Limit theorems for the measure of level sets of Gaussian random fields

Alexey Shashkin

MSU

Saint Petersburg
2013

Outline

Excursion sets and level sets

Let $X = \{X(s), s \in \mathbb{R}^d\}$ be a continuous random field.
Definition. *An excursion set* of a random field X at level $u \in \mathbb{R}$ is a random set

$$A_u = \{s \in \mathbb{R}^d : X(s) \geq u\}.$$

The level set of X determined by a level $u \in \mathbb{R}$ is the random set

$$B_u = \{s \in \mathbb{R}^d : X(s) = u\}.$$

Problem setup

Let $T \subset \mathbb{R}^d$ be a bounded observation window. Consider a bounded random set $A_u(X) \cap T$ (or $B_u(X) \cap T$). What can be said of the behavior of its geometric characteristics when T grows to infinity?

For example, if $T = [0, t]^d$, and $\mathcal{H}_d^k(B)$ is the k-dimensional Hausdorff measure of $B \subset \mathbb{R}^d$, then

$$V_t(u) = \mathcal{H}_d^d(A_u(X) \cap T) = \int_T \mathbb{I}\{X(s) \geq u\} ds \text{ is the volume of excursion set,}$$

$$N_t(u) = \mathcal{H}_d^{d-1}(B_u(X) \cap T) = \mathcal{H}_d^{d-1}\{s \in [0, t]^d : X(s) = u\} \text{ is the area of the level set.}$$

In what follows, all processes and fields (generating level and excursion sets) are Gaussian, with mean zero and variance one. The covariance function of a process or field X is denoted by R. The density of a random variable or vector η is p_η.

Problem setup

Let $T \subset \mathbb{R}^d$ be a bounded observation window. Consider a bounded random set $A_u(X) \cap T$ (or $B_u(X) \cap T$). What can be said of the behavior of its geometric characteristics when T grows to infinity?

For example, if $T = [0, t]^d$, and $\mathcal{H}_d^k(B)$ is the k-dimensional Hausdorff measure of $B \subset \mathbb{R}^d$, then

$$V_t(u) = \mathcal{H}_d^d(A_u(X) \cap T) = \int_T \mathbb{I}\{X(s) \geq u\} ds \text{ is the volume of excursion set,}$$

$$N_t(u) = \mathcal{H}_d^{d-1}(B_u(X) \cap T) = \mathcal{H}_d^{d-1}\{s \in [0, t]^d : X(s) = u\} \text{ is the area of the level set.}$$

In what follows, all processes and fields (generating level and excursion sets) are Gaussian, with mean zero and variance one. The covariance function of a process or field X is denoted by R. The density of a random variable or vector η is p_η.

Problem setup

Let $T \subset \mathbb{R}^d$ be a bounded observation window. Consider a bounded random set $A_u(X) \cap T$ (or $B_u(X) \cap T$). What can be said of the behavior of its geometric characteristics when T grows to infinity?

For example, if $T = [0, t]^d$, and $\mathcal{H}_d^k(B)$ is the k-dimensional Hausdorff measure of $B \subset \mathbb{R}^d$, then

$$V_t(u) = \mathcal{H}_d^d(A_u(X) \cap T) = \int_T \mathbb{I}\{X(s) \geq u\} ds \text{ is the volume of excursion set,}$$

$$N_t(u) = \mathcal{H}_d^{d-1}(B_u(X) \cap T) = \mathcal{H}_d^{d-1}\{s \in [0, t]^d : X(s) = u\} \text{ is the area of the level set.}$$

In what follows, all processes and fields (generating level and excursion sets) are Gaussian, with mean zero and variance one. The covariance function of a process or field X is denoted by R. The density of a random variable or vector η is p_η.

