Translation Invariant Statistical Experiments with Independent Increments

(joint work with Nino Kordzakhia and Alex Novikov)

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Outline

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1. **Introduction**
2. Representation of translation invariant experiments with independent increments
3. Large deviations and convergence results

Example 1

Observations: $X_1, \dots, X_n, \dots$ are i.i.d. random variables with density $f(x - \vartheta)$, $x \in \mathbb{R}$;
$f(x)$ is regular everywhere except $x = 0$, where it has a jump:

$$f(-0) = q, \quad f(+0) = p, \qquad p > 0, \quad q > 0, \quad p \neq q;$$

P_ϑ^n is the law of $X^{(n)} = (X_1, \dots, X_n)$;
$\varphi_n = n^{-1}$ is a normalization rate;
the normalized likelihood $Z_u^{n,\vartheta_0} = d\mathrm{P}^n_{\vartheta_0+\varphi_n u}/d\mathrm{P}^n_{\vartheta_0}$;

$$\mathrm{Law}\left((Z_u^{n,\vartheta_0})_{u\in\mathbb{R}} | \mathrm{P}^n_{\vartheta_0}\right) \underset{n\to\infty}{\Rightarrow} \mathrm{Law}\left((Z_u)_{u\in\mathbb{R}}\right),$$

where $Z_u = \exp(Y_u)$,

$$Y_u = \begin{cases} u(p-q) + \log(q/p)\pi^+(u), & u \geq 0, \\ -u(q-p) + \log(p/q)\pi^-(-u), & u \leq 0, \end{cases}$$

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Example 2: $q = 0$

What happens with the limiting process $Z_t = Z_t^{(p.q)}$ if $q \to 0$:

$$\mathrm{Law}\left((Z_u^{(p,q)})_{u\in\mathbb{R}}\right) \underset{q\to 0}{\Rightarrow} \mathrm{Law}\left((Z_u^{(p,0)})_{u\in\mathbb{R}}\right),$$

where

$$Z_u^{(p,0)} = e^{pu} 1_{\{u \le \tau\}},$$

where τ is a random variable with exponential distribution with parameter p.

For example, let $f(x) = e^{-x} 1_{\{x \ge 0\}}$ $(p = 1,\ q = 0)$, then

$$\frac{d\mathrm{P}_\vartheta^n}{d\lambda}(x_1, \dots, x_n) = \begin{cases} e^{n\vartheta - \sum_1^n x_i}, & x_1 \ge \vartheta, \dots, x_n \ge \vartheta, \\ 0, & \text{otherwise,} \end{cases}$$

hence

$$Z_u^{n,\vartheta_0} = d\mathrm{P}^n_{\vartheta_0 + \varphi_n u} / d\mathrm{P}^n_{\vartheta_0} = e^u 1_{\{x_1 \wedge \dots \wedge x_n \ge \vartheta_0 + u/n\}}(x_1, \dots, x_n) \overset{\mathrm{Law}}{=} Z_u^{(1,0)}.$$

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Observations: $X_1, \dots, X_n, \dots$ are independent uniform on $[\vartheta, \vartheta+1]$, i.e. $f(x) = 1_{[0,1]}(x)$.
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Example 4 (Ibragimov and Hasminskii 1975)

Observations:

$$dX(t) = S(t-\vartheta)\,dt + \varepsilon dW(t), \quad t \in [0,1],$$

$S(t)$ is regular everywhere except $t = 0$, where it has a jump:

$$S(-0) = q, \quad S(+0) = p, \qquad p - q = r \neq 0;$$

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where $Z_u = \exp(rB_u - \frac{1}{2}r^2|u|)$, and $B(u)$, $u \in \mathbb{R}$, is a two-sided standard Brownian motion.

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MLE and BE

Let $Z_u^0 = \exp(B_u - \frac{1}{2}|u|)$,

$$\xi_0 = \arg\max_{u \in \mathbb{R}} Z_u^0 \quad \text{and} \quad \zeta_0 = \frac{\int_{\mathbb{R}} u Z_u^0\, du}{\int_{\mathbb{R}} Z_u^0\, du}$$

be the maximum likelihood estimator and a generalized Bayesian (wrt quadratic loss) estimator respectively.

Terent'yev (1968): $\mathrm{Var}(\xi_0) = 26$.

Ibragimov and Khasminskii (1979/1981): numerical simulation of $\mathrm{Var}(\zeta_0) = 19.5 \pm 0.5$.

Golubev (1979): $\mathrm{Var}(\zeta_0)$ in terms of the second derivative of a certain improper integral.

Rubin and Song (1995): exact value $\mathrm{Var}(\zeta_0) = 16\zeta(3)$, where $\zeta(\cdot)$ is Riemann's zeta function.

