Hyperkähler geometry lecture 3

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Broom Bridge

"Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication

$$I^2 = J^2 = K^2 = IJK = -1$$

and cut it on a stone of this bridge."

Geometric structures

DEFINITION: A geometric structure (Elie Cartan, Charles Ehresmann) is an atlas on a manifold, with the differentials of all transition functions in a given subgroup $G \subset GL(n, \mathbb{R})$.

EXAMPLE: $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ ("the complex structure").

EXAMPLE: $Sp(n, \mathbb{R}) \subset GL(2n, \mathbb{R})$ ("the symplectic structure").

"Quaternionic structures" in the sense of Elie Cartan **don't exist.**

THEOREM: Let $f: \mathbb{H}^n \longrightarrow \mathbb{H}^m$ be a function, defined locally in some open subset of n-dimensional quaternion space $\mathbb{H}^n$. Suppose that the differential Df is $\mathbb{H}$-linear. **Then f is a linear map.**

Proof (a modern one): The graph of f is a hyperkähler ("trianalytic") submanifold in $\mathbb{H}^n \times \mathbb{H}^m$, hence geodesically complete, hence linear. ■

Complex manifolds (reminder)

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

REMARK: The "usual definition": complex structure is a geometric structure: an atlas on a manifold with differentials of all transition functions in $GL(n,\mathbb{C})$.

THEOREM: (Newlander-Nirenberg)
These two definitions are equivalent.

Kähler manifolds (reminder)

DEFINITION: An Riemannian metric g on an almost complex manifold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

DEFINITION: Let (M, g) be a Riemannian manifold. A connection ∇ is called **orthogonal** if $\nabla(g) = 0$. It is called **Levi-Civita** if it is orthogonal and torsion-free.

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) (M, I, g) **is Kähler**

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection.

Algebraic geometry over $\mathbb{H}$.

Over $\mathbb{C}$, we have **3 distinct notions of "algebraic geometry":**

1. Scheme over $\mathbb{C}$: locally ringed space with Zariski topology, each ring a quotient of a polynomical ring.

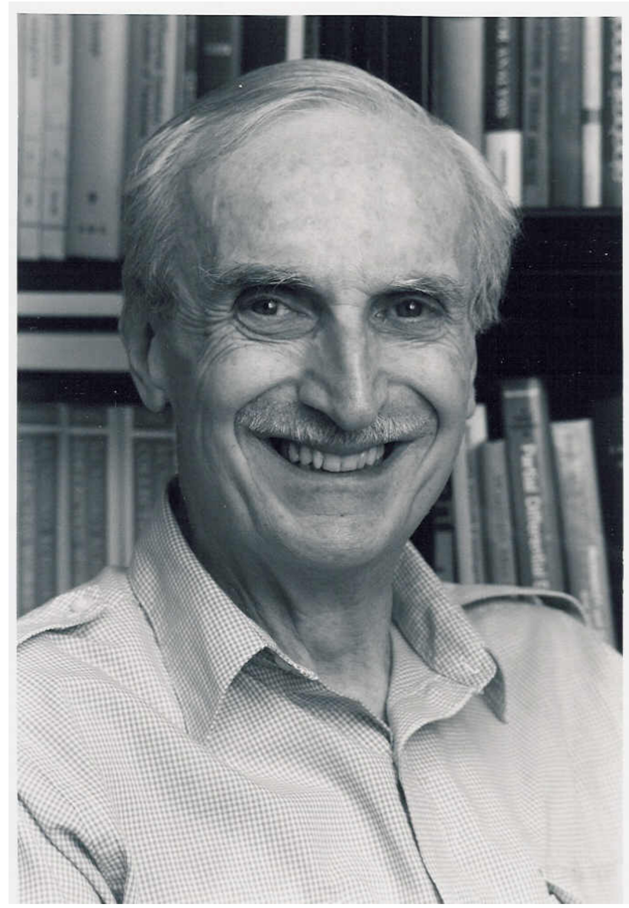
2. Complex manifold: manifold with holomorphic transition functions.

3. Kähler manifold: complex structure, metric, Levi-Civita connection, $\nabla I = 0$.

The first notion does not work for $\mathbb{H}$, because polynomial functions on $\mathbb{H}^n$ generate all real polynomials on $\mathbb{R}^4$. The second version does not work, because any quaternionic-differentiable function is linear. The third one works!

