Estimates for the concentration functions in the Littlewood–Offord problem

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2013, June 11

Saint-Petersburg

Let $X, X_1, \ldots, X_n$ be independent identically distributed random variables with common distribution $F = \mathcal{L}(X)$. In this talk we discuss the behavior of the concentration functions of the weighted sums $S_a = \sum_{k=1}^{n} a_k X_k$ with respect to the arithmetic structure of coefficients a_k. Such concentration results recently became important in connection with investigations about singular values of random matrices. In this context the problem is referred to as the Littlewood–Offord problem. We formulate some refinements of results of Friedland and Sodin (2007), Rudelson and Vershynin (2009), Vershynin (2011) which are proved in the recent preprints of Eliseeva and Zaitsev (2012), Eliseeva, Götze and Zaitsev (2012) and Eliseeva (2013).

The Lévy concentration function of a random variable X is defined by the equality

$$Q(F,\lambda)=\sup_{x\in\mathbf{R}} F\{[x,x+\lambda]\},\quad \lambda\geq 0.$$

Let $X,X_1,\dots,X_n$ be independent identically distributed random variables with common distribution $F=\mathcal{L}(X)$, $a=(a_1,\dots,a_n)\in\mathbf{R}^n$. In the sequel, let F_a denote the distribution of the sum $S_a=\sum_{k=1}^n a_k X_k$, and let G be the distribution of the symmetrized random variable $\widetilde{X}=X_1-X_2$. Let

$$M(\tau)=\tau^{-2}\int_{|x|\leq\tau} x^2\,G\{dx\}+\int_{|x|>\tau} G\{dx\}=\mathbf{E}\min\left\{\widetilde{X}^2/\tau^2,1\right\},\quad \tau>0. \quad (1)$$

The symbol c will be used for absolute positive constants. We shall write $A\ll B$ if $A\leq cB$. Also we shall write $A\asymp B$ if $A\ll B$ and $B\ll A$.

Theorem 1 (Eliseeva and Zaitsev (2012)). *Let $X, X_1, \ldots, X_n$ be i.i.d. random variables, and let $a = (a_1, \ldots, a_n) \in \mathbf{R}^n$. If, for some $D \geq \dfrac{1}{2\,\|a\|_\infty}$ and $\alpha > 0$,*

$$\|\,ta - m\| \geq \alpha, \quad \textit{for all} \;\; m \in \mathbf{Z}^n \;\textit{and}\;\; t \in \Big[\frac{1}{2\,\|a\|_\infty}, D\Big], \tag{2}$$

then, for any $\tau > 0$,

$$Q\Big(F_a, \frac{\tau}{D}\Big) \ll \frac{1}{\|a\| D\sqrt{M(\tau)}} + \exp(-c\,\alpha^2 M(\tau)). \tag{3}$$

Theorem 1 is an improvement of a result of Friedland and Sodin (2007).

Theorem 2 (Eliseeva and Zaitsev (2012)). *Let $X, X_1, \dots, X_n$ be i.i.d. random variables, $\alpha, D > 0$ and $\gamma \in (0,1)$. Assume that*

$$\| ta - m\| \geq \min\{\gamma t \, \|a\|, \alpha\}, \quad \textit{for all } m \in \mathbf{Z}^n \textit{ and } \ t \in [0, D]. \tag{4}$$

Then, for any $\tau > 0$,

$$Q\Big(F_a, \frac{\tau}{D}\Big) \ll \frac{1}{\|a\| D\gamma \sqrt{M(\tau)}} + \exp(-c\,\alpha^2 M(\tau)). \tag{5}$$

Theorem 2 is an improvement of a result of Rudelson and Vershynin (2009).

Let $L > 1$.

Theorem 3 (Eliseeva, Götze and Zaitsev (2012)). *Let $X, X_1, \ldots, X_n$ be i.i.d. random variables. Let $a = (a_1, \ldots, a_n) \in \mathbf{R}^n$. Assume that $D \geq \dfrac{1}{2\,\|a\|_\infty}$, $\tau > 0$ and $L^2 \geq 1/M(\tau)$,*

$$\| ta - m\| \geq f_L(t\|a\|) \ \text{ for all } m \in \mathbf{Z}^n \text{ and } \ t \in \Big[\frac{1}{2\,\|a\|_\infty}, D\Big], \tag{6}$$

where

$$f_L(t) = \begin{cases} t/6, & \text{for } 0 < t < eL, \\ L\sqrt{\log(t/L)}, & \text{for } t \geq eL. \end{cases} \tag{7}$$

Then

$$Q\Big(F_a, \frac{\tau}{D}\Big) \ll \frac{1}{\|a\| D\sqrt{M(\tau)}}. \tag{8}$$

Theorem 3 is an improvement of a result of Vershynin (2011).

Define the least common denominator $D^*(a)$ as

$$D^*(a) = \inf\Big\{t > 0 : \operatorname{dist}(ta, \mathbf{Z}^n) < f_L(t\|a\|)\Big\}, \tag{9}$$

Theorem 4 (Eliseeva, Götze and Zaitsev (2012)). *Let $X, X_1, \dots, X_n$ be i.i.d. random variables. Let $a = (a_1, \dots, a_n) \in \mathbf{R}^n$. Assume that $L^2 > 1/P$, where $P = \mathbf{P}(\widetilde{X} \neq 0) = \lim_{\tau \to 0} M(\tau)$. Then there exists a τ_0 such that $L^2 = 1/M(\tau_0)$. Moreover, the bound*

$$Q(F_a, \varepsilon) \ll \frac{1}{\|a\| D^*(a) \sqrt{M(\varepsilon\, D^*(a))}} \tag{10}$$

is valid for $0 < \varepsilon \le \varepsilon_0 = \tau_0 / D^*(a)$. Furthermore, for $\varepsilon \ge \varepsilon_0$, the bound

$$Q(F_a, \varepsilon) \ll \frac{\varepsilon L}{\varepsilon_0\, \|a\| D^*(a)} \tag{11}$$

holds.

