Lower Tail Probabilities and Normal Comparison Inequalities

In Memory of Wenbo V. Li's Contributions

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Outline

- Lower tail probabilities
    - Non-stationary processes
    - Stationary processes
- Special cases
    - Capture time of Brownian pursuits
    - Random polynomials
- Normal comparison inequalities

1. Lower Tail Probabilities

Let $\{X_t,\ t \in T\}$ be a real valued Gaussian process indexed by T with $\mathbb{E}\, X_t = 0$. Lower tail probability refers to

$$\mathbb{P}\Big(\sup_{t \in T} (X_t - X_{t_0}) \le x \Big) \ \text{as}\ \ x \to 0,\ t_0 \in T$$

or

$$\mathbb{P}\Big(\sup_{t \in T} X_t \le x \Big) \ \text{as}\ |T| \to \infty$$

- Examples:
  - (a) Capture time of Brownian pursuits
  - (b) Random polynomials

▶ A general result for non-stationary Gaussian processes

Let $X = \{X_t,\ t \in T\}$ be a real valued Gaussian random process indexed by T with mean zero. Define the L^2-metric

$$d(s,t) = (\mathbb{E}\,|X_s - X_t|^2)^{1/2}, \ \ s,t \in T.$$

For every $\varepsilon > 0$ and a subset A of T, let $N(A,\varepsilon)$ denote the minimal number of open balls of radius ε for the metric d that are necessary to cover A. For $t \in T$ and $h > 0$, let

$$B(t,h) = \{s \in T : d(t,s) \le h\}$$

and define

$$Q = \sup_{h>0} \sup_{t \in T} \int_0^\infty (\ln N(B(t,h),\varepsilon h))^{1/2} d\boldsymbol{e}$$

For $\theta = 1000(1+Q)$, define

$$\begin{aligned} \mathcal{A}_{-1} &= \{t \in T : d(t, t_0) \le \theta^{-1} x\}, \\ \mathcal{A}_k &= \{t \in T : \theta^{k-1} x < d(t, t_0) \le \theta^k x\}, \end{aligned}$$

where $0 \le k \le L$, $L = 1 + [\ln_\theta(D/x)]$ and $D = \sup_{t \in T} d(t, t_0)$.
Let

$$\begin{aligned} N_k(x) &= N(\mathcal{A}_k,\ \theta^{k-2} x) \ \text{ for } \ k = 0, 1, \cdots, L \\ N(x) &= 1 + \sum_{0 \le k \le L} N_k(x). \end{aligned}$$

Theorem

(Li and Shao (2004))

(i) *Assume that* $Q < \infty$ *and*

$$\mathbb{E}\left((X_s - X_{t_0})(X_t - X_{t_0})\right) \geq 0 \ \ for \ \ s,t \in T$$

Then

$$\mathbb{P}\left(\sup_{t \in T} X_t - X_{t_0} \leq x\right) \geq e^{-N(x)}$$

(ii) *For* $x > 0$, *let* $s_i \in T$, $i = 1, ..., M$ *be a sequence such that for every* i

$$\sum_{j=1}^{M} |Corr(X_{s_i} - X_{t_0}, X_{s_j} - X_{t_0})| \leq 5/4$$

and $d(s_i, t_0) = (\mathbb{E}\,|X_{s_i} - X_{t_0}|^2)^{1/2} \geq x/2$. *Then*

$$\mathbb{P}\left(\sup_{t \in T} X_t - X_{t_0} \leq x\right) \leq e^{-M/10}.$$

Special Cases:

- *Let* $\{X(t), t \in [0,1]^d\}$ *be a centered Gaussian process with* $X(0) = 0$ *and stationary increments, that is*

$$\forall\, t, s \in [0,1]^d, \;\; \mathbb{E}\,(X_t - X_s)^2 = \sigma^2(||t-s||).$$

If there are $0 < \alpha \le \beta < 1$ *such that*

$$\sigma(h)/h^{\alpha} \uparrow, \;\; \sigma(h)/h^{\beta} \downarrow$$

Then there exist $0 < c_1 \le c_2 < \infty$ *such that for* $0 < x < 1/2$

$$-c_2 \ln \frac{1}{x} \le \ln \mathbb{P}\Big(\sup_{t \in [0,1]^d} X(t) \le \sigma(x) \Big) \le -c_1 \ln \frac{1}{x}.$$