Hyperkähler manifolds.

Hyperkähler manifolds



Eugenio Calabi

DEFINITION: (E. Calabi, 1978)
Let (M,g) be a Riemannian manifold equipped with three complex structure operators $I,J,K: TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that I, J, K are Kähler. Then (M,I,J,K,g) is called **hyperkähler**.

Holomorphic symplectic geometry

CLAIM: A hyperkähler manifold (M, I, J, K) is **holomorphically symplectic** (equipped with a holomorphic, non-degenerate 2-form). Recall that M is equipped with 3 symplectic forms ω_I, ω_J, ω_K.

LEMMA: The form $\Omega := \omega_J + \sqrt{-1}\omega_K$ **is a holomorphic symplectic 2-form on (M, I).** ■

Converse is also true, as follows from the famous conjecture, made by Calabi in 1952.

THEOREM: (S.-T. Yau, 1978) Let M be a compact, holomorphically symplectic Kähler manifold. Then M **admits a hyperkähler metric,** which is uniquely determined by the cohomology class of its Kähler form ω_I.

Hyperkähler geometry is essentially the same as holomorphic symplectic geometry

"Hyperkähler algebraic geometry" is almost as good as the usual one.

Define **trianalytic subvarieties** as closed subsets which are complex analytic with respect to I, J, K.

0. **Trianalytic subvarieties are singular hyperkähler.**

1. Let L be a generic quaternion satisfying $L^2 = -1$. Then **all complex subvarieties of (M, L) are trianalytic.**

2. A normalization of a hyperkähler variety is smooth and hyperkähler. **This gives a desingularization** ("hyperkähler Hironaka").

3. A complex deformation of a trianalytic subvariety **is again trianalytic**, the corresponding moduli space is (singularly) hyperkähler.

4. Similar results are true for vector bundles which are holomorphic under I, J, K **("hyperholomorphic bundles")**

Examples of hyperkähler manifolds (reminder)

EXAMPLE: An even-dimensional complex vector space.

EXAMPLE: An even-dimensional complex torus.

EXAMPLE: **A non-compact example:** $T^*\mathbb{C}P^n$ (Calabi).

REMARK: $T^*\mathbb{C}P^1$ **is a resolution of a singularity** $\mathbb{C}^2/\pm 1$**.**

EXAMPLE: Take a 2-dimensional complex torus T, then the singular locus of $T/\pm 1$ is of form $(\mathbb{C}^2/\pm 1) \times T$. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

REMARK: Take a symmetric square $\operatorname{Sym}^2 T$, with a natural action of T, and let $T^{[2]}$ be a blow-up of a singular divisor. **Then $T^{[2]}$ is naturally isomorphic to the Kummer surface $\widetilde{T/\pm 1}$.**

DEFINITION: A complex surface is called **K3 surface** if it a deformation of the Kummer surface.

THEOREM: **(a special case of Enriques-Kodaira classification)**
Let M be a compact complex surface which is hyperkähler. **Then M is either a torus or a K3 surface.**

Hilbert schemes (reminder)

DEFINITION: A **Hilbert scheme** $M^{[n]}$ of a complex surface M is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension n over $\mathbb{C}$.

REMARK: A Hilbert scheme **is obtained as a resolution of singularities** of the symmetric power $\operatorname{Sym}^n M$.

THEOREM: (Fujiki, Beauville) **A Hilbert scheme of a hyperkähler surface is hyperkähler.**

EXAMPLE: **A Hilbert scheme of K3** is hyperkähler.

EXAMPLE: Let T be a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, it is called **a generalized Kummer variety**.

REMARK: There are 2 more "sporadic" examples of compact hyperkähler manifolds, constructed by K. O'Grady. **All known compact hyperkaehler manifolds are these 2 and the three series:** tori, Hilbert schemes of K3, and generalized Kummer.

Induced complex structures

LEMMA: Let ∇ be a torsion-free connection on a manifold, $I \in \operatorname{End} TM$ an almost complex structure, $\nabla I = 0$. **Then I is integrable.**

Proof: Let $X, Y \in T^{1,0}M$, then $[X,Y] = \nabla_X Y - \nabla_Y X \in T^{1,0}M$. ■

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors.

