory can be constructed by using the topological descent equations. As we have emphasized, these equations have a canonical solution given by the operator (6.17). Using again the $\mathcal{N}=2$ supersymmetry transformations, one can work out the action of G_μ on the different fields of the theory. The result is:

$$\begin{array}{rclrcl} [G_\mu,\phi] &=& \frac{1}{2\sqrt{2}}\psi_\mu, & [G,\overline{\phi}] &=& 0, \\ [G_\nu,A_\mu] &=& \frac{i}{2}g_{\mu\nu}\eta - i\chi_{\mu\nu}, & [G,F^+] &=& i\nabla\chi + \frac{3i}{2}*\nabla\eta, \\ [G,\eta] &=& -\frac{i\sqrt{2}}{4}\nabla\overline{\phi}, & \{G,\chi\} &=& -\frac{3i\sqrt{2}}{8}*\nabla\overline{\phi}, \\ \{G_\mu,\psi_\nu\} &=& -(F^-_{\mu\nu}+D^+_{\mu\nu}), & [G,D] &=& -\frac{3i}{4}*\nabla\eta + \frac{3i}{2}\nabla\chi. \end{array} \tag{7.8}$$

We can now construct the topological observables of the theory by using the descent equations. The starting point must be a gauge-invariant, $\overline{\mathcal{Q}}$-closed operator which is not $\overline{\mathcal{Q}}$-trivial. Since $[\overline{\mathcal{Q}},\phi]=0$, the simplest candidates are the operators

$$\mathcal{O}_n = \mathrm{Tr}(\phi^n), \quad n=2,\cdots,N. \tag{7.9}$$

Here we are going to restrict ourselves to $SU(2)$, therefore the starting point for the descent procedure will be the operator,

$$\mathcal{O} = \mathrm{Tr}(\phi^2). \tag{7.10}$$

It is easy to see that the following operators satisfy the descent equations (6.10):

$$\begin{array}{rcl} \mathcal{O}^{(1)} &=& \mathrm{Tr}\,(\frac{1}{\sqrt{2}}\phi\psi_\mu)dx^\mu, \\ \mathcal{O}^{(2)} &=& -\frac{1}{2}\mathrm{Tr}\,(\frac{1}{\sqrt{2}}\phi F_{\mu\nu} - \frac{1}{4}\psi_\mu\psi_\nu)dx^\mu\wedge dx^\nu, \\ \mathcal{O}^{(3)} &=& -\frac{1}{8}\mathrm{Tr}\,(\psi_\lambda F_{\mu\nu})dx^\lambda\wedge dx^\mu\wedge dx^\nu, \\ \mathcal{O}^{(4)} &=& \frac{1}{32}\mathrm{Tr}\,(F_{\lambda\tau}F_{\mu\nu})dx^\lambda\wedge dx^\tau\wedge dx^\mu\wedge dx^\nu. \end{array} \tag{7.11}$$

Notice for example that

$$\{\overline{\mathcal{Q}},\mathcal{O}^{(1)}\} = 2\mathrm{Tr}(\phi\nabla_\mu\phi)dx^\mu = d\mathcal{O}. \tag{7.12}$$

so the first descent equation is satisfied. This is, however, not the canonical solution to the descent equations provided by G, which in this case is a little bit more complicated.

Exercise 7.2. *Descent equations in topological Yang-Mills theory.* Show that (7.11) satisfy the descent equations. Compare with the canonical solution.

The observables

$$I_1(\delta) = \int_\delta \mathcal{O}^{(1)}, \qquad I_2(S) = \int_S \mathcal{O}^{(2)}, \tag{7.13}$$

where $\delta\in H_1(X)$, $S\in H_2(X)$, correspond to the differential forms on the moduli space of ASD connections that were introduced in (3.53) through the use of the Donaldson map (3.52) (and this is why we have used the same notation for both). Notice that the ghost number of the operators in (7.11) is in fact their degree as differential forms in moduli space. The operators (7.11) are naturally interpreted as the decomposition of the Pontriagin class of the universal bundle (3.51) with respect to the bigrading of $\Omega^*(\mathcal{B}^* \times X)$. In fact, the Grasmannian field ψ_μ can be interpreted as a $(1,1)$ form: a one-form in spacetime and also a one-form in the space $\mathcal{A}$. The operator $\overline{\mathcal{Q}}$ is then interpreted as the equivariant differential in $\mathcal{A}$ with respect to gauge transformations. This leads to a beautiful geometric interpretation of topological Yang-Mills theory in terms of equivariant cohomology [23] and the Mathai-Quillen formalism [3], which is reviewed in detail in [9].