Rice formula

- Rice, 1945: if a process X is C^1 and $N_t(u) = \mathcal{H}_1^0\{s \in [0,t] :\ X(s) = u\}$, then

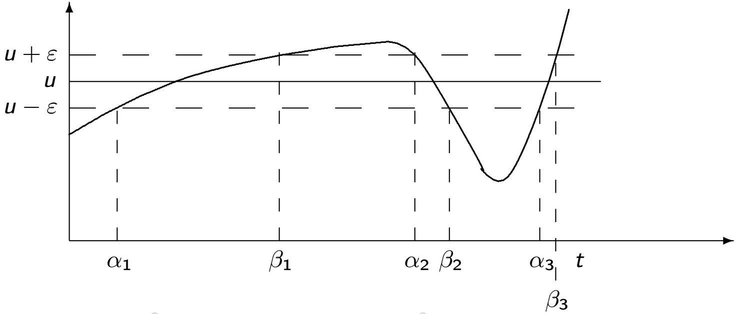
$$\mathsf{E}N_t(u) = \int_0^t \mathsf{E}\Big(|X'(s)|\Big|X(s) = u\Big) p_{X(s)}(u)ds.$$

For a stationary process this reduces to

$$\mathsf{E}N_t(u) = t\mathsf{E}|X'(0)|p_{X(0)}(u) = te^{-u^2/2}\frac{\sqrt{\mathrm{Var}X'(0)}}{\pi}.$$

Rice formula

Assume that there are no points s with $X(s) = u, X'(s) = 0$, and also that $X(0) \neq u, X(t) \neq u$ (this holds a.s.). Then, for $\varepsilon > 0$ small enough,



$$N_t(u) = \sum_{i=1}^{3} \frac{1}{2\varepsilon} |X(\beta_i) - X(\alpha_i)| = \sum_{i=1}^{3} \frac{1}{2\varepsilon} (X(\beta_i) - X(\alpha_i)) \operatorname{sgn}(X(\beta_i) - X(\alpha_i))$$

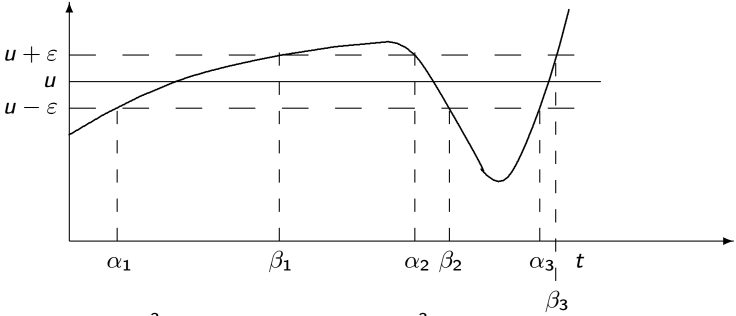
$$= \sum_{i=1}^{3} \frac{1}{2\varepsilon} \Big(\int_{\alpha_i}^{\beta_i} X'(s) ds \Big) \operatorname{sgn}(X'(s),\ s \in [\alpha_i, \beta_i]) = \sum_{i=1}^{3} \frac{1}{2\varepsilon} \int_{\alpha_i}^{\beta_i} |X'(s)| ds.$$

Hence

$$N_t(u) = \lim_{\varepsilon \to 0} \int_0^t |X'(s)| \frac{\mathbb{I}\{|X(s) - u| \leq \varepsilon\}}{2\varepsilon} ds.$$

Rice formula

Assume that there are no points s with $X(s) = u, X'(s) = 0$, and also that $X(0) \neq u, X(t) \neq u$ (this holds a.s.). Then, for $\varepsilon > 0$ small enough,



$$N_t(u) = \sum_{i=1}^{3} \frac{1}{2\varepsilon} |X(\beta_i) - X(\alpha_i)| = \sum_{i=1}^{3} \frac{1}{2\varepsilon} (X(\beta_i) - X(\alpha_i)) \operatorname{sgn}(X(\beta_i) - X(\alpha_i))$$

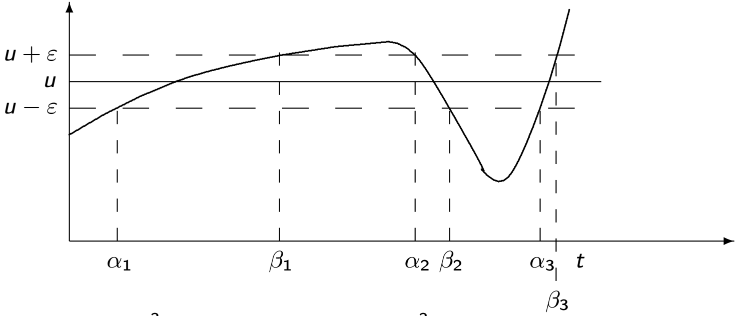
$$= \sum_{i=1}^{3} \frac{1}{2\varepsilon} \left(\int_{\alpha_i}^{\beta_i} X'(s) ds \right) \operatorname{sgn}(X'(s),\ s \in [\alpha_i, \beta_i]) = \sum_{i=1}^{3} \frac{1}{2\varepsilon} \int_{\alpha_i}^{\beta_i} |X'(s)| ds.$$

