# Von Mises statistics of measure preserving tranformations: elements of limit theory

After a joint paper with Manfrd Denker (Pennsylvania State University)

Mikhail Gordin
V.A. Steklov Mathematical Institute at Saint Petersburg

*Russian-Chinese Seminar on Asymptotic Methods in Probability Theory and Mathematical Statistics*

Saint Petersburg, June 10 -14, 2013

$(X, \mathcal{F}, \mu)$: a probability space
$T : X \to X$: a measure preserving transformation
Call random variables

$$x \mapsto \sum_{0 \le i_1 < n, \dots, 0 \le i_d < n} f(T^{i_1}x, \dots, T^{i_d}x), n = 1, 2, \dots, \quad (1)$$

**von Mises statistics** (or V-**statistics**)
constructed against the **transformation** $T$ and the **kernel** $f$.

The kernel $f$ will be assumed **symmetric**.

Our task: investigate asymptotic behavior of (1) as $n \to \infty$.

A particular case of the above definition:
random variables

$$x \mapsto \sum_{0 \le i_1 < n, \ldots, 0 \le i_d < n} F(\xi_{i_1}(x), \ldots, \xi_{i_d}(x)),\ n = 1, 2, \ldots,$$

constructed after a stationary process $\xi = (\xi_n)_{n \in \mathbb{Z}}$ and a function $F$.

If $(\xi_n)_{n \in \mathbb{Z}}$ is a sequence of i.i.d. variables one gets the original definition of von Mises (1947).

Turn back to measure preserving transformations. For $d = 1$ von Mises statistics reduce to well-known **Birkhoff's sums**

$$\sum_{0 \le i < n} f \circ T^i.$$

Classical ergodic theorems describe the behavior of **normalized Birkhoff's sums**

$$\frac{1}{n} \sum_{0 \le i < n} f \circ T^i$$

for **arbitrary** measure preserving **transformation** $T$ and **every function** $f \in L_1(\mu)$.

# Additional structures: equivariant filtrations

**Additional structures** are needed for more precise asymptotical results (like the Central Limit Theorem) to specify a class of suitable functions and conduct proofs.

An important type of such structures:

$T$-equivariant **filtrations** (increasing or decreasing sequences of $\sigma$-subfields of the $\sigma$-field $\mathcal{F}$ shifted by $T$).

A **decreasing equivariant filtration** for $(X, \mathcal{F}, \mu, T)$ is a sequence

$$\mathcal{F}_0 \supset T^{-1}\mathcal{F}_0 \supset T^{-2}\mathcal{F}_0 \cdots$$

of $\sigma$-subfields of $\mathcal{F}$.

For an invertible $T$ it is more natural to consider a **bilateral increasing equivariant filtration**

$$\cdots \subset T^{1}\mathcal{F}_0 \subset \mathcal{F}_0 \subset T^{-1}\mathcal{F}_0 \subset \cdots.$$

In presence of such a filtration one can establish limit theorems for Birkhoff's sums by means of the **martingale-coboundary decomposition, its extensions and modifications.** This approach is widely applied to Birkhoff's sums (but not to von Mises statistics).

**QUESTION**: Is it possible to investigate von Mises statistics for $d > 1$ by means of the same additional structures (filtrations) as in case $d = 1$ ?
If so what kind of **changes** in earlier known methods or **additions** to them have to be made in this situation ?

**ANSWER**: **Multiparameter generalization of the martingale-coboundary decomposition**, applied to **suitable spaces of finctions** on $(X^d, \mathcal{F}^{\otimes d}, \mu^d)$ and supplemented by the analysis of the **restriction operator**, can be taken as a basis for the asymptotic study of $V$-statistics in presence of equivariant filtrations.

## Initial assumptions, notation, facts and remarks I

Results we are going to discuss are established under the assumptions that $T$ is an **exact transformation** that is

$$\bigcap_{k\geq 0} T^{-k}\mathcal{F} = \mathcal{N},$$

where $\mathcal{N}$ is the trivial $\sigma$-field of the space $(X, \mathcal{F}, \mu)$. In the rest of the present talk we assume that $T$ satifies this condition.