REMARK: Because of the Lemma above, **induced complex structure operators are integrable.**

Hodge theory: graded vector spaces and algebras

DEFINITION: A **graded vector space** is a space $V^* = \bigoplus_{i\in\mathbb{Z}} V^i$.

REMARK: If V^* is graded, the endomorphisms space $\operatorname{End}(V^*) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}^i(V^*)$ is also graded, with $\operatorname{End}^i(V^*) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(V^j, V^{i+j})$

DEFINITION: A **graded algebra**(or "graded associative algebra") is an associative algebra $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$, with the product compatible with the grading: $A^i \cdot A^j \subset A^{i+j}$.

REMARK: A bilinear map of graded paces which satisfies $A^i \cdot A^j \subset A^{i+j}$ is called **graded**, or **compatible with grading**.

REMARK: The category of graded spaces can be defined as a **category of vector spaces with $U(1)$-action,** with the weight decomposition providing the grading. Then **a graded algebra is an associative algebra in the category of spaces with $U(1)$-action**.

Hodge theory: supercommutator

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a,b\} = ab - (-1)^{\tilde{a}\tilde{b}} ba.$

DEFINITION: A graded associative algebra is called **graded commutative** (or "supercommutative") if its supercommutator vanishes.

EXAMPLE: The Grassmann algebra is supercommutative.

DEFINITION: A graded Lie algebra (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\}: \ \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a,b\} = -(-1)^{\tilde{a}\tilde{b}}\{b,a\}$ and satisfies **the super Jacobi identity** $\{c,\{a,b\}\} = \{\{c,a\},b\} + (-1)^{\tilde{a}\tilde{c}}\{a,\{c,b\}\}$

EXAMPLE: Consider the algebra $\operatorname{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then $\operatorname{End}(A^*), \{\cdot,\cdot\}$ is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d,d\} = 0$, and L an even or odd element. **Then $\{\{L,d\},d\} = 0$.**

Proof: $0 = \{L,\{d,d\}\} = \{\{L,d\},d\} + (-1)^{\tilde{L}}\{d,\{L,d\}\} = 2\{\{L,d\},d\}.$ ■

Hodge theory: Laplacian

DEFINITION: Let M be an oriented Riemannian manifold. **The d^*-operator** is a Hermitian adjoint to d with respect to the product $\alpha, \beta \longrightarrow \int_M (\alpha, \beta) \operatorname{Vol}_M$.

DEFINITION: The anticommutator $\Delta := \{d, d^*\} = dd^* + d^*d$ is called **the Laplacian** of M. It is self-adjoint and positive definite: $(\Delta x, x) = (dx, dx) + (d^*x, d^*x)$. Also, Δ commutes with d and d^* (Lemma 1).

THEOREM: (The main theorem of Hodge theory)
There is a basis in the Hilbert space $L^2(\Lambda^*(M))$ consisting of eigenvectors of Δ.

THEOREM: ("Elliptic regularity for Δ") Let $\alpha \in L^2(\Lambda^k(M))$ be an eigenvector of Δ. **Then α is a smooth k-form.**

Hodge theory: harmonic forms

DEFINITION: The space $H^i(M) := \frac{\ker d\big|_{\Lambda^i M}}{d(\Lambda^{i-1}M)}$ is called **the de Rham cohomology of M**.

DEFINITION: A form α is called **harmonic** if $\Delta(\alpha) = 0$.

REMARK: Let α be a harmonic form. **Then $(\Delta x, x) = (dx, dx) + (d^* x, d^* x)$,** hence $\alpha \in \ker d \cap \ker d^*$

REMARK: The projection $\mathcal{H}^i(M) \longrightarrow H^i(M)$ from harmonic forms to cohomology is injective. Indeed, a form α lies in the kernel of such projection if $\alpha = d\beta$, but then $(\alpha, \alpha) = (\alpha, d\beta) = (d^*\alpha, \beta) = 0$.

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism (see the next page).

Hodge theory and the cohomology

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism.