Hence

$$N_t(u) = \lim_{\varepsilon \to 0} \int_0^t |X'(s)| \frac{\mathbb{I}\{|X(s) - u| \leq \varepsilon\}}{2\varepsilon} ds.$$

Rice formula

Assume that there are no points s with $X(s) = u, X'(s) = 0$, and also that $X(0) \neq u, X(t) \neq u$ (this holds a.s.). Then, for $\varepsilon > 0$ small enough,



$$N_t(u) = \sum_{i=1}^{3} \frac{1}{2\varepsilon}|X(\beta_i) - X(\alpha_i)| = \sum_{i=1}^{3} \frac{1}{2\varepsilon}(X(\beta_i) - X(\alpha_i))\mathrm{sgn}(X(\beta_i) - X(\alpha_i))$$

$$= \sum_{i=1}^{3} \frac{1}{2\varepsilon}\Big(\int_{\alpha_i}^{\beta_i} X'(s)ds\Big)\mathrm{sgn}(X'(s),\ s \in [\alpha_i, \beta_i]) = \sum_{i=1}^{3} \frac{1}{2\varepsilon}\int_{\alpha_i}^{\beta_i} |X'(s)|ds.$$

Hence

$$N_t(u) = \lim_{\varepsilon \to 0} \int_0^t |X'(s)| \frac{\mathbb{I}\{|X(s) - u| \leq \varepsilon\}}{2\varepsilon} ds.$$

Higher moments

- Cramer and Leadbetter, 1967: if for any $s_1 \neq s_2$ the vector $(X(s_1), X(s_2))$ is nondegenerate, then

$$\mathsf{E}N_t(u)(N_t(u)-1) = \int_0^t \int_0^t \mathsf{E}\Big(|X'(s_1)X'(s_2)| \Big| X(s_1) = X(s_2) = u\Big) p_{X(s_1),X(s_2)}(u,u)ds.$$

  If a process is stationary and $L(t) = (R''(t) - R''(0))/t \in L^1([0,\delta], \mathrm{Leb})$ (*Geman condition*), then $\mathsf{E}N_t^2(u) < \infty$.
- Geman, 1972: the converse is true.
- Belyaev, 1967: moments of higher order, as well as conditions for their finiteness.

Higher moments

- Cramer and Leadbetter, 1967: if for any $s_1 \neq s_2$ the vector $(X(s_1), X(s_2))$ is nondegenerate, then

$$\mathsf{E}N_t(u)(N_t(u)-1) = \int_0^t \int_0^t \mathsf{E}\Big(|X'(s_1)X'(s_2)|\Big|X(s_1) = X(s_2) = u\Big) p_{X(s_1),X(s_2)}(u,u)ds.$$

  If a process is stationary and $L(t) = (R''(t) - R''(0))/t \in L^1([0,\delta], \mathrm{Leb})$ (*Geman condition*), then $\mathsf{E}N_t^2(u) < \infty$.

- Geman, 1972: the converse is true.
- Belyaev, 1967: moments of higher order, as well as conditions for their finiteness.

Higher moments

- Cramer and Leadbetter, 1967: if for any $s_1 \neq s_2$ the vector $(X(s_1), X(s_2))$ is nondegenerate, then

$$\mathsf{E}N_t(u)(N_t(u)-1) = \int_0^t \int_0^t \mathsf{E}\Big(|X'(s_1)X'(s_2)|\Big| X(s_1) = X(s_2) = u\Big) p_{X(s_1),X(s_2)}(u,u)ds.$$

  If a process is stationary and $L(t) = (R''(t) - R''(0))/t \in L^1([0,\delta], \mathrm{Leb})$ (*Geman condition*), then $\mathsf{E}N_t^2(u) < \infty$.
- Geman, 1972: the converse is true.
- Belyaev, 1967: moments of higher order, as well as conditions for their finiteness.

Random fields ($d > 1$)

Set $N_t(u) = \mathcal{H}_d^{d-1}\{s \in [0,t]^d : X(s) = u\}$.