An exact transformation $T$ defines on an nonatomic $(X, \mathcal{F}, \mu)$ a **strictly decreasing filtration** $(T^{-k}\mathcal{F})_{k\geq 0} = \mathcal{F} \supset T^{-1}\mathcal{F} \supset \cdots$ .

# Initial assumptions, notation, facts and remarks II

The **dynamical operator** $V : f \mapsto f \circ T$, $f \in L_p$, acts in every space $L_p = L_p(X, \mathcal{F}, \mu)$ $(p \in [1, \infty])$. The (pre)adjoint operator $V^*$ acts in the same scale of the $L_p$-spaces.

The symbols $V^{(k_1,\dots,k_m)}$ $(V^{*(k_1,\dots,k_m)})$ denote operators acting coordinate-wise as $V^{k_1}, \dots, V^{k_m}$ (or as $V^{*k_1}, \dots, V^{*k_m}$) in various spaces of functions on $X^m$ (in particular, in $m$-th tensor powers of the spaces $L_p$).

**Example.**

$$(V^{(k_1,k_2)}f)(x_1, x_2) = f(T^{k_1}x_1, T^{k_2}x_2) = \left((V^{k_1} \otimes V^{k_2})f\right)(x_1, x_2).$$

# Approach and tools used

1. **Restriction to the principal diagonal**
   The function $x \mapsto f(T^{k_1}x, \dots, T^{k_d}x)$ is the restriction of the function $(x_1, \dots, x_d) \mapsto f(T^{k_1}x_1, \dots, T^{k_d}x_d)$ to the principal diagonal of the space $(X^d, \mathcal{F}^{\otimes d}, \mu^d)$.
   Constructing a well-defined restriction looks impossible within the $L_p$ spaces: the diagonal is of product measure zero.
   However: for suitable $p$ и $q$ the **restriction operator** $D_d$ **has a finite norm** as an operator whose domain is the $d$-th **projective tensor degree** $L_{p,\pi}(\mu^d)$ of the space $L_p$ and the range is $L_q(\mu)$
2. **Hoeffding's decomposition**
3. ***d*-parametric extension of the martingale-coboundary decomposition**

## Hoeffding's decomposition of symmetric kernels

Let

- $(X_l, \mathcal{F}_l, \mu_l)$, $l = 1, \ldots, d$, be copies of the space $(X, \mathcal{F}, \mu)$,
- for every $m \in \{1, \ldots, d\}$ $\quad L_p^{sym}(\mu^m)$ be the subspace of symmetric elements of the space $L_p(\mu^m) \stackrel{\text{def}}{=} L_p(X^m, \mathcal{F}^{\otimes m}, \mu^m)$,
- $\mathcal{S}_d^m$ be the set of all $m$-subsets of the set $\{1, ..., d\}$ and, for every $S \in \mathcal{S}_d^m$,

$$\pi_S : X^d \to X^m$$

be the projection map which only holds coordinates with indices in $S$.

**Symmetric Hoeffding decomposition:** there exist such operators $R_m : L_p^{sym}(\mu^d) \to L_p^{sym}(\mu^m)$ that every $f \in L_p^{sym}(\mu^d)$ can be represented in the form

$$f = \sum_{m=0}^{d} \sum_{S \in \mathcal{S}_d^m} (R_m f) \circ \pi_S$$

and, moreover, integrating out with respect to the measure $\mu$ any of $m$ arguments of the function $R_m f$ returns 0 (such a function is called **canonical**). Analogous decomposition and the same notation will be applied later to the spaces $L_{p,\pi}(\mu^d)$.

**Example.** Let $d = 2$ and $f \in L_p^{sym}(\mu^2)$. Then

$$f(x_1, x_2) = f_0 + f_1(x_1) + f_1(x_2) + f_2(x_1, x_2)$$

where

$$f_0 = \int_{X^2} f(z_1, z_2)\mu(dz_1)\mu(dz_2),$$

$$f_1(x) = \int_X f(x, z)\mu(dz) - f_0 \left( = \int_X f(z, x)\mu(dz) - f_0 \right),$$

$$f_2(x_1, x_2) = f(x_1, x_2) - f_1(x_1) - f_1(x_2) - f_0.$$

Here $\int_X f_1(z)\mu(dz) = 0$, $f_2 \in L_p^{sym}(\mu^2)$ and for almost all $x \in X$