In particular, for the fractional Levy's Brownian motion $L_\alpha(t)$ of order α, i.e. $L_\alpha(0) = 0$ and

$$\mathbb{E}\,(L_\alpha(t) - L_\alpha(s))^2 = ||t-s||^{\alpha},$$

$$\ln \mathbb{P}\Big(\sup_{t \in [0,1]^d} L_\alpha(t) \le x \Big) \approx -\ln \frac{1}{x}.$$

- Let $\{X(t), t \in [0,1]^d\}$ be a centered Gaussian process with $X(0) = 0$ and

$$\mathbb{E}\left(X_t X_s\right) = \prod_{i=1}^{d} \frac{1}{2}(\sigma^2(t_i) + \sigma^2(s_i) - \sigma^2(|t_i - s_i|)).$$

If there are $0 < \alpha \leq \beta < 1$ such that

$$\sigma(h)/h^\alpha \uparrow, \ \ \sigma(h)/h^\beta \downarrow$$

Then

$$\ln \mathbb{P}\left(\sup_{t \in [0,1]^d} X(t) \leq \sigma^d(x) \right) \approx -\ln^d \frac{1}{x}.$$

In particular, for d-dimensional fractional Brownian sheet $B_\alpha(t)$ (i.e., $\sigma(h) = h^\alpha$)

$$\ln \mathbb{P}\left(\sup_{t \in [0,1]^d} B_\alpha(t) \leq x \right) \approx -\ln^d \frac{1}{x}$$

▶ Lower tail probabilities for stationary Gaussian processes

Let $\{B(t), t \geq 0\}$ be the Brownian motion and $\{U(t), t \geq 0\}$ be the Ornstein-Uhlenbeck process. It is known that $\{U(t), t \geq 0\}$ and $\{B(e^t)/e^{t/2}, t \geq 0\}$ have the same distribution. Moreover

$$\mathbb{P}\Big(\sup_{0 \leq t \leq 1} B(t) \leq x\Big) = \mathbb{P}\Big(|B(1)| \leq x\Big) \sim (2/\pi)^{1/2} x$$

as $x \to 0$ and

$$\mathbb{P}\Big(\sup_{0 \leq t \leq T} U(t) \leq 0\Big) = \exp(-T/2 + o(T))$$

as $T \to \infty$.

Is there a connection between these two types of lower tail probabilities ?

Let $\{X_t, t \geq 0\}$ be a Gaussian process with $X_0 = 0$, $\mathbb{E}\, X_t = 0$. Assume that

(A1) $\mathbb{E}\, X_s X_t \geq 0$ and $\mathbb{E}\, X_t^2 = t^\alpha$ for $\alpha > 0$;

(A2) $\{Y_t = X(e^t)/e^{\alpha/2}, t \geq 0\}$ is a stationary Gaussian process;

(A3) $\{X_{at}, 0 \leq t \leq 1\}$ and $\{a^{\alpha/2} X_t, 0 \leq t \leq 1\}$ have the same distribution for each fixed $a > 0$.

(A4) $\rho(t) := \mathbb{E}\, Y_t Y_0$ is decreasing and

$$a_{h,\theta}^2 := \inf_{0<t\leq h} \frac{\rho(\theta t) - \rho(t)}{1 - \rho(t)} > 0$$

for every $0 < h < \infty$ and $0 < \theta < 1$

By subadditivity and the Slepian lemma,

$$c_{\alpha,Y} := -\lim_{T\to\infty}\frac{1}{T}\ln\mathbb{P}\Big(\sup_{0\le t\le T} Y_t \le 0\Big) = -\sup_{T>0}\frac{1}{T}\mathbb{P}\Big(\sup_{0\le t\le T} Y_t \le 0\Big)$$

exists. Next result shows that the constant c is closely related to the rate of the lower tail probability $\mathbb{P}\Big(\sup_{0\le t\le 1} X_t \le x\Big)$.

- Li and Shao (2004):
  Under conditions $(A1) - A(4)$, we have

  $$P(\sup_{0\le t\le 1} X_t \le x) = x^{2c_{\alpha,Y}/\alpha + o(1)}$$

  as $x \to 0$.
- Molchan (1999): For fractional Brownian motion $\{B_\alpha(t), t\ge 0\}$ of order α $(0<\alpha<1)$

  $$P(\sup_{0\le t\le 1} B_\alpha(t) \le x) = x^{(1-\alpha)/\alpha + o(1)}$$

  and hence $c_\alpha = 1-\alpha$.