- Wschebor, 1982; Ibragimov and Zaporozhets, 2010

$$\mathsf{E}N_t(u) = \int_{[0,t]^d} \mathsf{E}\Big(\|\nabla X(s)\| \Big| X(s) = u\Big) p_{X(s)}(u)\,ds,$$

$$\mathsf{E}N_t^2(u) = \int\limits_{[0,t]^d \times [0,t]^d} \mathsf{E}\Big(\|\nabla X(s_1)\| \|\nabla X(s_2)\| \Big| X(s_1) = X(s_2) = u\Big) p_{X(s_1),X(s_2)}(u,u)\,ds.$$

-

CLT: random processes

$X = \{X(s), s \in \mathbb{R}\}$ — stationary random process

- Malevich, 1969: Spectral density $f(\lambda) \searrow 0$ as $|\lambda| \to \infty$, $\int_{\mathbb{R}} (\lambda^4 f^2(\lambda) + f^3(\lambda) + \lambda^2 f(\lambda) \log(1+|\lambda|)^{1+a}) d\lambda < \infty$ $(a > 0)$, $\mathrm{Var} N_t(0)/t \to \sigma^2 \Rightarrow (N_t(0) - \mathrm{E} N_t(0))/\sqrt{t} \to N(0,\sigma^2)$, $t \to \infty$
- Cuzick, 1976: $R \in C^2 \cap L^2(\mathbb{R})$, $R'' \in L^2(\mathbb{R})$, $\mathrm{Var} N_t(0)/t \to \sigma^2 \Rightarrow (N_t(0) - \mathrm{E} N_t(0))/\sqrt{t} \to N(0,\sigma^2)$, $t \to \infty$
- Piterbarg, 1978 Geman condition, $|R| + |R''| \in L^1(\mathbb{R}) \Rightarrow \mathrm{Var} N_t(u) \sim \sigma^2(u) t$ and $(N_t(u) - \mathrm{E} N_t(u))/\sqrt{t} \to N(0,\sigma^2)$
- Slud, 1994 $\mathrm{Var} N_t(u) < \infty$, $R \in C^2 \cap L^1(\mathbb{R}) \Rightarrow (N_t(u) - \mathrm{E} N_t(u))/\sqrt{t} \to N(0,\sigma^2)$

CLT: random processes

$X = \{X(s), s \in \mathbb{R}\}$ — stationary random process

- Malevich, 1969: Spectral density $f(\lambda) \searrow 0$ as $|\lambda| \to \infty$,
  $\int_{\mathbb{R}} (\lambda^4 f^2(\lambda) + f^3(\lambda) + \lambda^2 f(\lambda) \log(1 + |\lambda|)^{1+a}) d\lambda < \infty$ $(a > 0)$,
  $\mathrm{Var} N_t(0)/t \to \sigma^2 \Rightarrow (N_t(0) - \mathrm{E} N_t(0))/\sqrt{t} \to N(0, \sigma^2),\ t \to \infty$

- Cuzick, 1976:
  $R \in C^2 \cap L^2(\mathbb{R}),\ R'' \in L^2(\mathbb{R})$,
  $\mathrm{Var} N_t(0)/t \to \sigma^2 \Rightarrow (N_t(0) - \mathrm{E} N_t(0))/\sqrt{t} \to N(0, \sigma^2),\ t \to \infty$

- Piterbarg, 1978
  Geman condition, $|R| + |R''| \in L^1(\mathbb{R}) \Rightarrow \mathrm{Var} N_t(u) \sim \sigma^2(u) t$ and
  $(N_t(u) - \mathrm{E} N_t(u))/\sqrt{t} \to N(0, \sigma^2)$

- Slud, 1994
  $\mathrm{Var} N_t(u) < \infty,\ R \in C^2 \cap L^1(\mathbb{R}) \Rightarrow (N_t(u) - \mathrm{E} N_t(u))/\sqrt{t} \to N(0, \sigma^2)$

CLT: random processes

$X = \{X(s), s \in \mathbb{R}\}$ — stationary random process

- Malevich, 1969: Spectral density $f(\lambda) \searrow 0$ as $|\lambda| \to \infty$,
  $\int_{\mathbb{R}} (\lambda^4 f^2(\lambda) + f^3(\lambda) + \lambda^2 f(\lambda) \log(1+|\lambda|)^{1+a}) d\lambda < \infty$ $(a > 0)$,
  $\mathrm{Var} N_t(0)/t \to \sigma^2 \Rightarrow (N_t(0) - \mathrm{E}N_t(0))/\sqrt{t} \to N(0,\sigma^2), \ t \to \infty$