Proof. Step 1: Since $d^2 = 0$ and $(d^*)^2 = 0$, one has $\{d, \Delta\} = 0$. This means that **Δ commutes with the de Rham differential.**

Step 2: Consider the eigenspace decomposition $\Lambda^*(M) \hat{=} \bigoplus_\alpha \mathcal{H}^*_\alpha(M)$, where α runs through all eigenvalues of Δ, and $\mathcal{H}^*_\alpha(M)$ is the corresponding eigenspace. **For each α, de Rham differential defines a complex**

$$\mathcal{H}^0_\alpha(M) \xrightarrow{d} \mathcal{H}^1_\alpha(M) \xrightarrow{d} \mathcal{H}^2_\alpha(M) \xrightarrow{d} \ldots$$

Step 3: On $\mathcal{H}^*_\alpha(M)$, one has $dd^* + d^*d = \alpha$. When $\alpha \neq 0$, and η closed, this implies $dd^*(\eta) + d^*d(\eta) = dd^*\eta = \alpha\eta$, hence $\eta = d\xi$, with $\xi := \alpha^{-1} d^*\eta$. This implies that **the complexes $(\mathcal{H}^*_\alpha(M), d)$ don't contribute to cohomology.**

Step 4: We have proven that

$$H^*(\Lambda^* M, d) = \bigoplus_\alpha H^*(\mathcal{H}^*_\alpha(M), d) = H^*(\mathcal{H}^*_0(M), d) = \mathcal{H}^*(M).$$

■

The Hodge decomposition in linear algebra (reminder)

DEFINITION: The Hodge decomposition $V\otimes_{\mathbb R}\mathbb C := V^{1,0}\oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: Let $V_{\mathbb C} := V\otimes_{\mathbb R}\mathbb C$. The Grassmann algebra of skew-symmetric forms $\Lambda^n V_{\mathbb C} := \Lambda^n_{\mathbb R} V\otimes_{\mathbb R} C$ admits a decomposition

$$\Lambda^n V_{\mathbb C} = \bigoplus_{p+q=n} \Lambda^p V^{1,0}\otimes \Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0}\otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q}V$. The resulting decomposition $\Lambda^n V_{\mathbb C} = \bigoplus_{p+q=n}\Lambda^{p,q}V$ is called **the Hodge decomposition of the Grassmann algebra**.

REMARK: The operator I induces $U(1)$-action on V by the formula $\rho(t)(v) = \cos t\cdot v + \sin t\cdot I(v)$. We extend this action on the tensor spaces by muptiplicativity.

$U(1)$-representations and the weight decomposition

REMARK: Any complex representation W of $U(1)$ is written as a sum of 1-dimensional representations $W_i(p)$, with $U(1)$ acting on each $W_i(p)$ as $\rho(t)(v) = e^{\sqrt{-1}\,pt}(v)$. The 1-dimensional representations are called **weight p representations of $U(1)$.**

DEFINITION: A **weight decomposition** of a $U(1)$-representation W is a decomposition $W = \oplus W^p$, where each $W^p = \oplus_i W_i(p)$ is a sum of 1-dimensional representations of weight p.

REMARK: The Hodge decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is a weight decomposition, with $\Lambda^{p,q} V$ being a weight $p-q$-component of $\Lambda^n V_{\mathbb{C}}$.

REMARK: $V^{p,p}$ is the space of $U(1)$-invariant vectors in $\Lambda^{2p} V$.

Hodge decomposition on cohomology

THEOREM: (Bott)
Let M be a Riemannian manifold, and $V: TM \longrightarrow TM$ an endomorphism satisfying $\nabla V = 0$. **Then $[V, \Delta] = 0$.** In particular, if M is compact, V acts on cohomology of M.

COROLLARY: The Weil operator $W\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$ acts on cohomology of a compact Kähler manifold, giving **the Hodge decomposition:** $H^*(M) = \bigoplus H^{p,q}(M)$.

COROLLARY: For any hyperkähler manifold, the group $SU(2)$ of unitary quaternions **defines $SU(2)$-action on cohomology**.

REMARK: For each induced complex structure L, we have an embedding $U(1) \subset SU(2)$. Therefore, **the Hodge decomposition for $L = aI + bJ + cK$ is induced by the $SU(2)$-action.**

$SU(2)$-action on the cohomology and its applications

DEFINITION: Trianalytic subvarieties are closed subsets which are complex analytic with respect to I, J, K.