7.3 Evaluation of the path integral

We now consider the topological theory defined by the topological Yang-Mills action, $S_{\rm TYM} = \{\overline{\mathcal{Q}}, V\}$, where V is defined in (7.7). The evaluation of the path integral of the theory defined by the Donaldson-Witten action can be drastically simplified by taking into account the following fact. The (unnormalized) correlation functions of the theory are defined by

$$\langle \phi_1 \cdots \phi_n \rangle = \int \mathcal{D}\phi \, \phi_1 \cdots \phi_n \, {\rm e}^{-\frac{1}{g^2} S_{\rm TYM}}, \tag{7.14}$$

where $\phi_1, \cdots, \phi_n$ are generic fields, and g is the coupling constant. Since $S_{\rm TYM}$ is $\overline{\mathcal{Q}}$-exact, one has:

$$\frac{\partial}{\partial g} \langle \phi_1 \cdots \phi_n \rangle = \frac{2}{g^3} \langle \phi_1 \cdots \phi_n S_{\rm TYM} \rangle = 0, \tag{7.15}$$

where we have used the fact that $\overline{\mathcal{Q}}$ is a symmetry of the theory, and therefore the insertion of a $\overline{\mathcal{Q}}$-exact operator in the path integral gives zero. The above result is remarkable: it says that, in a topological field theory in which the action is $\overline{\mathcal{Q}}$-exact, the computations do not depend on the value of the coupling constant. In particular, *the semiclassical approximation is exact!* [43]. We can then evaluate the path integral in the saddle-point approximation as follows: first, we look at zero modes, *i.e.* classical configurations that minimize the action. Then we look at nonzero modes, *i.e.* we consider quantum fluctuations around these configurations. Since the saddle-point approximation is exact, it is enough to consider the quadratic fluctuations. The integral over the zero modes gives a finite integral over the space of bosonic collective coordinates, and a finite Grassmannian integral over the zero modes of the fermi fields. The integral over the quadratic fluctuations gives a bunch of determinants. Since the theory has a bose-fermi $\overline{\mathcal{Q}}$ symmetry, it is easy to see that the determinants cancel (up to a sign), as in supersymmetric theories.

Let us then analyze the bosonic and fermionic zero modes. A quick way to find the bosonic zero modes is to look for supersymmetric configurations. These are classical configurations such that $\{\overline{\mathcal{Q}}, {\rm fermi}\} = 0$ for all Fermi fields in the theory, and they give minima of the Lagrangian. Indeed, it was shown in [46] that in topological field theories with a fermionic symmetry $\overline{\mathcal{Q}}$ one can compute by localization on the fixed points of this symmetry. In this case, by looking at $\{\overline{\mathcal{Q}}, \chi\} = 0$, one finds

$$F^+ = D^+. \tag{7.16}$$

But on-shell $D^+ = 0$, and therefore (7.16) reduces to the usual ASD equations. The zero modes of the gauge field are then instanton configurations. In addition, by looking at $\{\overline{\mathcal{Q}}, \psi\} = 0$, we find the equation of motion for the ϕ field,

$$\nabla_A \phi = 0. \tag{7.17}$$

This equation is also familiar: as we saw in section 3, its nontrivial solutions correspond to reducible connections. Let us assume for simplicity that we are in a situation in which no reducible solutions occur, so that $\phi = 0$. In that case, (7.16) tells us that the integral over the collective coordinates reduces to an integral over the instanton moduli space $\mathcal{M}_{\rm ASD}$.

Let us now look at the fermionic zero modes in the background of an instanton. The kinetic terms for the ψ, χ and η fermions fit precisely into the instanton deformation complex (3.44). Therefore, using the index theorem we can compute:

$$N_\psi - N_\chi = \dim \mathcal{M}_{\rm ASD}, \tag{7.18}$$

where $N_{\psi,\chi}$ denotes the number of zeromodes of the corresponding fields, and we have used the fact that the connection A is irreducible, so that η (which is a scalar) has no zero modes (in other words, $\nabla_A \eta = 0$ only has the trivial solution). Finally, if we assume that the connection is regular, then one has that $\mathrm{Coker}\, p^+ \nabla_A = 0$, and there are no χ zero modes. In this situation, the number of ψ zero modes is simply the dimension of the moduli space of ASD instantons. If we denote the bosonic and the fermionic zero modes by da_i, $d\psi_i$, respectively, where $i = 1, \cdots, D$ and $D = \dim \mathcal{M}_{\rm ASD}$, then the zero-mode measure becomes:

$$\prod_{i=1}^{D} da_i d\psi_i. \tag{7.19}$$

This is in fact the natural measure for integration of differential forms on $\mathcal{M}_{\rm ASD}$, and the Grassmannian variables ψ_i are then interpreted as a basis of one-forms on $\mathcal{M}_{\rm ASD}$.