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Example 5: Chigansky and Kutoyants (2012)

Observations: nonlinear (threshold autoregressive) TAR(1) model

$$X_{j+1} = h(X_j)1_{\{X_j<\vartheta\}} + g(X_j)1_{\{X_j\ge\vartheta\}} + \varepsilon_{j+1}, \quad j = 0, \dots, n, \dots,$$

where h and g are known functions, ε_j are i.i.d. random variables with a known density $f(x) > 0$. Put $\delta(u) = g(u) - h(u)$.

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Dachian and Negri 2011

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$$Z_u^{\gamma,f} = \begin{cases} \exp\Big(\sum_{i=1}^{\pi^+(u)} \log \frac{f(\varepsilon_i^+ + \gamma)}{f(\varepsilon_i^+)}\Big), & u \geq 0, \\ \exp\Big(\sum_{i=1}^{\pi^-(-u)} \log \frac{f(\varepsilon_i^- - \gamma)}{f(\varepsilon_i^-)}\Big), & u \leq 0, \end{cases}$$

where $\gamma > 0$, $\pi^+(u)$ and $\pi^-(u)$, $u \geq 0$, are independent Poisson processes with intensity 1, $\varepsilon^\pm$ are i.i.d. random variables with density $f > 0$ which are also independent of $\pi^\pm$. It is also assumed that f has zero mean and variance 1 and quadratic mean differentiable with the Fisher information $I > 0$.

Then

$$\mathrm{Law}\left((Z^{\gamma,f}_{u/(I\gamma^2)})_{u\in\mathbb{R}}\right) \underset{\gamma\to 0}{\Rightarrow} \mathrm{Law}\left((Z^0_u)_{u\in\mathbb{R}}\right),$$

and

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Dachian and Negri 2011

Let

$$Z_u^{\gamma,f} = \begin{cases} \exp\Big(\sum_{i=1}^{\pi^+(u)} \log \frac{f(\varepsilon_i^+ + \gamma)}{f(\varepsilon_i^+)}\Big), & u \geq 0, \\ \exp\Big(\sum_{i=1}^{\pi^-(-u)} \log \frac{f(\varepsilon_i^- - \gamma)}{f(\varepsilon_i^-)}\Big), & u \leq 0, \end{cases}$$

where $\gamma > 0$, $\pi^+(u)$ and $\pi^-(u)$, $u \geq 0$, are independent Poisson processes with intensity 1, $\varepsilon^\pm$ are i.i.d. random variables with density $f > 0$ which are also independent of $\pi^\pm$. It is also assumed that f has zero mean and variance 1 and quadratic mean differentiable with the Fisher information $I > 0$.

Then

$$\mathrm{Law}\left((Z_{u/(I\gamma^2)}^{\gamma,f})_{u\in\mathbb{R}}\right) \underset{\gamma\to 0}{\Rightarrow} \mathrm{Law}\left((Z_u^0)_{u\in\mathbb{R}}\right),$$

and

$$\mathrm{Law}\left((Z_u^{\gamma,f})_{u\in\mathbb{R}}\right) \underset{\gamma\to\infty}{\Rightarrow} \mathrm{Law}\left((Z_u^\infty)_{u\in\mathbb{R}}\right).$$

Outline

Experiments

A statistical experiment is a triple $\mathbb{E} = \left(\Omega, \mathscr{F}, \left(\mathsf{P}_\vartheta\right)_{\vartheta\in\Theta}\right)$ consisting of a measurable space $(\Omega, \mathscr{F})$ and a family $\left(\mathsf{P}_\vartheta\right)_{\vartheta\in\Theta}$ of probability measures on $(\Omega, \mathscr{F})$.

We consider experiments only with $\Theta = \mathbb{R}$ or their finite subexperiments $\mathbb{E} = \left(\Omega, \mathscr{F}, \left(\mathsf{P}_\vartheta\right)_{\vartheta\in I}\right)$, $I = \{\vartheta_0, \vartheta_1, \dots, \vartheta_k\} \subset \mathbb{R}$, which are denoted simply as $\left(\mathsf{P}_{\vartheta_0}, \mathsf{P}_{\vartheta_1}, \dots, \mathsf{P}_{\vartheta_k}\right)$.