REMARK: Trianalytic subvarieties are hyperkähler submanifolds outside of their singularities.

REMARK: Let $[Z]$ be a fundamental class of a complex subvariety Z on a Kähler manifold. **Then Z is $U(1)$-invariant.**

COROLLARY: A fundamental class of a trianalytic subvariety is $SU(2)$-invariant.

THEOREM: Let M be a hyperkähler manifold. Then there exists a countable subset $S \subset \mathbb{C}P^1$, such that for any induced complex structure $L \in \mathbb{C}P^1 \backslash S$, **all compact complex subvarieties of (M, L) are trianalytic.**

Its proof is based on **Wirtinger's inequality**.

Wirtinger's inequality

PROPOSITION: (Wirtinger's inequality)
Let $V \subset W$ be a real $2d$-dimensional subspace in a complex Hermitian vector space (W, I, g), and ω its Hermitian form. **Then $\operatorname{Vol}_g V \geqslant \frac{1}{2^d d!}\omega^d|_V$, and the equality is reached only if V is a complex subspace.**

COROLLARY: Let (M, I, ω, g) be a Kähler manifold, and $Z \subset M$ its real subvariety of dimension $2d$. Then $\int_Z \operatorname{Vol}_Z \geqslant \frac{1}{2^d d!}\int_Z \omega^d$**, and the equality is reached only if Z is a complex subvariety.**

REMARK: Notice that $\int_Z \omega^d$ is a (co)homology invariant of Z, and stays constant if we deform Z. Therefore, **complex subvarieties minimize the Riemannian volume in its deformation class**.

Wirtinger's inequality for hyperkähler manifolds

DEFINITION: Let (M, I, J, K, g) be a hyperkähler manifold, and $Z \subset M$ a real $2d$-dimensional subvariety. Given an induced complex structure $L = aI + bJ + cK$, define **the degree** $\deg_L(Z) := \frac{1}{2^d d!}\int_Z \omega_L^d$, where $\omega_L(x,y) = g(x, Ly)$, which gives $\omega_L = a\omega_I + b\omega_J + c\omega_K$.

Proposition 1: Let $Z \subset (M, L)$ be a complex analytic subvariety of (M, L). (a) Then $\deg_L(Z)$ **has maximum at L.** (b) Moreover, this maximum is absolute and **strict, unless $\deg_L(Z)$ is constant as a function of L.** (c) In the latter case, Z is trianalytic.

Proof. Step 1: By Wirtinger's inequality, $\operatorname{Vol}_g Z \geqslant \deg_L(Z)$, and the equality is reached if and only if Z is complex analytic in (M, L). This proves (a).

Step 2: If the maximum is not strict, there are two quaternions L and L' such that Z is complex analytic with respect to L and L'. This means that **TZ is preserved by the algebra of quaternions generated by L and L',** hence **Z is trianalytic, and $\deg_L(Z)$ constant**. This proves (b) and (c). ■

Trianalytic subvarieties in generic induced complex structures

THEOREM: Let M be a hyperkähler manifold. Then there exists a countable subset $S \subset \mathbb{C}P^1$, such that for any induced complex structure $L \in \mathbb{C}P^1 \backslash S$, **all compact complex subvarieties of (M, L) are trianalytic.**

Proof. Step 1: Let $R \subset H^2(M, \mathbb{Z})$ be the set of all integer cohomology classes $[Z]$, for which the function $\deg_L([Z]) = \int_{[Z]} \omega_L^d$ is not constant, and S the set of all strict maxima of the function $\deg_L([Z])$ for all $[Z] \in R$. **Then S is countable.** Indeed, $\deg_L([Z])$ is a polynomial function.

Step 2: Now, let $L \in \mathbb{C}P^1 \backslash S$. For all complex subvarieties $Z \subset (M, L)$, $\deg_L([Z])$ cannot have strict maximum in L. By Proposition 1 (c), **this implies that Z is trianalytic.** ■

DEFINITION: A divisor on a complex manifold is a complex subvariety of codimension 1.

COROLLARY: For M compact and hyperkähler, and $L \in \mathbb{C}P^1$ generic, **the manifold (M, L) has no complex divisors.** In particular, **it is non-algebraic.**