$$\int_X f_2(z, x)\mu(dz) = \int_X f_2(x, z)\mu(dz) = 0.$$

# Limit distributions of "nondegenerate" $V$-statistics

**Теорема 2.** *Let $f \in L_2^{sym}(\mu^d)$ be a real-valued kernel. Let for every $m=1,\dots,d$ $\quad R_m f \in L_{2m,\pi}^{sym}(\mu^m)$ and the series*

$$\sum_{0\le k_1,\dots,k_m<\infty} V^{*(k_1,\dots,k_m)} R_m f \left( \stackrel{\text{def}}{=} \lim_{n\to\infty} \sum_{0\le k_1,\dots,k_m<n} V^{*(k_1,\dots,k_m)} R_m f \right)$$

*converges in $L_{2m,\pi}(\mu^m)$. Then the sequence*

$$V_n^{(d)} f \stackrel{\text{def}}{=} \frac{1}{n^{d-1/2}} \sum_{0\le k_1,\dots,k_d<n} D_d\, V^{(k_1,\dots,k_d)}(f - R_0 f)$$

*tends in distribution, along with its second moments, to the centered Gaussian random variable with the variance $d^2\sigma^2(f) \ge 0$ where*

$$\sigma^2(f) = \left| \sum_{k=0}^{\infty} V^{*\,k} R_1 f \right|_2^2 - \left| \sum_{k=1}^{\infty} V^{*k} R_1 f \right|_2^2 \ge 0.$$

# Limit distributions of canonical kernels of degree 2

**Theorem 4.** *Let $f \in L^{sym}_{2,\pi}(\mu^2)$ be such a real-valued canonical kernel that in $L_{2,\pi}(\mu^2)$ there exists the limit*

$$\lim_{n_1, n_2 \to \infty} \sum_{\substack{0 \le i_1 \le n_1 - 1 \\ 0 \le i_2 \le n_2 - 1}} V^{*(i_1, i_2)} f .$$

*Then as $n \to \infty$ the sequence of random variables*

$$\frac{1}{n} \sum_{0 \le i_1, i_2 \le n-1} D_2 V^{(i_1, i_2)} f , \quad n \ge 1,$$

*converges in distribution, along with its first moments, to $\xi \stackrel{\mathrm{def}}{=} \sum_{m=1}^{\infty} \lambda_m \eta_m^2$ where $(\eta_m)_{m=1}^{\infty}$ is a sequence of independent standard Gaussian random variables and $(\lambda_m)$ are the eigenvalues of the kernel $g^{\emptyset}$ (see the next slide).*

# Martingale-coboundary decomposition (d=2)

**Proposition**. *Let $f(=f_2)\in L^{sym}_{2,\pi}(\mu^2)$ be a canonical kertnel of degree 2. If the limit*

$$g \stackrel{\text{def}}{=} \lim_{n_1,\, n_2\to\infty} \sum_{\substack{0\le i_1\le n_1-1\\ 0\le i_2\le n_2-1}} V^{*(i_1,\, i_2)} f,$$

*there exists in $L_{2,\pi}(\mu^2)$, then $f$ admits a unique representation of the form*

$$f=g^{\emptyset}+(V^{(1,0)}-I)g^{\{1\}}+(V^{(0,1)}-I)g^{\{2\}}+(V^{(1,0)}-I)(V^{(0,1)}-I)g^{\{1,2\}}$$

*where*

$$E(g^{\emptyset}|(T^{-1}\mathcal{F}^{(1)})\otimes\mathcal{F}^{(2)})=0,\qquad E(g^{\emptyset}|\mathcal{F}^{(1)}\otimes T^{-1}\mathcal{F}^{(2)})=0,$$

$$E(g^{\{1\}}|\mathcal{F}^{(1)}\otimes T^{-1}\mathcal{F}^{(2)})=0,\quad E(g^{\{2\}}|(T^{-1}\mathcal{F}^{(1)})\otimes\mathcal{F}^{(2)})=0$$