Hellinger transform

For $k \in \mathbb{N}$, let

$$S_k = \{\alpha = (\alpha_0, \alpha_1, \dots, \alpha_k)\colon \alpha_i > 0,\ i = 0, \dots, k,\ \sum_{i=0}^{k} \alpha_i = 1\}.$$

The Hellinger transform $H(\alpha; \mathrm{P}_{\vartheta_0}, \mathrm{P}_{\vartheta_1}, \dots, \mathrm{P}_{\vartheta_k})$, $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_k) \in S_k$, is defined by

$$H(\alpha; \mathrm{P}_{\vartheta_0}, \mathrm{P}_{\vartheta_1}, \dots, \mathrm{P}_{\vartheta_k}) = \int \prod_{i=0}^{k} \Big(\frac{dP_{\vartheta_i}}{d\mu}\Big)^{\alpha_i}\, d\mu,$$

where μ is an arbitrary σ-finite measure dominating $\mathrm{P}_{\vartheta_0}, \mathrm{P}_{\vartheta_1}, \dots, \mathrm{P}_{\vartheta_k}$. In the binary case $k = 1$, we write $H(\alpha; \mathrm{P}_{\vartheta_0}, \mathrm{P}_{\vartheta_1})$ instead of $H((\alpha, 1-\alpha); \mathrm{P}_{\vartheta_0}, \mathrm{P}_{\vartheta_1})$.

The likelihood processes

For an experiment $\mathbb{E} = \left(\Omega, \mathscr{F}, (\mathsf{P}_\vartheta)_{\vartheta\in\Theta}\right)$ the likelihood process of $\mathbb{E}$ with base $\eta \in \Theta$ is

$$\left(\frac{d\mathsf{P}_\vartheta}{d\mathsf{P}_\eta}\right)_{\vartheta\in\Theta},$$

where $d\mathsf{P}_\vartheta/d\mathsf{P}_\eta$ stands for the Radon–Nikodym derivative of the P_η-absolutely continuous component of P_ϑ with respect to P_η. The distribution of the likelihood process with base η is always taken with respect to P_η.

Equivalent experiments

Two experiments $\mathbb{E} = \big(\Omega, \mathscr{F}, (\mathsf{P}_\vartheta)_{\vartheta\in\Theta}\big)$ and $\mathbb{E}' = \big(\Omega', \mathscr{F}', (\mathsf{P}'_\vartheta)_{\vartheta\in\Theta}\big)$ with the same parameter set are equivalent ($\mathbb{E} \sim \mathbb{E}'$) if

$$\mathrm{Law}\left(\left(\frac{d\mathsf{P}_\vartheta}{d\mathsf{P}_\eta}\right)_{\vartheta\in\Theta}\Big|\mathsf{P}_\eta\right) = \mathrm{Law}\left(\left(\frac{d\mathsf{P}'_\vartheta}{d\mathsf{P}'_\eta}\right)_{\vartheta\in\Theta}\Big|\mathsf{P}'_\eta\right) \quad \text{for every} \quad \eta\in\Theta.$$

$\mathbb{E} \sim \mathbb{E}'$ if and only if

$$H(\alpha; \mathsf{P}_{\vartheta_0}, \mathsf{P}_{\vartheta_1}, \dots, \mathsf{P}_{\vartheta_k}) = H(\alpha; \mathsf{P}'_{\vartheta_0}, \mathsf{P}'_{\vartheta_1}, \dots, \mathsf{P}'_{\vartheta_k})$$

for every $I = \{\vartheta_0, \vartheta_1, \dots, \vartheta_k\}$ with $k \in \mathbb{N}$ and every $\alpha \in S_k$.

Weak convergence

The weak convergence of experiments is understood as the weak convergence of finite-dimensional distributions of likelihood processes and is equivalent to the convergence of the Hellinger transforms. It is denoted by $\xrightarrow{w}$.

An experiment $\mathbb{E} = \left(\Omega, \mathscr{F}, \left(\mathsf{P}_\vartheta\right)_{\vartheta\in\Theta}\right)$ is called totally non-informative if $\mathsf{P}_\vartheta = \mathsf{P}_\eta$ for all $\vartheta, \eta \in \Theta$.

The product $\mathbb{E} \otimes \mathbb{E}'$ of experiments $\mathbb{E} = \left(\Omega, \mathscr{F}, \left(\mathsf{P}_\vartheta\right)_{\vartheta\in\Theta}\right)$ and $\mathbb{E}' = \left(\Omega', \mathscr{F}', \left(\mathsf{P}'_\vartheta\right)_{\vartheta\in\Theta}\right)$ is defined as

$$\mathbb{E} \otimes \mathbb{E}' = \left(\Omega \times \Omega', \mathscr{F} \otimes \mathscr{F}', \left(\mathsf{P}_\vartheta \times \mathsf{P}'_\vartheta\right)_{\vartheta\in\Theta}\right).$$

The Hellinger transforms of the product are obtained as the product of the corresponding Hellinger transforms. In an obvious manner, the nth power $\mathbb{E}^{\otimes n}$ of $\mathbb{E}$ is defined.