▶ Explicit bounds of lower tail probabilities (Li and Xiao (2013))

Let $B(t)$ be the Brownian motion.

- For $0 < \theta < 1$

$$P\big(\sup_{0\le t\le 1}(B(t) - \theta B(1)) \le x\big) \sim \frac{1}{3\theta^2(1-\theta)^2\sqrt{2\pi}}x^3,$$

$$P\big(\sup_{0\le t\le 1}(B(t) - \theta t B(1)) \le x\big) \sim (1-\theta)\sqrt{2/\pi}\,x$$

and

$$\ln P\big(\sup_{0\le t\le 1} B(t) - \int_0^1 B(s)ds \le x\big) \sim -x^2\,c$$

where c is a specified constant.

- Capture time of Brownian pursuits

Let $B_0, B_1, \cdots, B_n$ be independent standard Brownian motions. Define

$$\tau_n = \inf\left\{t > 0 : \max_{1\le k\le n} B_k(t) = B_0(t) + 1\right\}.$$

When is $\mathbb{E}(\tau_n)$ finite?

Note that for any $a > 0$, by Brownian scaling,

$$\begin{aligned} &\mathbb{P}(\tau_n > t) \\ &= \mathbb{P}\Big(\max_{1\le k\le n}\sup_{0\le s\le t}(B_k(s) - B_0(s)) < 1\Big) \\ &= \mathbb{P}\Big(\max_{1\le k\le n}\sup_{0\le s\le 1}(B_k(s) - B_0(s)) < t^{-1/2}\Big). \end{aligned}$$

Thus, the estimate is reduced to a lower tail probability problem.

- DeBlassie (1987):

$$\mathbb{P}\{\tau_n > t\} \sim ct^{-\gamma_n} \quad \text{as} \quad t \to \infty.$$

- Bramson and Griffeath (1991): $\mathbb{E}\,\tau_3 = \infty$
- Li and Shao (2001): $\mathbb{E}\,\tau_5 < \infty$.
- Ratzkin and Treibergs (2009): $\mathbb{E}\,\tau_4 < \infty$.

What is the asymptotic behavior of γ_n?

- Kesten (1992):

$$0 < \liminf_{n\to\infty} \gamma_n / \ln n \le \limsup_{n\to\infty} \gamma_n / \ln n \le 1/4$$

Conjecture: $\lim_{n\to\infty} \gamma_n / \ln n$ exists.

- Li and Shao (2002):

$$\lim_{n\to\infty} \gamma_n / \ln n = 1/4$$

Li and Shao (2002) also consider the capture time of the fractional Brownian motion pursuit. Let $\{B_{k,\alpha}(t);\ t \geq 0\}(k = 0, 1, 2, \ldots, n)$ be independent fractional Brownian motions of order $\alpha \in (0, 1)$. Put

$$\tau_n := \tau_{n,\alpha} = \inf \left\{ t > 0 : \max_{1 \leq k \leq n} B_{k,\alpha}(t) = B_{0,\alpha}(t) + 1 \right\}.$$

Let

$$X_{k,\alpha}(t) = e^{-t\alpha} B_{k,\alpha}(e^t),\ k = 0, 1, \cdots, n$$

and

$$\gamma_{n,\alpha} := -\lim_{T\to\infty} \frac{1}{T} \ln \mathbb{P}\Big(\sup_{0\leq t\leq T} \max_{1\leq k\leq n} X_{k,\alpha}(t) \leq 0 \Big)$$

Li and Shao (2002):

$$\frac{1}{d_\alpha} \leq \liminf_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} \leq \limsup_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} < \infty,$$

where $d_\alpha = 2\int_0^\infty (e^{2\alpha x} + e^{-2\alpha x} - (e^x - e^{-x})^{2\alpha})dx$.