*and $g^{\emptyset}$, $g^{\{1\}}$, $g^{\{2\}}$, $g^{\{1,2\}}$ are canonical. Moreover, $g^{\emptyset}, g^{\{1,2\}}\in L^{sym}_{2,\pi}(\mu^2)$, $g^{\{1\}}, g^{\{2\}}\in L_{2,\pi}(\mu^2)$, $g^{\{1\}}\circ\theta=g^{\{2\}}$ and $g^{\{2\}}\circ\theta=g^{\{1\}}$ (here $\theta(x_1,x_2)=(x_2,x_1), x_1,x_2\in X$)).*

# Projective tensor products of Banach spaces

Let $B_1, \dots, B_d$ be Banach spaces with norms $|\cdot|_{B_1}, \dots, |\cdot|_{B_d}$, and let $B_1 \otimes \dots \otimes B_d$ be their algebraic tensor prpduct. Elements of $B_1 \otimes \dots \otimes B_d$ representable in the form $f_1 \otimes \dots \otimes f_d$ are called **elementary tensors**. **The projective tensor product** $B_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi B_d$ of Banach spaces $B_1, \dots, B_d$ is defined as the completion of $B_1 \otimes \dots \otimes B_d$ relative to the **projective norm**. The latter is defined as the supremum of all norms on $B_1 \otimes \dots \otimes B_d$, which are equal to $\prod_{i=1}^d |f_i|_{B_i}$ for every elementary tensor $f_1 \otimes \dots \otimes f_d$.

**Example**. The space $L_{2,\pi}(\mu^2)$ can be identified with the space of nuclear opeartors from $L_2(\mu)^*$ to $L_2(\mu)$.

# Embedding of the projective product $L_{p,\pi}(\mu^d)$ to $L_p(\mu^d)$

Let $(X_i, \mathcal{F}_i, \mu_i)$ $(i = 1, \dots, d)$ be copies of $(X, \mathcal{F}, \mu)$. Set for $p \in [1, \infty]$

$$L_{p,\pi}(\mu^d) \stackrel{\text{def}}{=} L_p(X_1, \mathcal{F}_1, \mu_1)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi L_p(X_d, \mathcal{F}_d, \mu_d).$$

Let $|\cdot|_{p,d,\pi}$ be the norm of the space $L_{p,\pi}(\mu^d)$. It is shown in the following lemma that $L_{p,\pi}(\mu^d)$ can be regarded as a subspace of $L_p(\mu^d)$.

**Lemma**. *For every $p \in [1, \infty]$ there exists a unique linear mapping $J_d : L_{p,\pi}(\mu^d) \to L_p(\mu^d)$ of norm 1 which sends every elementary tensor $f_1 \otimes \cdots \otimes f_d$ to the function $(x_1, \dots, x_d) \mapsto f_1(x_1) \cdots f_d(x_d)$. Furthermore, $J_d$ maps $L_{p,\pi}(\mu^d)$ to $L_p(\mu^d)$ injectively. For every $p \in [1, \infty)$ the subspace $J_d(L_{p,\pi}(\mu^d))$ is dense in $L_p(\mu^d)$.*

# Restriction to the diagonal

**Proposition.** *Let $p \in [1, \infty]$ and $r = p/d$. Then*

1. *the mapping $\mathcal{D}$ which sends every $d$-tuple $(f_1, \dots, f_d) \in L_p(\mu) \times \dots \times L_p(\mu)$ to the function $x \mapsto f_1(x) \cdots f_d(x)$ is a $d$-linear mapping of norm 1 from $L_p(\mu) \times \cdots \times L_p(\mu)$ to $L_r(\mu)$;*
2. *there exists such a unique linear mapping (of norm 1) $D_d : L_{p,\pi}(\mu^d) \to L_r(\mu)$ that for every $d$-tuple $(f_1, \dots, f_d) \in L_{p_1}(\mu) \times \cdots \times L_{p_d}(\mu)$ $D_d(f_1 \otimes \cdots \otimes f_d) = \mathcal{D}(f_1, \dots, f_d)$.*

The existence of embedding of $L_{p,\pi}(\mu^d)$ to $L_p(\mu^d)$ allows us to interpret elements of $L_{p,\pi}(\mu^d)$ as functions defined on $X^d$. Then $D_d f$ plays the role of the restriction of $f$ to the principal diagonal $\{(x_1, \dots, x_d) : x_1 = \cdots = x_d)\} \subset X^d$.