Infinitely divisible experiments

An experiment $\mathbb{E}$ is called infinitely divisible if, for every $n \in \mathbb{N}$, there is an experiment $\mathbb{E}_n$ such that $\mathbb{E} \sim \mathbb{E}_n^{\otimes n}$.
An experiment $\mathbb{E} = \left(\Omega, \mathscr{F}, \left(\mathsf{P}_\vartheta\right)_{\vartheta \in \mathbb{R}}\right)$ is continuous if $\lim_{\eta \to \vartheta} \|\mathsf{P}_\eta - \mathsf{P}_\vartheta\| = 0$ for every $\vartheta \in \Theta$.
If $\lambda \geqslant 0$ and $t \in \mathbb{R}$, $\mathbb{E} = \left(\Omega, \mathscr{F}, \left(\mathsf{P}_\vartheta\right)_{\vartheta \in \mathbb{R}}\right)$, then $U_\lambda \mathbb{E}$ and $T_t \mathbb{E}$ are defined as

$$U_\lambda \mathbb{E} = \left(\Omega, \mathscr{F}, \left(\mathsf{P}_{\lambda\vartheta}\right)_{\vartheta \in \mathbb{R}}\right)$$

and

$$T_t \mathbb{E} = \left(\Omega, \mathscr{F}, \left(\mathsf{P}_{\vartheta + t}\right)_{\vartheta \in \mathbb{R}}\right).$$

Translation invariance, stability, independent increments

An experiment $\mathbb{E} = \big(\Omega, \mathscr{F}, (\mathsf{P}_\vartheta)_{\vartheta\in\mathbb{R}}\big)$ is said

- to be translation invariant if $\mathbb{E} \sim T_t\mathbb{E}$ for every $t \in \mathbb{R}$;
- to be stable if it is either totally non-informative or if it is continuous and there is some $p > 0$ (called the exponent of stability) such that $\mathbb{E}^{\otimes n} \sim U_{n^{1/p}}\mathbb{E}$ for every $n \in \mathbb{N}$;
- to have independent increments if, for any $\vartheta_0 < \vartheta_1 < \cdots < \vartheta_k$,

$$\begin{aligned}&\big(\mathsf{P}_{\vartheta_0}, \mathsf{P}_{\vartheta_1}, \dots, \mathsf{P}_{\vartheta_k}\big)\\ &\sim \big(\mathsf{P}_{\vartheta_0}, \mathsf{P}_{\vartheta_1}, \dots, \mathsf{P}_{\vartheta_1}\big) \otimes \big(\mathsf{P}_{\vartheta_1}, \mathsf{P}_{\vartheta_1}, \mathsf{P}_{\vartheta_2}, \dots, \mathsf{P}_{\vartheta_2}\big) \otimes \dots\\ &\qquad \otimes \big(\mathsf{P}_{\vartheta_{k-1}}, \dots \mathsf{P}_{\vartheta_{k-1}}, \mathsf{P}_{\vartheta_k}\big).\end{aligned}$$

Translation invariance, stability

Le Cam (1973) showed that weak limits of experiments $U_{\delta_n} T_{\vartheta_0} \mathbb{E}_n$, $\delta_n \to 0$, are "often" translation invariant.
Strasser (1985a) proved that an experiment $\mathbb{F}$ is stable if and only if it is the weak limit of a sequence $\left(U_{\delta_n} T_{\vartheta_0} \mathbb{E}\right)^{\otimes n} = U_{\delta_n} T_{\vartheta_0} \mathbb{E}^{\otimes n}$, $\delta_n \downarrow 0$, of product experiments (under a mild additional assumption on this sequence). Moreover, if $\mathbb{F}$ is not totally non-informative, then, necessarily,

$$\delta_n = n^{-1/p} a_n,$$

where p is the exponent of stability of $\mathbb{F}$ and (a_n) is a slowly varying sequence, i.e.

$$\lim_n \frac{a_{nm}}{a_n} = 1 \quad \text{for every } m \in \mathbb{N}.$$

Experiments with independent increments

The notion of an experiment with independent increments was introduced by Strasser (1985b), where he gave necessary and sufficient conditions for a weakly convergent sequence $\mathbb{E}_n = \mathbb{E}_{n,1} \otimes \cdots \otimes \mathbb{E}_{n,k_n}$, $k_n \to \infty$, of product experiments to have an experiment with independent increments in the limit. Experiments with independent increments often arise as limiting ones in non-regular models. To illustrate this, let us recall that every continuous translation invariant experiment with independent increments is stable with exponent $p = 1$, see Strasser (1985b). This means, in particular, that if $\mathbb{E}_n$ corresponds to n independent observations with a density $f(\cdot - \vartheta)$ and $U_{\delta_n}\mathbb{E}_n \xrightarrow{w} \mathbb{F}$, where $\mathbb{F}$ is an experiment with independent increments, then the rate of convergence is n up to a slowly varying sequence.