- Cuzick, 1976:
  $R \in C^2 \cap L^2(\mathbb{R})$, $R'' \in L^2(\mathbb{R})$,
  $\mathrm{Var} N_t(0)/t \to \sigma^2 \Rightarrow (N_t(0) - \mathrm{E}N_t(0))/\sqrt{t} \to N(0,\sigma^2), \ t \to \infty$

- Piterbarg, 1978
  Geman condition, $|R| + |R''| \in L^1(\mathbb{R}) \Rightarrow \mathrm{Var} N_t(u) \sim \sigma^2(u) t$ and
  $(N_t(u) - \mathrm{E}N_t(u))/\sqrt{t} \to N(0,\sigma^2)$

- Slud, 1994
  $\mathrm{Var} N_t(u) < \infty$, $R \in C^2 \cap L^1(\mathbb{R}) \Rightarrow (N_t(u) - \mathrm{E}N_t(u))/\sqrt{t} \to N(0,\sigma^2)$

CLT: random processes

$X = \{X(s), s \in \mathbb{R}\}$ — stationary random process

- Malevich, 1969: Spectral density $f(\lambda) \searrow 0$ as $|\lambda| \to \infty$,
  $\int_{\mathbb{R}} (\lambda^4 f^2(\lambda) + f^3(\lambda) + \lambda^2 f(\lambda) \log(1+|\lambda|)^{1+a}) d\lambda < \infty$ $(a > 0)$,
  $\mathrm{Var} N_t(0)/t \to \sigma^2 \Rightarrow (N_t(0) - \mathrm{E} N_t(0))/\sqrt{t} \to N(0, \sigma^2), \ t \to \infty$

- Cuzick, 1976:
  $R \in C^2 \cap L^2(\mathbb{R})$, $R'' \in L^2(\mathbb{R})$,
  $\mathrm{Var} N_t(0)/t \to \sigma^2 \Rightarrow (N_t(0) - \mathrm{E} N_t(0))/\sqrt{t} \to N(0, \sigma^2), \ t \to \infty$

- Piterbarg, 1978
  Geman condition, $|R| + |R''| \in L^1(\mathbb{R}) \Rightarrow \mathrm{Var} N_t(u) \sim \sigma^2(u) t$ and
  $(N_t(u) - \mathrm{E} N_t(u))/\sqrt{t} \to N(0, \sigma^2)$

- Slud, 1994
  $\mathrm{Var} N_t(u) < \infty$, $R \in C^2 \cap L^1(\mathbb{R}) \Rightarrow (N_t(u) - \mathrm{E} N_t(u))/\sqrt{t} \to N(0, \sigma^2)$

CLT: random fields

$X = \{X(s), s \in \mathbb{R}^2\}$ — stationary isotropic fields

- Kratz and Leon, 2001
  $R \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and $\partial R/\partial s_j \in L^2(\mathbb{R}^2)$, $j = 1, \dots, d$,
  $\Rightarrow (N_t(u) - \mathsf{E} N_t(u))/t \to N(0, \sigma^2)$, $t \to \infty$.

Non-Gaussian fields

- Iribarren, 1989
- Adler, Taylor, Samorodnitsky, 2010
- Bulinski, Spodarev, Timmermann, 2011

Functional limit theorems

Let γ be, as before, one of geometric functionals of an excursion set or level set determined by a level u.

Question number 1. It is possible to say something of the properties of the **random process** $\{\gamma(A_u(X) \cap T),\ u \in \mathbb{R}\}$ (resp. $\{\gamma(B_u(X) \cap T),\ u \in \mathbb{R}\}$)?

Question number 2. If this random process is an element of a good metric space (say $C(\mathbb{R})$), can one prove something about the asymptotics of its distribution, when T grows to infinity?

Let us start with the volume:

$$V_t(u) = \int_0^t \mathbb{I}\{X(s) \geq u\} ds, \quad Y_t(u) := t^{-1/2}(V_t(u) - \mathsf{E} V_t(u)).$$

Functional limit theorems

Let γ be, as before, one of geometric functionals of an excursion set or level set determined by a level u.