If we consider the special case, where $D = 1/2\,\|a\|_\infty$, then no assumptions on the arithmetic structure of the vector a are made, and Theorems 1–3 imply the bound

$$Q(F_a, \tau\,\|a\|_\infty) \ll \frac{\|a\|_\infty}{\|a\|\sqrt{M(\tau)}}. \tag{12}$$

This result follows from Esséen's (1968) inequality applied to the sum of non-identically distributed random variables $Y_k = a_k X_k$. For $a_1 = a_2 = \cdots = a_n = n^{-1/2}$, inequality (12) turns into the well-known particular case:

$$Q(F^{*n}, \tau) \ll \frac{1}{\sqrt{n\,M(\tau)}}. \tag{13}$$

Inequality (13) implies as well the Kolmogorov–Rogozin inequality for i.i.d. random variables:

$$Q(F^{*n}, \tau) \ll \frac{1}{\sqrt{n\,(1 - Q(F, \tau))}}.$$

Inequality (12) can not yield bound of better order than $O(n^{-1/2})$, since the right-hand side of (12) is at least $n^{-1/2}$. The results stated above are more interesting if D is essentially larger than $1/2\,\|a\|_\infty$. In this case one can expect the estimates of smaller order than $O(n^{-1/2})$. Such estimates of $Q(F_a, \lambda)$ are required to study the distributions of eigenvalues of random matrices.

Now we formulate the multidimensional generalizations of Theorem 1 and Theorem 2.

The concentration function of a $\mathbf{R}^d$–valued vector Y with the distribution $F = \mathcal{L}(Y)$ is defined by the equality

$$Q(F, \lambda) = \sup_{x \in \mathbf{R}^d} \mathbf{P}(Y \in x + \lambda B), \quad \lambda > 0,$$

where $B = \{x \in \mathbf{R}^n : \|x\| \le 1\}$. Let $X, X_1, \ldots, X_n$ be i.i.d. random variables, $a = (a_1, \ldots, a_n) \ne 0$, where $a_k = (a_{k1}, \ldots, a_{kd}) \in \mathbf{R}^d$, $k = 1, \ldots, n$. We write $A \ll_d B$ if $|A| \le c^d B$ and $B > 0$. Note that $\ll_d$ allows constants to be exponential with respect to d.

Theorem 5 (Eliseeva (2013)). *Let $X, X_1, \ldots, X_n$ be i.i.d. random variables, $a = (a_1, \ldots, a_n)$, $a_k = (a_{k1}, \ldots, a_{kd}) \in \mathbf{R}^d$, $k = 1, \ldots, n$. If for some $\alpha > 0$ and $D > 0$*

$$\sum_{k=1}^{n} (\langle t, a_k \rangle - m_k)^2 \geq \alpha^2 \text{ for all } m_1, \ldots, m_n \in \mathbf{Z},\ t \in \mathbf{R}^d \text{ such that}$$

$$\max_k |\langle t, a_k \rangle| \geq 1/2, \ \|t\| \leq D, \quad (14)$$

then for any $\tau > 0$

$$Q\Big(F_a, \frac{d\,\tau}{D}\Big) \ll_d \exp\left(-c\,\alpha^2 M(\tau)\right) + \Big(\frac{\sqrt{d}}{D\sqrt{M(\tau)}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}},$$

where

$$\mathbb{N} = \sum_{k=1}^{n} \mathbb{N}_k, \quad \mathbb{N}_k = \begin{pmatrix} a_{k1}^2 & a_{k1}a_{k2} & \ldots & a_{k1}a_{kd} \\ a_{k2}a_{k1} & a_{k2}^2 & \ldots & a_{k2}a_{kd} \\ \ldots & \ldots & \ldots & \ldots \\ a_{kd}a_{k1} & \ldots & \ldots & a_{kd}^2 \end{pmatrix}. \quad (15)$$

Theorem 6 (Eliseeva (2013)).*Let $X, X_1, \ldots, X_n$ be i.i.d. random variables. Let $a = (a_1, \ldots, a_n)$, $a_k \in \mathbf{R}^d$, $\alpha > 0$, $D > 0$, $\gamma \in (0,1)$, and*

$$\Big(\sum_{k=1}^{n}(\langle t, a_k\rangle - m_k)^2\Big)^{1/2} \geq \min\{\gamma\|t\cdot a\|, \alpha\} \text{ for all } m_1, \ldots, m_n \in \mathbf{Z} \text{ and } \|t\| \leq D. \tag{16}$$

Then for any $\tau > 0$

$$Q\Big(F_a, \frac{d\,\tau}{D}\Big) \ll_d \Big(\frac{\sqrt{d}}{D\,\gamma\,\sqrt{M(\tau)}}\Big)^d \frac{1}{\sqrt{\det \mathbb{N}}} + \exp\left(-c\,\alpha^2 M(\tau)\right).$$

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