We can already discuss how to compute correlation functions of the operators $\mathcal{O}$, $I_2(S)$, and $I_1(\delta)$. These operators contain the fields ψ, A_μ and ϕ. In evaluating the path integral, it is enough to replace ψ and A_μ by their zero modes, and the field ϕ (with no zero modes) by its quantum fluctuations, that we then integrate out at quadratic order. Further corrections are higher order in the coupling constant and do not contribute to the saddle-point approximation, which in this case is exact. We have then to compute the one-point function $\langle \phi^a \rangle$. The relevant terms in the action are

$$S(\phi, \phi^\dagger) = \int d^4x \mathrm{Tr}\{\nabla_\mu \phi \nabla^\mu \phi^\dagger - \frac{i}{\sqrt{2}} \phi^\dagger [\psi_\mu, \psi^\mu]\}, \tag{7.20}$$

since we are only considering quadratic terms. We then have to compute

$$\langle \phi^a(x) \rangle = \int \mathcal{D}\phi \mathcal{D}\phi^\dagger \phi^a(x) \exp -S(\phi, \phi^\dagger). \tag{7.21}$$

If we take into account that

$$\langle \phi^a(x) \phi^{b\dagger}(y) \rangle = -G^{ab}(x-y), \tag{7.22}$$

where $G^{ab}(x-y)$ is the Green's function of the Laplacian $\nabla_\mu \nabla^\mu$, we find:

$$\langle \phi^a(x) \rangle = -\frac{i}{\sqrt{2}} \int d^4y \sqrt{g} G^{ab}(x,y)[\psi(x)_\mu, \psi(y)^\mu]_b. \tag{7.23}$$

This expresses ϕ in terms of zero modes. It turns out that this is precisely (up to a multiplicative constant) the component along $\mathcal{B}^*$ of the curvature $K_{\mathbf{P}}$ of the universal bundle (see for example [13], p. 196). This is in perfect agreement with the correspondence between the observables (7.11) and the differential forms on moduli space (3.53) constructed in Donaldson theory.

The main conclusion of this analysis is that, up to possible normalizations,

$$\langle \mathcal{O}^\ell I_2(S_{i_1}) \cdots I_2(S_{i_p}) I_1(\delta_{j_1}) \cdots \ I_1(\delta_{j_q}) \rangle \tag{7.24}$$
$$= \int_{\mathcal{M}_{\rm ASD}} \mathcal{O}^\ell \wedge I_2(S_{i_1}) \wedge \cdots \wedge I_2(S_{i_p}) \wedge I_1(\delta_{j_1}) \wedge \cdots \wedge \ I_1(\delta_{j_q}), \tag{7.25}$$

i.e. the correlation function of the observables of twisted $\mathcal{N} = 2$ Yang-Mills theory is precisely the corresponding Donaldson invariant. The requirement that the differential form in the r.h.s. has top degree (otherwise the invariant is zero) corresponds, in the field theory side, to the requirement that the correlation has ghost number equal to $\dim \mathcal{M}_{\rm ASD}$, *i.e.* that the operator in the correlation function soaks up all the fermionic zero modes, which is the well-known 't Hooft rule [38]. (7.24) was one of the most important results of Witten's seminal work [43], and it opened a completely different approach to Donaldson theory by means of topological quantum field theory.

8 Conclusions and further developments

What we have covered in these lectures is just the beginning of a very beautiful story that we can only summarize at this concluding section. Since we have a quantum field theory realization of Donaldson-Witten theory, one could imagine that knowledge of the physics of this theory would be extremely useful in learning about the deep mathematics of Donaldson invariants. A first step in that direction was taken by Witten in [47], but a much more ambitious picture appeared after the classical work of Seiberg and Witten on the low-energy effective action of $\mathcal{N} = 2$ super Yang-Mills theory [35]. This led to the introduction [48] of the Seiberg-Witten monopole equations and the Seiberg-Witten invariants. It turns out that the unknown constants a_s in (3.62) are essentially Seiberg-Witten invariants, and the basic classes of Kronheimer and Mrowka can be reformulated in terms of a finite set of Spin_c structures called Seiberg-Witten classes. Moreover, one can evaluate using quantum field theory techniques the Donaldson-Witten generating functional and write it in terms of some universal functions and Seiberg-Witten invariants [32]. In one word, quantum field theory leads to a complete solution of the basic problem in Donaldson theory (the evaluation of Donaldson invariants). To learn about this, we recommend the references [14, 28, 33] (from a mathematical point of view) and [12, 29, 37] from the quantum field theory viewpoint. A systematic account of these developments can be found in [30].