Question number 1. It is possible to say something of the properties of the **random process** $\{\gamma(A_u(X) \cap T),\ u \in \mathbb{R}\}$ (resp. $\{\gamma(B_u(X) \cap T),\ u \in \mathbb{R}\}$)?

Question number 2. If this random process is an element of a good metric space (say $C(\mathbb{R})$), can one prove something about the asymptotics of its distribution, when T grows to infinity?

Let us start with the volume:

$$V_t(u) = \int_0^t \mathbb{I}\{X(s) \geq u\} ds, \quad Y_t(u) := t^{-1/2}(V_t(u) - \mathsf{E} V_t(u)).$$

the process of volumes of excursion sets

Elizarov, 1984:
$X = \{X(t), t \in \mathbb{R}\}$ is stationary, $1 - R(t) \sim |t|^\alpha$ $(t \to 0)$ for some $0 < \alpha \leq 2$, $R \in L^1(\mathbb{R})$. Then the processes $\{Y_t(\cdot), t > 0\}$ converge in distribution in $C(\mathbb{R})$ to a centered Gaussian process.

Local times

A similar statement is true for the local times:

$$L_t(u) = \lim_{\delta \to 0} \frac{1}{2\delta}(V_t(u-\delta) - V_t(u+\delta)).$$

Theorem 2. If $\alpha \leq 1$, then the processes $\{t^{-1/2}(L_t(\cdot) - \mathsf{E}L(\cdot)), t > 0\}$ converge in distribution in $C(\mathbb{R})$ to a centered Gaussian process.

What can be said not for local time but for the level set area?

Local times

A similar statement is true for the local times:

$$L_t(u) = \lim_{\delta\to 0} \frac{1}{2\delta}(V_t(u-\delta) - V_t(u+\delta)).$$

Theorem 2. If $\alpha \le 1$, then the processes $\{t^{-1/2}(L_t(\cdot) - \mathsf{E}L(\cdot)), t > 0\}$ converge in distribution in $C(\mathbb{R})$ to a centered Gaussian process.
What can be said not for local time but for the level set area?

Assumptions

Let $d \geq 3$, the random field $X = \{X(s), s \in \mathbb{R}^d\}$ with C^1 realizations be stationary and isotropic,

$$N_t(u) = \mathcal{H}_d^{d-1}\{s \in [0,t]^d : X(s) = u\},\ Z_t(u) := t^{-d/2}(N_t(u) - \mathsf{E}N_t(u)).$$

We may and will always assume that

$$\mathsf{E}X(0) = 0,\ \ \mathrm{Var}X(0) = 1,\ \ \mathrm{Var}\frac{\partial X(0)}{\partial s_1} = -\frac{\partial^2 R(0)}{\partial s_1^2} = 1.$$

Assume also that

1) $\mathsf{P}(\mathcal{H}_{d-1}(\{s \in \mathbb{R}^d : \nabla X(s) = 0\}) > 0) = 0;$
2) $\mathsf{P}(X(s) = u, \nabla X(s) = 0 \text{ for all } s \in \mathbb{R}^d) = 0$ with any $u \in \mathbb{R}$.

Both last requirements are true, e.g., if the realizations of X are C^2 a.s.

Assumptions

Let $d \geq 3$, the random field $X = \{X(s), s \in \mathbb{R}^d\}$ with C^1 realizations be stationary and isotropic,

$$N_t(u) = \mathcal{H}_d^{d-1}\{s \in [0,t]^d : X(s) = u\}, \ Z_t(u) := t^{-d/2}(N_t(u) - \mathsf{E}N_t(u)).$$

We may and will always assume that

$$\mathsf{E}X(0) = 0, \ \ \mathrm{Var}X(0) = 1, \ \ \mathrm{Var}\frac{\partial X(0)}{\partial s_1} = -\frac{\partial^2 R(0)}{\partial s_1^2} = 1.$$

Assume also that

1) $\mathsf{P}(\mathcal{H}_{d-1}(\{s \in \mathbb{R}^d : \nabla X(s) = 0\}) > 0) = 0$;
2) $\mathsf{P}(X(s) = u, \nabla X(s) = 0 \text{ for all } s \in \mathbb{R}^d) = 0$ with any $u \in \mathbb{R}$.

Both last requirements are true, e.g., if the realizations of X are C^2 a.s.