Lévy processes

Recall that the law of a real-valued Lévy process $X = (X_t)_{t\geq 0}$ is uniquely characterized by a triple (b, c, F), where $b \in \mathbb{R}$, $c \geq 0$, and F is a measure on $\mathbb{R} \setminus \{0\}$ satisfying $\int (1 \wedge x^2)\, F(dx) < \infty$; for every $t \geqslant 0$ and $\lambda \in \mathbb{R}$,

$$\mathsf{E}\left[e^{i\lambda X_t}\right] = \exp[t\Psi(\lambda)],$$

where

$$\Psi(\lambda) = i\lambda b - \frac{\lambda^2}{2}c + \int \left(e^{i\lambda x} - 1 - i\lambda h(x)\right) F(dx) \qquad (1)$$

and $h\colon \mathbb{R} \to \mathbb{R}$ is a truncation function. The function Ψ is called the cumulant of X.

A set $\mathscr{Q}$ of triples

Denote by $\mathscr{Q}$ the set of all triples (a, c, Π), where $a \geq 0$, $c \geq 0$, Π is a measure on $[-1, \infty)$ such that $\int (|x| \wedge x^2)\, \Pi(dx) < \infty$. Given such a triple $(a, c, \Pi) \in \mathscr{Q}$, let

$$\Psi_{a,c,\Pi}(\lambda) = -i\lambda a - \frac{\lambda^2}{2} c + \int \left(e^{i\lambda x} - 1 - i\lambda x\right) \Pi(dx). \qquad (2)$$

It is easy to see that (2) reduces to (1) if we put $b = -a - \int (x - h(x))\, \Pi(dx)$, $F = \Pi$. Therefore, there exists a Lévy process X with the cumulant $\Psi_{a,c,\Pi}$.

Stochastic exponential

Recall that the stochastic exponential $Z = \mathscr{E}(X)$ of a semimartingale X is defined as a solution Z to the stochastic differential equation

$$dZ_t = Z_{t-}\, dX_t, \quad , Z_0 = 1,$$

which is understood as

$$Z_t = 1 + \int_{(0,t]} Z_{s-}\, dX_s.$$

A solution always exists and is unique. $\mathscr{E}(X) \geq 0$ iff $\Delta X \geq -1$. $\mathscr{E}(X) > 0$ and $\mathscr{E}(X)_- > 0$ iff $\Delta X > -1$.

Exponentials and stochastic exponentials of Lévy processes

If Y is a real-valued Lévy process, then $e^Y = \mathscr{E}(X)$, where X is also a Lévy process with $\Delta X > -1$. If conversely X is a Lévy process with $\Delta X > -1$, then $Y = \log \mathscr{E}(X)$ is also a Lévy process. Furthermore the triples (b_X, c_X, F_X) and (b_Y, c_Y, F_Y) of X and Y respectively can be expressed via each other by

- $b_X = b_Y + \frac{c_Y}{2} + \int [h(e^x - 1) - h(x)]\, F_X(dx)$,
- $c_Y = c_X$,
- F_X is the image of F_Y under the mapping $x \rightsquigarrow e^x - 1$,

and

- $b_Y = b_X - \frac{c_X}{2} + \int [h(\log(1+x)) - h(x)]\, F_Y(dx)$,
- $c_X = c_Y$,
- F_Y is the image of F_X under the mapping $x \rightsquigarrow \log(1+x)$.

Lévy processes with the cumulant $\Psi_{a,c,\Pi}$, $(a, c, \Pi) \in \mathscr{Q}$

Proposition

Let X be a Lévy process. The following conditions are equivalent:

- $\mathscr{E}(X) \geq 0$, $\mathsf{E}\mathscr{E}(X)_t \leq 1$ *for some* $t > 0$;
- *the cumulant of X is $\Psi_{a,c,\Pi}$ for some $(a, c, \Pi) \in \mathscr{Q}$.*

If these conditions are satisfied, then

- $\mathsf{E}\mathscr{E}(X)_t = e^{-at}$; $e^{at}\mathscr{E}(X)_t$ *is a martingale*;
- $\mathscr{E}(X)$ *is represented in the form*

$$\mathscr{E}(X)_t = e^{Y_t} e^{-at} V_t,$$

where Y is a Lévy process, $V \equiv 1$ if $\Pi(\{-1\}) = 0$ and $V_t = e^{\Pi(\{-1\})t} 1_{\{t<\tau\}}$ otherwise, where $\tau = \inf\{t \geq 0 \colon \Delta X_t = -1\}$ is a random variable independent of Y with the exponential distribution with mean $1/\Pi(\{-1\})$.

Decomposition

The above decomposition can be also written in the form

$$\mathscr{E}(X) = \mathscr{E}(X^{(1)})\mathscr{E}(X^{(2)})\mathscr{E}(X^{(3)}),$$

where $X^{(1)}$, $X^{(2)}$, $X^{(3)}$ are independent Lévy processes with the cumulants $\Psi_{0,c,1_{(-1,\infty)}\cdot\Pi}$, $\Psi_{0,0,1_{\{-1\}}\cdot\Pi}$, $\Psi_{a,0,0}$, respectively.

