# Large deviations and moderate large deviations for general renewal processes

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Let

$$\{(\tau_i, \xi_i);\ i = 1, 2, \cdots\}$$

be a sequence of i.i.d. random vectors,

$$\mathbf{P}(\tau_1 > 0) = 1.$$

Put $for \quad n \geq 1$

$$T_n := \tau_1 + \cdots + \tau_n, \quad S_n := \xi_1 + \cdots + \xi_n.$$

We study large deviations (LD) and moderate large deviations (MLD) for the renewal process

$$Z(t) := S_{\eta(t)}, \quad t \geq 0,$$

where

$$\eta(t) := \min\{m \geq 0 : \ T_{m+1} \geq t\}.$$

$[\mathbf{C}_0]$. (Cramér condition) *For some* $\delta > 0$

$$\mathbf{E}e^{\delta(\tau+|\xi|)} < \infty.$$

Put

$$A(\mu,\nu) := \ln \mathbf{E} e^{\mu\tau+\nu\xi},$$

$$\mathcal{A}_{\le 0} := \{(\mu,\nu) \in \mathbb{R}^2 : \quad A(\mu,\nu) \le 0\}.$$

*The second deviation (rate) function* for a vector $(\tau,\xi)$:

$$D(u,\alpha) := \sup_{(\mu,\nu)\in\mathcal{A}_{\le 0}} \{\mu u + \nu\alpha\}.$$

See in [1] (Borovkov A.A., Mogulskii A.A. Siberian Math. J.—1996).

The deviation (rate) function for general renewal process $Z(t)$:

$$G(\alpha) := \inf_{0\le u\le 1} \{D(1-u,\alpha)+u\lambda_{\tau+}\}, \quad \alpha \in \mathbb{R},$$

where $\lambda_{\tau+} := \sup\{\lambda : \ \mathbf{E}e^{\lambda\tau} < \infty\}$.

See [2] (Mogulskii A.A., to appear),

[3] (Borovkov A.A., to appear)

Two conditions:
$(I)$.

$$\ln \mathbf{P}(\tau \geq t) \sim \lambda_{\mathcal{T}+}, \quad t \to \infty;$$

$(II)$.

$$\lambda_{\mathcal{T}+} \geq \mu_{\mathcal{T}+}$$

where

$$\mu_{\mathcal{T}+} := \sup_{(\mu,\nu)\in\mathcal{A}_{\leq 0}} \mu.$$

**Theorem 1.** (Local LD for $Z(T)$)
*Under $(I)$ or $(II)$ we have*

$$\lim_{\varepsilon\to 0} \lim_{T\to\infty} \frac{1}{T} \ln \mathbf{P}(|\frac{1}{T}Z(T) - \alpha| < \varepsilon) = -G(\alpha).$$

**Corollary 1.** *For any Borel set $B \subset \mathbb{R}$ we have*

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \ln \mathbf{P}\left(\frac{1}{T} Z(T) \in B\right) \le - \inf_{\alpha\in[B]} G(\alpha),$$

$$\underline{\lim_{T\to\infty}} \frac{1}{T} \ln \mathbf{P}\left(\frac{1}{T} Z(T) \in B\right) \ge - \inf_{\alpha\in(B)} G(\alpha),$$

*where $[B]$, $(B)$ is the closure, the interior of a set $B$, respectively.*

In the domain of moderate large deviations similar results for $Z(T)$ are obtained.

In the lattice case $\mathbf{P}((\tau, \xi) \in \mathbb{Z}^2) = 1$, the sharp asymptotics of large, moderate large and normal deviation probabilities for $Z(n)$ has been studed:

$$\mathbf{P}(Z(n) = k) \sim ?$$

$[\overline{\mathbf{C}}_\mathbf{0}]$.

$$\lambda_{\tau+} > \mu_{\tau+} \quad and \quad \mathcal{A}_{\leq 0} \subset (\mathcal{A}_{<\infty}),$$

*where* $\mathcal{A}_{<\infty} := \{(\mu, \nu) : \ A(\mu, \nu) < \infty\}$, *and* $(\mathcal{A}_{<\infty})$ *is the interior of* $\mathcal{A}_{<\infty}$.