Assumptions

Let $d \geq 3$, the random field $X = \{X(s), s \in \mathbb{R}^d\}$ with C^1 realizations be stationary and isotropic,

$$N_t(u) = \mathcal{H}_d^{d-1}\{s \in [0,t]^d : X(s) = u\},\ Z_t(u) := t^{-d/2}(N_t(u) - \mathsf{E}N_t(u)).$$

We may and will always assume that

$$\mathsf{E}X(0) = 0,\ \ \mathrm{Var}X(0) = 1,\ \ \mathrm{Var}\frac{\partial X(0)}{\partial s_1} = -\frac{\partial^2 R(0)}{\partial s_1^2} = 1.$$

Assume also that

1) $\mathsf{P}(\mathcal{H}_{d-1}(\{s \in \mathbb{R}^d : \nabla X(s) = 0\}) > 0) = 0$;
2) $\mathsf{P}(X(s) = u, \nabla X(s) = 0 \text{ for all } s \in \mathbb{R}^d) = 0$ with any $u \in \mathbb{R}$.

Both last requirements are true, e.g., if the realizations of X are C^2 a.s.

Assumptions

Let $d \geq 3$, the random field $X = \{X(s), s \in \mathbb{R}^d\}$ with C^1 realizations be stationary and isotropic,

$$N_t(u) = \mathcal{H}_d^{d-1}\{s \in [0,t]^d : X(s) = u\},\ Z_t(u) := t^{-d/2}(N_t(u) - \mathsf{E}N_t(u)).$$

We may and will always assume that

$$\mathsf{E}X(0) = 0,\ \ \mathrm{Var}X(0) = 1,\ \ \mathrm{Var}\frac{\partial X(0)}{\partial s_1} = -\frac{\partial^2 R(0)}{\partial s_1^2} = 1.$$

Assume also that

1) $\mathsf{P}(\mathcal{H}_{d-1}(\{s \in \mathbb{R}^d : \nabla X(s) = 0\}) > 0) = 0;$
2) $\mathsf{P}(X(s) = u, \nabla X(s) = 0 \text{ for all } s \in \mathbb{R}^d) = 0$ with any $u \in \mathbb{R}$.

Both last requirements are true, e.g., if the realizations of X are C^2 a.s.

Continuity of paths

Let $A \subset \mathbb{R}^d$ be a block, i.e. $A = (a_1, b_1) \times \ldots \times (a_d, b_d)$ with some $a_i < b_i$, $i = 1, \ldots, d$.

Theorem 3 (A.Sh., 2013). There exists an event Ω_0 with $\mathrm{P}(\Omega_0 = 1)$, on which for any $u \in \mathbb{R}$ the set function $N_X(D, u) := \mathcal{H}_d^{d-1}(D \cap B_u(X))$ defines a measure on Borel subsets of A. On the same event, for any continuous function $f : \mathbb{R}^d \to \mathbb{R}$ the map

$$u \mapsto \int_{B_u(X) \cap A} f(s) N_X(ds, u)$$

is well-defined and continuous on $\mathbb{R}$.
With $f \equiv 1$ one obtains the continuity of $N_t(u)$ in u.

Functional central limit theorem in $L^2(\mathbb{R})$

Let μ be a standard Gaussian measure on $\mathbb{R}$.

Theorem 4 (D.Meschenmoser, A.Sh., 2012). Assume that the conditions of previous theorem hold and, in addition, there exists a bounded continuous function $g : \mathbb{R}^d \to \mathbb{R}$ such that

- $g(s) \to 0$ при $\|s\| \to \infty$,
- $\int_{\mathbb{R}^d} \sqrt{g(s)} ds < \infty$,
- $$|R(s)| + \sum_{j=1}^{d} \left| \frac{\partial R(s)}{\partial s_j} \right| + \sum_{j,q=1}^{d} \left| \frac{\partial^2 R(s)}{\partial s_j \partial s_q} \right| < g(s)$$

  as $s \neq 0$.

Then the random processes

$$Z_t := t^{-d/2}(N_t - \mathsf{E} N_t)$$

converge in distribution in $L^2(\mathbb{R}, \mu)$, as $t \to \infty$ to a Gaussian random element Z with covariance operator

$$\mathrm{Var}(Z, f)_{L^2(\mathbb{R},\mu)} = \frac{1}{2\pi} \int_{\mathbb{R}^d} cov \left(f(X(0)) e^{-X(0)^2/2} \|\nabla X(0)\|, f(X(s)) e^{-X(s)^2/2} \|\nabla X(s)\| \right) ds,$$

here $f \in L^2(\mathbb{R}, \mu)$.