Conjugate triple

For a triple $(a, c, \Pi) \in \mathscr{Q}$ define a conjugate triple $(\hat{a}, \hat{c}, \hat{\Pi})$:

$$\hat{a} = \Pi(\{-1\}), \quad \hat{c} = c, \quad \hat{\Pi}(\{-1\}) = a,$$

$$\hat{\Pi}(G) = \int_{\{x>-1\}} (1+x) 1\Big(-\frac{x}{1+x} \in G\Big)\, \Pi(dx), \quad G \in \mathscr{B}(-1, \infty).$$

It is easy to check that $(\hat{a}, \hat{c}, \hat{\Pi}) \in \mathscr{Q}$.
Note that the conjugate triple to $(\hat{a}, \hat{c}, \hat{\Pi})$ is (a, c, Π).

Constructing a two-sided process

Therefore, we can take a Lévy process $\hat{X}$ with the cumulant $\Psi_{\hat{a},\hat{c},\hat{\Pi}}$ and independent of X. Starting from $\hat{X}$, we construct as above the processes $\hat{X}^{(1)}$, $\hat{X}^{(2)}$, $\hat{X}^{(3)}$, $\hat{Y}$ and a random variable $\hat{T}$.
Finally, for $-\infty < t < \infty$, define

$$Z_t = \begin{cases} \mathscr{E}(X)_t, & \text{if } t \geq 0, \\ \mathscr{E}(\hat{X})_{|t|}, & \text{if } t \leq 0, \end{cases}$$
$$= \begin{cases} e^{Y_t + (\hat{a}-a)t} 1_{\{t \leq T\}}, & \text{if } t \geq 0, \\ e^{\hat{Y}_{|t|} + (a-\hat{a})|t|} 1_{\{|t| \leq \hat{T}\}}, & \text{if } t \leq 0, \end{cases}$$

Main theorems (1/2)

Theorem

Let $\mathbb{E} = \left(\Omega, \mathscr{F}, (\mathsf{P}_t)_{t\in\mathbb{R}}\right)$ *be a continuous translation invariant experiment with independent increments. Then there is a triple* $(a, c, \Pi) \in \mathscr{Q}$ *such that*

$$\mathrm{Law}\left(\left(\frac{d\mathsf{P}_t}{d\mathsf{P}_0}\right)_{t\in\mathbb{R}}\middle|\mathsf{P}_0\right) = \mathrm{Law}\left((Z_t)_{t\in\mathbb{R}}\right),$$

where $Z = (Z_t)_{t\in\mathbb{R}}$ *is defined by*

$$Z_t = \begin{cases} \mathscr{E}(X)_t, & \text{if } t \geq 0, \\ \mathscr{E}(\hat{X})_{|t|}, & \text{if } t \leq 0, \end{cases}$$

Main theorems (2/2)

Theorem

Assume that $(a, c, \Pi) \in \mathscr{Q}$, X *and* $\hat{X}$ *are independent Lévy processes with the cumulants* $\Psi_{a,c,\Pi}$ *and* $\Psi_{\hat{a},\hat{c},\hat{\Pi}}$, *respectively, and* $Z = (Z_t)_{t\in\mathbb{R}}$ *is defined as in the previous theorem. Then there is a continuous translation invariant experiment with independent increments* $\mathbb{E} = \left(\Omega, \mathscr{F}, \left(\mathsf{P}_t\right)_{t\in\mathbb{R}}\right)$ *such that*

$$\mathrm{Law}\left(\left(\frac{d\mathsf{P}_t}{d\mathsf{P}_0}\right)_{t\in\mathbb{R}}\middle|\mathsf{P}_0\right) = \mathrm{Law}\left((Z_t)_{t\in\mathbb{R}}\right).$$

Outline

Preliminaries

$\mathbb{E} = \left(\Omega, \mathscr{F}, (\mathsf{P}_t)_{t\in\mathbb{R}}\right)$ is a continuous translation invariant experiment with independent increments,

$$Z_t = \frac{d\mathsf{P}_t}{d\mathsf{P}_0}, \quad t \in \mathbb{R},$$

and

$$\begin{aligned} Z_t &= \begin{cases} \mathscr{E}(X)_t, & \text{if } t \geq 0, \\ \mathscr{E}(\hat{X})_{|t|}, & \text{if } t < 0, \end{cases} \\ &= \begin{cases} e^{Y_t + (\hat{a}-a)t} 1_{\{t \leq T\}}, & \text{if } t \geq 0, \\ e^{\hat{Y}_{|t|} + (a-\hat{a})|t|} 1_{\{|t| \leq \hat{T}\}}, & \text{if } t < 0, \end{cases} \end{aligned}$$

where X and $\hat{X}$ are independent Lévy processes with cumulants $(a, c, \Pi) \in \mathscr{Q}$ and the conjugate triple $(\hat{a}, \hat{c}, \hat{\Pi})$, respectively, Y and T are obtained from X and $\hat{Y}$ and $\hat{T}$ are obtained from $\hat{X}$ as described above.