# On the proof of Theorem 4. I

We will use the above martingale-coboundary decomposition (propositition following Theorem 4). It follows from inequalites

$$\Big| \sum_{0\le i_1, i_2 \le n-1} D_2 V^{(i_1,i_2)}\big((V^{(1,0)}-I)g^{\{1\}} + (V^{(0,1)}-I)g^{\{2\}}\big)\Big|_1 \le 2C_{2,2,1}\sqrt{n}\,|g|_{2,2,\pi},$$

$$\Big| \sum_{0\le i_1, i_2 \le n-1} D_2 V^{(i_1,i_2)}\big((V^{(1,0)}-I)(V^{(0,1)}-I)g^{\{1,2\}}\big)\Big|_1 \le C_{2,2,2}\,|g|_{2,2,\pi}.$$

and the proposition on martingale-coboundary decomposition that

$$\Big|\frac{1}{n}\sum_{0\le i_1, i_2 \le n-1} D_2 V^{(i_1,i_2)}(f-g^{\emptyset})\Big|_1 \underset{n\to\infty}{\to} 0.$$

This reduces the proof to the special case of the "orthomartingale kernel" $g^{\emptyset}$.

# On the proof of Theorem 4. II

The real-valued symmetric function $g^{\emptyset}$ is a kernel of an symmetric integral nuclear operator in $L_2(\mu)$. Hence, it expands over the eigenfunctions:

$$g^{\emptyset}(x_1, x_2) = \sum_{m=1}^{\infty} \lambda_m \varphi_m(x_1) \varphi_m(x_2) \qquad (2)$$

where $(\varphi_m)_{m\geq 1}$ is a real-valued orthonormal sequence $L_2(\mu)$ (possibly incomplete) and $(\lambda_m)_{m\geq 1}$ is a sequence of reals for which $\sum_{m=1}^{\infty} |\lambda_m| < \infty$. We assume that $\lambda_m \neq 0$ for all $m \geq 1$.

# On the proof of Theorem 4. III

Let us show that for every $m \geq 1$

$$E(\varphi_m | T^{-1}\mathcal{F}) = 0.$$

We have $\mu \times \mu$-almost surely

$$0 = E(g^{\emptyset} | T^{-(0,1)}\mathcal{F}^{\otimes 2})(x_1, x_2) = \sum_{l=1}^{\infty} \lambda_l \varphi_l(x_1) E(\varphi_l | T^{-1}\mathcal{F})(x_2)$$

For every $m \geq 1$, multiplying this identity by $\varphi_m(x_1)$ and integrating it in $x_1$ with respect to $\mu$, we obtain that

$$\lambda_m E(\varphi_m | T^{-1}\mathcal{F})(x_2) = 0$$

which implies the assertion.

# On the proof of Theorem 4. IV

Let us set for $N \geq 1$

$$\xi_N = \sum_{m=1}^{N} \lambda_m \eta_m^2, \quad g_N^{\emptyset}(x_1, x_2) = \sum_{m=1}^{N} \lambda_m \varphi_m(x_1) \varphi_m(x_2).$$

Notice that for every $N$ the assertions of the theorem on the convergence in distribution and the convergence of the first moments will hold if we substitute $f$ and $\xi$ in these assertions with $g_N^{\emptyset}$ and $\xi_N$, respectively. Indeed, by means of the Cramer–Wold device the Billingsley–Ibragimov theorem extends to $\mathbb{R}^N$-valued martingale-differences. Hence, the random vectors

$$\left(\frac{1}{\sqrt{n}} \sum_{k=o}^{n-1} \varphi_1 \circ T^k, \dots, \frac{1}{\sqrt{n}} \sum_{k=o}^{n-1} \varphi_N \circ T^k\right)$$

converge in distribution to $(\eta_1, \dots, \eta_N)$ as $n \to \infty$.

# On the proof of Theorem 4. V

Therefore, random variables

$$\frac{1}{n}\sum_{0\le i_1,\, i_2\le n-1} D_2 V^{(i_1,\, i_2)} g_{\emptyset}^{(N)} = \sum_{m=1}^{N} \lambda_m \left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} \varphi_m \circ T^k\right)^2$$

comverge in distribution to $\sum_{m=1}^{N} \lambda_m \eta_m^2$ as $n \to \infty$. Convergence of the first moments follows now from the convergence of the second moments in the CLT for martingale differences. Observe that

$$|\,\xi - \xi_N|_1 = \Big|\sum_{m=N+1}^{\infty} \lambda_m\, \eta_m^2\Big|_1 \le \sum_{m=N+1}^{\infty} |\,\lambda_m| \underset{N\to\infty}{\to} 0.$$

Consequently, $(\xi_n)_{n\ge 1}$ converges in distribution to $\xi$ along with the first moments.