Functional central limit theorem in $C(\mathbb{R})$

Theorem 5. (A.Sh., 2013). Assume that X satisfies the conditions of Theorem 3, and, moreover, the covariance function of X is integrable over $\mathbb{R}^d$, together with its partial derivatives of order 1 and 2. Then the random processes $\{Z_n(\cdot), n \in \mathbb{N}\}$ processes converge in distribution in $C(\mathbb{R})$, as $n \to \infty$, to a centered Gaussian process Z with covariance function

$$\mathsf{E}Z(u)Z(v) = \int\limits_{\mathbb{R}^d} \Big(\mathsf{E}(\|\nabla X(0)\| \|\nabla X(s)\| | X(0) = u, X(s) = v) p_{X(0),X(s)}(u,v)$$

$$- \mathsf{E}(\|\nabla X(0)\|)^2 p_{X(0)}(u) p_{X(s)}(v) \Big) ds.$$

Applications: integrals of (BL, θ)-dependent random fields

Definition (A.Bulinski, 2010). A square-integrable random field $\xi = \{\xi(t), t \in \mathbb{R}^d\}$ is called (BL, θ)-dependent if there exist a non-increasing function $\theta_\xi : \mathbb{R}_+ \to \mathbb{R}_+$, $\theta_\xi(r) \to 0$ as $r \to \infty$, such that for any $\Delta > 0$ large enough, any disjoint finite $I, J \subset T(\Delta)$ and all bounded Lipschitz functions $f : \mathbb{R}^{|I|} \to \mathbb{R}$, $g : \mathbb{R}^{|J|} \to \mathbb{R}$ one has

$$|cov(f(\xi_I), g(\xi_J))| \leq Lip(f)Lip(g)(|I| \wedge |J|)\Delta^d \theta_\xi(r).$$

Here $T(\Delta) = \{j/\Delta \in \mathbb{R}^d : j \in \mathbb{Z}^d\}$, the notation $\xi_I = (\xi_i, i \in I)$ is employed, $|M|$ is the cardinality of a finite M, r is the distance between I and J, and the Lipschitz constants are with respect to the norm $\|z\|_1$.

Functional central limit theorem for the integrals

Suppose that X is as in Theorem 5. Assume also that Y is a strictly stationary (BL, θ)-dependent random field, independent from X. Define

$$Z_n(u) := n^{-d/2} \int_{[0,n]^d} (Y(s) N_X(ds, u) - \mathsf{E}Y(0)\mathsf{E}\|\nabla X(0)\| p_{X(0)}(u)\, ds), \; n \in \mathbb{N}, \; u \in \mathbb{R}.$$

Theorem 6 (A.Sh., 2013). These processes converge in distribution in $C(\mathbb{R})$, as $n \to \infty$, to a centered Gaussian process Z with covariance function

$$\mathsf{E}Z(u)Z(v) = \int_{\mathbb{R}^d} \Big(\mathsf{E}Y(0)\mathsf{E}Y(s)\mathsf{E}(\|\nabla X(0)\| \|\nabla X(s)\| | X(0) = u, X(s) = v) p_{X(0),X(s)}(u, v)$$

$$- (\mathsf{E}Y(0))^2 (\mathsf{E}(\|\nabla X(0)\|)^2 p_{X(0)}(u) p_{X(s)}(v) \Big)\, ds.$$

References

- J.M.Azais, M.Wschebor. Level sets and extrema of random processes and fields. Wiley, 2009.
- A.I. Elizarov. Central limit theorem for the sojourn time and local time of a stationary process. Theory Probab. Appl., 1988, V. 33, p. 161–164.
- I.A.Ibragimov, D.N.Zaporozhets. On random surface area. Zap. Nauchn. Sem. POMI, 2010, V. 384, p. 154–175.
- M. Kratz. Level crossings and other level functionals of stationary Gaussian processes. Probab. Surv., 2006, V. 3, p. 230–288.
- D. Meschenmoser, A. Shashkin. A functional CLT for the surface measures of a Gaussian random field. Theory Probab. Appl., V. 57, No. 2.
- A. Shashkin. A functional CLT for the level measure of a Gaussian random field. Statist. Probab. Lett., 2013, V. 83, p. 637–643.