Integrability property

The trivial case $a = c = 0$, $\Pi = 0$ is tacitly excluded from the consideration. Then

$$\kappa = -\log \mathrm{E}_0 \sqrt{Z_1} = \frac{c}{8} + \int_{x \geq -1} \left(1 + \frac{x}{2} - \sqrt{1+x}\right) \Pi(dx) + \frac{a}{2} > 0.$$

Lemma

For any $p \geq 0$,

$$\int_{-\infty}^{\infty} Z_t (1 \vee |t|^p)\, dt < \infty \quad \mathrm{P}_0\text{-a.s.}$$

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Posterior distribution

In particular, there exists a posterior distribution (with respect to the Lebesgue measure on $\mathbb{R}$ as a prior) $F = F(\omega, B)$, $\omega \in \Omega$, $B \in \mathscr{B}$, satisfying

$$F(B) = \int_B q_t \, dt, \quad B \in \mathscr{B}, \quad \mathsf{P}_0\text{-a.s.},$$

where

$$q_t = \frac{Z_t}{\int_{\mathbb{R}} Z_t \, dt}.$$

Bayesian estimators

Let $W\colon \mathbb{R} \to [0, \infty)$ be a loss function. Here we shall assume that $W(0) = 0$, W is strictly convex and $W(t) \le C(1 \vee |t|^p)$, $t \in \mathbb{R}$, for some $p \ge 1$ and $C > 0$.

A generalized Bayesian estimator with respect to the uniform prior (i.e. the Lebesgue measure) on $\mathbb{R}$ and the loss function W. or a Pitman estimator, denoted by ζ, exists, is translation invariant (i.e. $\mathrm{Law}\,(\zeta \mid P_0) = \mathrm{Law}\,(\zeta - t \mid P_t)$ for every $t \in \mathbb{R}$) and minimax, and satisfies

$$\zeta = \arg\min_{x\in\mathbb{R}} \int_{\mathbb{R}} W(x-t) q_t \, dt = \arg\min_{x\in\mathbb{R}} \int_{\mathbb{R}} W(x-t) Z_t \, dt \quad \mathrm{P}_0\text{-a.s.}$$

In particular, if $W(x) = x^2$,

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Large deviations for the posterior distribution

Theorem

There exist positive constants C_0, C_1, R_0, and C such that, for all $R > R_0$,

$$\mathrm{P}_0\Big(\int_{|t|\geq R} q_t\,dt \geq e^{-C_1 R}\Big) \leq Ce^{-C_0 R}.$$

Remark

One can arbitrarily choose C_0 from the interval $(0, \kappa/8)$ and then C_1 from the interval $(0, 2(\kappa - 8C_0)/5)$, and R_0 is any number satisfying $R_0 \geq 2$ and $e^{-\gamma R_0} \leq 1 - e^{-\gamma}$, where $\gamma = 2(\kappa - 8C_0)/5 - C_1$.

The proof follows the arguments used in the proof of Theorem 1.5.2 in Ibragimov and Has'minskii (1979/1981).

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Notation

It follows from the definitions that the class of continuous translation invariant experiments with independent incrementsis closed with respect to the weak convergence. Here our aim is to study conditions for the weak convergence. All necessary tools can be found in Coquet and Jacod (1990).

For $(a, c, F) \in \mathscr{Q}$ and $\alpha \in (0, 1)$, put

$$g_{a,c,F}(\alpha) = \alpha a + \frac{\alpha(1-\alpha)}{2} c + \int_{x \geq -1} \left(1 + \alpha x - (1+x)^{\alpha}\right) \Pi(dx).$$

It is easy to check that $g_{a,c,F}(\alpha) = -\log H(1-\alpha; \mathrm{P}_0, \mathrm{P}_1)$.

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Closedness

The following proposition is taken from Coquet and Jacod (1990). It also follows from our previous considerations.

Proposition

Let $(a_n, c_n, F_n) \in \mathscr{Q}$ for any n. If g_{a_n,c_n,F_n} converges pointwise to g, then there is $(a, c, F) \in \mathscr{Q}$ such that $g = g_{a,c,F}$.

Convergence in terms of characteristics

Let $h\colon \mathbb{R} \to \mathbb{R}$ be a truncation function.