**Theorem 2.** *Under condition* $[\overline{\mathbf{C_0}}]$ *we have*

$$\mathbf{P}(Z(n) = k) \sim \frac{C(\alpha)}{\sqrt{n}} e^{-nG(\frac{k}{n})},$$

*where* $k = k_n \in \mathbb{Z}$, $\frac{k}{n} \to \alpha$ *as* $n \to \infty$ *and the function* $C(\alpha)$ *is known in an explicit form.*

*If*

$$|k - an| = o(n^{2/3}), \quad as \quad n \to \infty,$$

*then*

$$\mathbf{P}(Z(n) = k) \sim \frac{1}{\sqrt{2\pi n \sigma}} e^{-\frac{(k-an)^2}{2n\sigma^2}},$$

*where* $a := \frac{\mathbf{E}\xi}{\mathbf{E}\tau}$, $\sigma^2 := \frac{\mathbf{E}(\xi - a\tau)^2}{\mathbf{E}\tau}$.

Put

$$z_T(t) := \frac{1}{x} Z(tT),\ 0 \le t \le 1,$$

where a function $x = x_T > 0$ is such that $x \sim T$ as $T \to \infty$. Large deviation principle (extended ) for $\{z_T(\cdot);\ T > 0\}$ was obtained:

$$\ln \mathbf{P}(z_T(\cdot) \in B) \sim -T \inf_{f \in B} I(f).$$

The definition of the extended LDP see in [4] (Borovkov A.A., Mogulskii A.A. Siberian Math. J.—2010).

Let $\mathbb{V}$ be the metric space of function $f = f(t);\ 0 \le t \le 1,\ f(0) = 0$, with finite variation $\mathrm{Var}(f) < \infty$. Let $\rho = \rho(f, g)$ be the metric $\rho_{\mathbb{V}} = \rho_{\mathbb{V}}(f, g)$ (see [5] Borovkov A.A., Mogulskii A.A. Theory Probab. Appli. (2011-2013)).

Denote

$$g_{\pm} := \lim_{\alpha \to \infty} \frac{G(\pm\alpha)}{\alpha}.$$

For

$$f = f_a + f_s^+ - f_s^- \in \mathbb{V}, \quad f_s^+(0) = f_s^-(0) = 0$$

put

$$I(f) := \int_0^1 G(f_a'(t))dt + g_+ f_s^+(1) + g_- f_s^-(1).$$

The properties of $I(f)$ see in

[6] (Borovkov A.A., Mogulskii A.A., Siberian Math. J.—2011) and

[7] (Mogulskii A.A., Siberian Adv. Math.—2012.)

**Theorem 3.** (Local LDP for $\zeta_T(\cdot)$) *If*

$$\lambda_{\tau+} \geq \mu_{\tau+},$$

*then for any* $f \in \mathbb{V}$

$$\lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \ln \mathbf{P}(z_T(\cdot) \in (f)_\varepsilon) = -I(f),$$

*where*

$$(f)_\varepsilon := \{g \in \mathbb{V} : \ \rho(f, g) < \varepsilon\}.$$

For a Borel set $B \subset \mathbb{V}$ put

$$I(B) := \inf_{f \in B} I(f), \quad I(B+) := \lim_{\varepsilon \to 0} I((B)_\varepsilon).$$

**Theorem 4.** (Extended LDP for $\zeta_T(\cdot)$)
*If*

$$\lambda_{\tau+} \geq \mu_{\tau+},$$

*then*

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \ln \mathbf{P}(z_T(\cdot) \in B) \leq -I(B+);$$

$$\underline{\lim_{T\to\infty}} \frac{1}{T} \ln \mathbf{P}(z_T(\cdot) \in B) \geq -I((B)).$$

In the domain of moderate latge deviations similar results for $z_T(\cdot)$ are obtained.

# References


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[7] *Mogulskii A. A.* The composite theorem for deviation integral// Siberian Adv. Math.—2012.—Vol 15.—P.127—145.