Conjecture:

$$\lim_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} = \frac{1}{d_\alpha}.$$

▶ The probability that a random polynomial has no real root

Dembo, Poonen, Shao and Zeitouni (2002)

$$\mathbb{P}\Big(\sum_{i=0}^{n} Z_i x^i < 0 \;\forall\, x \in \mathbb{R}^1\Big) = n^{-b+o(1)}$$

where n is even, Z_i are i.i.d. $N(0,1)$, and

$$b = -4 \lim_{T\to\infty} \frac{1}{T} \ln \mathbb{P}\Big(\sup_{0\le t\le T} X_t \le 0\Big)$$

where X_t is a centered stationary Gaussian process with

$$\mathbb{E}\, X_s X_t = \frac{2e^{-|t-s|/2}}{1+e^{-|t-s|}}$$

- Dembo, Poonen, Shao and Zeitouni (2002): $0.4 < b < 1.29$.
  Simulation: $b = 0.76 \pm 0.03$
- Li and Shao (2002): $0.5 < b < 1$

Let $\{B(t), t \geq 0\}$ be the Brownian motion and put $B_0(t) = B(t)$,

$$B_m(t) = \int_0^t B_{m-1}(s)ds$$

Li and Shao (2003+)

$$P(\sup_{0 \leq t \leq 1} B_m(t) \leq x) = x^{r_m + o(1)}$$

and

$$b \leq 2r_m(2m+1), \;\; 2r_m(2m+1) \to b$$

Open questions:

- What is the value of b?
- If $\{X_t, t \geq 0\}$ is a differentiable stationary Gaussian process with positive correlation, what is the limit

$$\lim_{T \to \infty} \frac{1}{T} \ln P\Big(\sup_{0 \leq t \leq T} X_t \leq 0\Big) \; ?$$

Let $n \geq 2$, and let $(\xi_j, 1 \leq j \leq n)$ be standard normal random variables with correlation matrix $R^1 = (r_{ij}^1)$, let $(\eta_j, 1 \leq j \leq n)$ be standard normal random variables with correlation matrix $R^0 = (r_{ij}^0)$. Set $\rho_{ij} = \max(|r_{ij}^1|, |r_{ij}^0|)$.

Slepian's Lemma: If $r_{ij}^1 \geq r_{ij}^0$, then

$$\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \leq u_j\}\Big) \geq \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \leq u_j\}\Big)$$

Normal comparison inequality:

- Berman (1964,1971), Cramer and Leadbetter (1967)):

$$\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \le u_j\}\Big) - \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \le u_j\}\Big)$$
$$\le \frac{1}{2\pi}\sum_{1\le i<j\le n}(r_{ij}^{1}-r_{ij}^{0})^{+}(1-\rho_{ij}^{2})^{-1/2}\exp\Big(-\frac{u_i^2+u_j^2}{2(1+\rho_{ij})}\Big)$$

- Li and Shao (2002):

$$\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \le u_j\}\Big) - \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \le u_j\}\Big)$$
$$\le \frac{1}{4}\sum_{1\le i<j\le n}(r_{ij}^{1}-r_{ij}^{0})^{+}\exp\Big(-\frac{u_i^2+u_j^2}{2(1+\rho_{ij})}\Big)$$

- Li and Shao (2002): If

$$r_{ij}^1 \geq r_{ij}^0 \geq 0 \ \text{ for all } \ 1 \leq i,j \leq n$$

Then

$$\begin{aligned}
&\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \leq u_j\}\Big) \\
\leq \ & \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \leq u_j\}\Big) \leq \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \leq u_j\}\Big) \\
& \exp\Big\{ \sum_{1\leq i<j\leq n} \ln\Big(\frac{\pi - 2\arcsin(r_{ij}^0)}{\pi - 2\arcsin(r_{ij}^1)}\Big) \exp\Big(- \frac{(u_i^2 + u_j^2)}{2(1 + r_{ij}^1)}\Big)\Big\}
\end{aligned}$$

for any $u_i \geq 0, i = 1, 2, \cdots, n$ satisfying

$$(r_{ki}^l - r_{ij}^l r_{kj}^l)u_i + (r_{kj}^l - r_{ij}^l r_{ki}^l)u_j \geq 0 \qquad (*)$$

for $l = 0, 1$ and for all $1 \leq i, j, k \leq n$.

- Note: Condition (**) is satisfied if $u_i = u \geq 0$.

Open questions:

- Does the result remain valid without assuming (*)? Yan (2009) gave a partial answer.
- Can a sharper bound be obtained?

Thank you!!

Spaseeba!!!!