Theorem (Coquet and Jacod (1990))

Let $(a_n, c_n, F_n) \in \mathscr{Q}$ for any n, $(a, c, F) \in \mathscr{Q}$. The following statements are equivalent:

(i) g_{a_n,c_n,F_n} *converges pointwise to* $g_{a,c,F}$;

(ii)

- $-a_n - \int (x - h(x))\,\Pi_n(dx) \to -a - \int (x - h(x))\,\Pi(dx)$;
- $c_n + \int h^2(x)\,\Pi_n(dx) \to c + \int h^2(x)\,\Pi(dx)$;
- $\int \chi(x)\,\Pi_n(dx) \to \int \chi(x)\,\Pi_n(dx)$ *for any bounded* $\chi\colon \mathbb{R} \to \mathbb{R}$ *which equals zero in a neighborhood of zero.*

Weak convergence (preliminaries)

Let now $\mathbb{E}^n = \left(\Omega^n, \mathscr{F}^n, \left(\mathsf{P}^n_\vartheta\right)_{\vartheta\in\mathbb{R}}\right)$, $n \geqslant 1$, and $\mathbb{E} = \left(\Omega, \mathscr{F}, \left(\mathsf{P}_\vartheta\right)_{\vartheta\in\mathbb{R}}\right)$ be continuous translation invariant experiments with independent increments. The corresponding Lévy processes on $\mathbb{R}_+$ are denoted by X^n and X, their cumulants are Ψ_{a_n,c_n,F_n} and $\Psi_{a,c,F}$ respectively. The likelihood processes are denoted by Z^n and Z respectively. The next theorem is also mainly due to Coquet and Jacod (1990).

Weak convergence (main result)

Theorem

The statements (i) *and* (ii) *of the previous theorem are equivalent to any one of the following statements*:

(i) $\mathrm{Law}\left(dP_1^n/dP_0^n \middle| P_0^n\right) \Rightarrow \mathrm{Law}\left(dP_1/dP_0 \middle| P_0\right)$;

(ii) $\mathbb{E}^n \xrightarrow{w} \mathbb{E}$.

Moreover, if these conditions are satisfied, then

- X^n *converge in distribution to* X *in the Skorokhod space* $D[0,\infty)$;
- Z^n *converge in distribution to* Z *in the Skorokhod space* $D(-\infty,\infty)$.

Convergence of Bayesian estimators

Let a sequence $\mathbb{E}^n$ of continuous translation invariant experiments with independent increments weakly converges to $\mathbb{E}$. If $\mathbb{E}$ is not totally noninformative, then we automatically have the weak convergence of Bayesian estimators together with convergence of all their moments.

Example: $Z_{\gamma,f}$, $\gamma \to 0$ (1/2)

Recall the notation

$$Z_u^{\gamma,f} = \begin{cases} \exp\Big(\sum_{i=1}^{\pi^+(u)} \log \frac{f(\varepsilon_i^+ + \gamma)}{f(\varepsilon_i^+)}\Big), & u \geq 0, \\ \exp\Big(\sum_{i=1}^{\pi^-(-u)} \log \frac{f(\varepsilon_i^- - \gamma)}{f(\varepsilon_i^-)}\Big), & u \leq 0, \end{cases}$$

where $\gamma > 0$, $\pi^+(u)$ and $\pi^-(u)$, $u \geq 0$, are independent Poisson processes with intensity 1, $\varepsilon^\pm$ are i.i.d. random variables with density $f > 0$ which are also independent of $\pi^\pm$.

Example: $Z_{\gamma,f}$, $\gamma \to 0$ (2/2)

Now let

$$f(x) = C(\alpha)e^{-|x|^{\alpha}}, \quad x \in \mathbb{R},$$

where $\alpha > 0$. In the regular case $\alpha > 1/2$ Dachian and Negri (2011) showed that

$$\mathrm{Law}\left((Z^{\gamma,f}_{u/(c(\alpha)\gamma^2)})_{u\in\mathbb{R}}\right) \underset{\gamma\to 0}{\Rightarrow} \mathrm{Law}\left((Z^0_u)_{u\in\mathbb{R}}\right).$$

It follows easily from our results that

$$\mathrm{Law}\left((Z^{\gamma,f}_{u/(c(\alpha)\gamma^2\log(1/\gamma))})_{u\in\mathbb{R}}\right) \underset{\gamma\to 0}{\Rightarrow} \mathrm{Law}\left((Z^0_u)_{u\in\mathbb{R}}\right), \quad \alpha = 1/2,$$

(almost regular case), and

$$\mathrm{Law}\left((Z^{\gamma,f}_{u/(c(\alpha)\gamma^{1+2\alpha})})_{u\in\mathbb{R}}\right) \underset{\gamma\to 0}{\Rightarrow} \mathrm{Law}\left((Z^0_u)_{u\in\mathbb{R}}\right), \quad \alpha < 1/2.$$

Thank you
for your attention!