# On the proof of Theorem 4. VI

This convegernce and the fact that the relation

$$\left|\frac{1}{n}\sum_{0\le i_1,\, i_2\le n-1} D_2 V^{(i_1,\, i_2)} g^{\emptyset} - \frac{1}{n}\sum_{0\le i_1,\, i_2\le n-1} D_2 V^{(i_1,\, i_2)} g^{\emptyset}_N\right|_1$$
$$\le \left|\sum_{m=N+1}^{\infty}\lambda_m\left(\frac{1}{\sqrt{n}}\sum_{0\le i\le n-1}\varphi_m\circ T^i\right)\otimes\left(\frac{1}{\sqrt{n}}\sum_{0\le i\le n-1}\varphi_m\circ T^i\right)\right|_{2,2,\pi}$$
$$\le \sum_{m=N+1}^{\infty}|\lambda_m| \underset{N\to\infty}{\to} 0$$

holds uniformly in $n$ (we use here that the functions $(\varphi_m\circ T^i)_{1\le m,\,1\le i}$ are orthonormal) complete the proof.

# Example: the doubling transformation

Let $X = \{z \in \mathbb{C} : |z| = 1\}$, $\mu$ be the normalized Haar measure on $X$, $Tz = z^2$ for $z \in X$. We have

$$(Vf)(x) = f(x^2), \ (V^* f)(x) = 1/2 \sum_{\{u:\, u^2 = x\}} f(u).$$

It is known that $T$ is an exact transformation. If $f_1 \in L^2(\mu)$ and $\int_X f_1(x)\mu(dx) = 0$ then the setries

$$\sum_{k \geq 0} V^{*k} f_1$$

converges in $L^2(\mu)$ if, for example,

$$\sum_{k \geq 0} w^{(2)}(f_1, 2^{-k}) < \infty.$$

Here $w^{(2)}(f_1, \cdot)$ is the continuity modulus of $f_1$ in $L^2(\mu)$.

# Example (continued). Translation-invariant kernels

Let now $f \in L^2(\mu^2)$ be such that $f(x_1, x_2) = g(x_1 x_2^{-1})$ with some $g(x) = \sum_{k\in\mathbb{Z}} g_k x^k \in L^2(\mu)$. Assume that $f = f_2$ (that is $f$ is canonical), real-valued and symmetric. This means that $g_0 = 0$, $g_k$ are real and such that $g_{-k} = g_k$ for every $k \in \mathbb{Z}$. Next, let $f_2 \in L^{sym}_{2,\pi}(\mu^2)$ which is equivalent in our conditions to the relation

$$\sum_{k\in\mathbb{Z}} |g_k| < \infty.$$

Furthermore, if $C > 0$ and $\delta > 0$

$$|g_k| \leq \frac{C}{|k|(\log|k|)^{1+\delta}}, k \in \mathbb{Z}, k \neq 0,$$

then Theorem 4 applies to $f$.

# Example (continued). General kernels

Consider now a general kernel $f \in L_2(X^2, \mathcal{F}^{\otimes 2}, \mu^2)$ with the Fourier expansion

$$f(x_1, x_2) = \sum_{k_1,\, k_2 \in \mathbb{Z}} f_{k_1,\, k_2} x_1^{k_1} x_2^{k_2}, \quad x_1, x_2 \in X.$$