A Conventions for spinors

In this appendix we collect our conventions for spinors (both in Minkowski and Euclidean space). We follow almost strictly [1].

The Minkowski flat metric is $\eta_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$. We raise and lower spinor indices with the antisymmetric tensor $\epsilon_{\alpha\beta}$, $\epsilon_{\dot\alpha\dot\beta}$:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta,$$

where the ϵ tensor is chosen as follows:

$$\epsilon_{21} = \epsilon^{12} = -\epsilon_{12} = -\epsilon^{21} = 1,$$

Contractions satisfy the perverse rule:

$$\psi^\alpha\phi_\alpha = -\psi_\alpha\phi^\alpha.$$

We define the matrices:

$$(\sigma^\mu)_{\alpha\dot\alpha} \equiv (1,\vec\sigma),$$

where $\vec\sigma$ are the Pauli matrices, and after raising indices we find

$$(\bar\sigma^\mu)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta} = (1,-\vec\sigma).$$

The continuation from Minkowski space is made via $x^0 = -ix^4$, $p_0 = ip_4$. The conventions for Euclidean spinors are as follows:

$$\sigma^\mu_{\alpha\dot\beta} = (i,\vec\sigma), \quad \bar\sigma^{\mu\dot\alpha\beta} = (i,-\vec\sigma).$$

The following identities are useful:

$$\begin{aligned} \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu &= -2g^{\mu\nu}\delta^{\dot\alpha}{}_{\dot\beta}, \\ (\bar\sigma^\mu)^{\dot\alpha\beta}(\sigma^\mu)_{\gamma\dot\rho} &= -2\delta^\beta{}_\gamma\delta^{\dot\alpha}{}_{\dot\rho}. \end{aligned} \tag{A.1}$$

The (A)SD projectors are

$$\begin{aligned} \sigma^{\mu\nu} &= \frac{1}{4}\big(\sigma^\mu\overline\sigma^\nu - \sigma^\nu\overline\sigma^\mu\big), \\ \overline\sigma^{\mu\nu} &= \frac{1}{4}\big(\overline\sigma^\mu\sigma^\nu - \overline\sigma^\nu\sigma^\mu\big), \end{aligned} \tag{A.2}$$

where $\sigma^{\mu\nu}_{\alpha\beta}$ is ASD, while $\overline\sigma^{\mu\nu}_{\dot\alpha\dot\beta}$ is SD. We have, explicitly:

$$\begin{aligned} \sigma^{\mu\nu} &= \begin{pmatrix} 0 & -\frac{i}{2}\sigma^3 & \frac{i}{2}\sigma^2 & \frac{i}{2}\sigma^1 \\ \frac{i}{2}\sigma^3 & 0 & -\frac{i}{2}\sigma^1 & \frac{i}{2}\sigma^2 \\ -\frac{i}{2}\sigma^2 & \frac{i}{2}\sigma^1 & 0 & \frac{i}{2}\sigma^3 \\ -\frac{i}{2}\sigma^1 & -\frac{i}{2}\sigma^2 & -\frac{i}{2}\sigma^3 & 0 \end{pmatrix} \\ \overline\sigma^{\mu\nu} &= \begin{pmatrix} 0 & -\frac{i}{2}\sigma^3 & \frac{i}{2}\sigma^2 & -\frac{i}{2}\sigma^1 \\ \frac{i}{2}\sigma^3 & 0 & -\frac{i}{2}\sigma^1 & -\frac{i}{2}\sigma^2 \\ -\frac{i}{2}\sigma^2 & \frac{i}{2}\sigma^1 & 0 & -\frac{i}{2}\sigma^3 \\ \frac{i}{2}\sigma^1 & \frac{i}{2}\sigma^2 & \frac{i}{2}\sigma^3 & 0 \end{pmatrix} \end{aligned} \tag{A.3}$$

We finally define:

$$v_{\dot\alpha\dot\beta} = \overline{\sigma}^{\mu\nu}_{\dot\alpha\dot\beta} v^+_{\mu\nu}.$$

This implies that, if we consider a self-dual tensor

$$F^+_{\mu\nu} = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{pmatrix}$$

one has

$$F_{\dot\alpha\dot\beta} = 2i \begin{pmatrix} c - ib & -a \\ -a & -c - ib \end{pmatrix}.$$

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