Assume that the kernel $f$ is real-valued and symmetric, that is $f_{-k_1,\, -k_2} = \overline{f}_{k_1,\, k_2}$ and $f_{k_2,\, k_1} = f_{k_1,\, k_2}$ for $k_1,\ k_2 \in \mathbb{Z}$. The summands of Hoeffding's decomposition are $f_0 = f_{0,\,0}$, $f_1(x) = \sum_{k \in \mathbb{Z}\setminus\{0\}} f_{k,\,0}\, x^k$, $f_2(x_1, x_2) = \sum_{k_1,\, k_2 \in \mathbb{Z}\setminus\{0\}} f_{k_1,\, k_2}\, x_1^{k_1} x_2^{k_2}$.
The kernel $f$ satisfies all conditions of Theorems 2 and 4 whenever $\sum_{n_1,\, n_2 \geq 0} \ \sum_{k_1,\, k_2 \in\, \mathbb{Z}\setminus\{0\}} |f_{2^{n_1} k_1,\, 2^{n_2} k_2}| < \infty$ and $\sum_{n \geq 0} \Big( \sum_{k \in \mathbb{Z}\setminus\{0\}} |f_{2^n k,\, 0}|_2^2 \Big)^{1/2} < \infty$ (for Theorem 1 only), $f_0 = 0$, $f_1(\cdot) = 0$ (for Theorem 4 only).

# How to verify our assumptions ?

A more general approach can be developed on the basis of the *transfer operator* ($V^*$ in our setting) restricted to some spaces of nice (smooth, Hölder or Sobolev) functions.
We assume now that $T$ acts on a compact smooth manifold as an expanding map preserving a measure $\mu$ so that some rate of for convergence of $V^{*n}f$ for nice $f$ is known. Expansion of a kernel into an **absolutely convergent series whose summands are products of nice functions in separate variables** is natural in the context of the limit theory of $V$-statistics. Neither uniqueness of the representation, nor linear independence of these functions is assumed.

**Proposition**. Let, for some $p \in [1, \infty]$, $(e_k)_{k=0}^{\infty}$ be a sequence of functions such that $e_0 \equiv 1$ and for every $k \geq 1$ $e_k \in L_p(\mu)$ with $\int_X e_k(x)\mu(dx) = 0$. Assume that for every $k \geq 1$

$$C_{p\,,\,k} \stackrel{\text{def}}{=} \sum_{n \geq 0} |\, V^{*n} e_k \,|_p < \infty.$$

Suppose that $f \in L_p(\,\mu^m)$ admits a representation

$$f(x_1, \ldots, x_m) = \sum_{\mathbf{0} < \mathbf{k} < \boldsymbol{\infty}} \lambda_{\mathbf{k}}(f)\, e_{k_1}(x_1) \cdots e_{k_m}(x_m) \tag{3}$$

where $(\lambda_{\mathbf{k}}(f))_{\mathbf{0} < \mathbf{k} < \boldsymbol{\infty}}$ is a family of constants satisfying

$$C_p(f) \stackrel{\text{def}}{=} \sum_{\mathbf{0} < \mathbf{k} < \boldsymbol{\infty}} |\, \lambda_{\mathbf{k}}\,(f)|\, C_{p\,,\,k_1} \cdots\, C_{p\,,\,k_m} < \infty. \tag{4}$$

Then $f$ is a canonical kernel of degree $m$, $f \in L_{p,\pi}(\mu^m)$, the series in

$$g = \sum_{\mathbf{0} \le \mathbf{k} < \infty} V^{*\mathbf{k}} f \left( \stackrel{\text{def}}{=} \lim_{\substack{n_1 \to \infty \\ \cdots \\ n_m \to \infty}} \sum_{\mathbf{0} \le \mathbf{k} < \mathbf{n}} V^{*\mathbf{k}} f \right) \tag{5}$$

converges in $L_{p,\pi}(\mu^m)$ and its sum $g$ satisfies the inequality

$$|g|_{p,m,\pi} \le C_p(f). \tag{6}$$

# QUESTIONS, PROBLEMS

1. Prove the Functional CLT (doable)
2. Develop a non-adapted version (doable)
3. Incorporate the development after 2000 in the field of martingale approximation (Maxwell-Woodroofe, Peligrad-Utev) (doable)
4. Investigate large deviations and local CLT (perturbation of some kind of transfer operator ?)
5. Substitute the nuclearity assumption by a weaker requirement (serious functional-theoretic problem)

The preprint

by M. Denker and the speaker

**arXiv:1109.0635v2 [math.DS]**

covers the present talk but contains some missprints.

A corrected version can be requested from the speaker via E-mail:

**gordin@pdmi.ras.ru**

# Thank you !