Some moment inequalities and moment estimates for characteristic functions

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Introduction

In 1900–1901 A. M. Lyapounov proposed a powerful tool for proving limit theorems — the method of **characteristic functions**. Convergence conditions in many limit theorems can be expressed in terms of **moments**. In this talk the attention is focused on:

- moment estimates for characteristic function;
- moment inequalities;
- applications to the central limit theorem.

Let X be a r.v. with $\mathsf{E}|X|^n < \infty$ for some $n \in \mathbf{N}$. Denote

• characteristic function (ch.f.):

$$f(t) = \mathsf{E}e^{itX}, \ \ t \in \mathbf{R},$$

• moments (algebraic and absolute):

$$\alpha_k = \mathsf{E}X^k, \quad \beta_k = \mathsf{E}|X|^k, \quad |\alpha_k| \leqslant \beta_k, \quad k = 1, 2, \ldots, n.$$

Estimates for ch.f. in the vicinity of zero. I

Behavior of the ch.f. in the vicinity of zero determines the rate of convergence, e.g., in limit theorems for sums of independent random variables.

Denote

$$R_n(t) = f(t) - \sum_{k=0}^{n-1} \frac{\alpha_k (it)^k}{k!} = \mathsf{E}\left(e^{itX} - \sum_{k=0}^{n-1} \frac{(itX)^k}{k!}\right), \quad t \in \mathbf{R}.$$

Well-known estimate:

$$|R_n(t)| \leqslant \frac{\beta_n |t|^n}{n!}, \quad t \in \mathbf{R}.$$

Theorem

If $\beta_n < \infty$, then for all $t \in \mathbf{R}$

$$|R_n(t)| \leqslant 2\beta_{n-1} \int_0^{|t|} \int_0^{t_{n-1}} \cdots \int_0^{t_2} \sin\left(\frac{\beta_n t_1}{2\beta_{n-1}} \wedge \frac{\pi}{2}\right) dt_1 \cdots dt_{n-2}\, dt_{n-1}.$$

Estimates for ch.f. in the vicinity of zero. II

Well-known estimate:

$$|R_n(t)| \equiv \left| f(t) - \sum_{k=0}^{n-1} \frac{\alpha_k (it)^k}{k!} \right| \leqslant \frac{\beta_n |t|^n}{n!}, \quad t \in \mathbf{R}.$$

Theorem

If $\beta_n < \infty$, then for all $t \in \mathbf{R}$

$$|R_n(t)| \leqslant \inf_{0 \leqslant \lambda < 1/2} (\lambda |\alpha_n| + q_n(\lambda)\beta_n) \frac{|t|^n}{n!}, \quad t \in \mathbf{R},$$

$$\left| \frac{d^m R_n(t)}{dt^m} \right| \leqslant \inf_{0 \leqslant \lambda < 1/2} (\lambda |\alpha_n| + q_{n-m}(\lambda)\beta_n) \frac{|t|^{n-m}}{(n-m)!}, \quad m = 1, \ldots, n.$$

where

$$q_n(\lambda) = n! \sup_{x>0} x^{-n} \left| e^{ix} - \sum_{k=0}^{n-1} \frac{(ix)^k}{k!} - \lambda \frac{(ix)^n}{n!} \right|, \quad 0 \leqslant \lambda < \frac{1}{2}.$$

(Prawitz, 1991): $\quad R_n(t) \leqslant \dfrac{n|\alpha_n| + (n+2)\beta_n}{2(n+1)} \cdot \dfrac{|t|^n}{n!} \leqslant \dfrac{\beta_n |t|^n}{n!}, \quad t \in \mathbf{R}.$

Example for $n = 3$

Jensen's inequality: $|\alpha_3| \leqslant \beta_3$.

Theorem

For all $b \geqslant 1$ and any r.v. X with $\alpha_1 = 0$, $\alpha_2 = 1$

$$|\alpha_3| \leqslant A(\beta_3)\beta_3,$$

where

$$A(b) = \sqrt{\frac{1}{2}\sqrt{1+8b^{-2}} + \frac{1}{2} - 2b^{-2}} < 1, \;\; b \geqslant 1,$$

with the equality attained for each value of $\beta_3 = b \geqslant 1$ at the distribution

$$\mathsf{P}\left(X = \frac{1}{2}\left(b \pm \sqrt{b^2+4}\right)\right) = \frac{2 + b\big(b \mp \sqrt{b^2+4}\,\big)/2}{b^2+4}.$$

In particular, if $\beta_3 = 1$, then $\alpha_3 = 0$.

Example for $n = 3$

Well-known estimate:

$$|R_3(t)| \equiv \left|f(t) - 1 - i\alpha_1 t + \alpha_2 t^2/2\right| \leqslant \frac{\beta_3|t|^3}{6} \approx 0.1667 \cdot \beta_3|t|^3, \quad t \in \mathbf{R}.$$

Prawitz' estimate:

$$|R_3(t)| \leqslant \left(\frac{3}{8}|\alpha_3| + \frac{5}{8}\beta_3\right)\frac{|t|^3}{6} \approx \left(0.0625 \cdot |\alpha_3| + 0.1042 \cdot \beta_3\right)|t|^3.$$

By considering symmetric three-point distributions (for which $\alpha_3 = 0$) Prawitz also noticed that the factor $\frac{5}{48} \approx 0.1042$ cannot be less than

$$\varkappa_3 \equiv \sup_{x>0} x^{-3}(\cos x - 1 + x^2/2) \approx 0.0992.$$

Corollary

1°. *If* $\alpha_3 = 0$, *then*

$$|R_3(t)| \leqslant \varkappa_3 \cdot \beta_3|t|^3, \quad t \in \mathbf{R}.$$

2°. *If* $\alpha_1 = 0$, $\alpha_2 = 1$, *then for all* $t \in \mathbf{R}$

$$|R_3(t)| \leqslant \inf_{\lambda \geqslant 0}(\lambda|\alpha_3| + q_3(\lambda)\beta_3)|t|^3/6 \leqslant$$

$$\leqslant \inf_{\lambda \geqslant 0}(\lambda A(b) + q_3(\lambda)\beta_3)\frac{|t|^3}{6} \leqslant \begin{cases} 0.0992 \cdot \beta_3|t|^3, & \beta_3 = 1, \\ 0.1110 \cdot \beta_3|t|^3, & \beta_3 \leqslant 1.01, \\ 0.1328 \cdot \beta_3|t|^3, & \beta_3 \leqslant 1.1, \\ 0.1556 \cdot \beta_3|t|^3, & \beta_3 \leqslant 1.5, \\ 0.1667 \cdot \beta_3|t|^3 & \forall\beta_3. \end{cases}$$

Square bias transformation

If $\mathrm{E}X^2 < \infty$, then $\dfrac{f'(t) - f'(0)}{tf''(0)}$, $\dfrac{f''(t)}{f''(0)}$ are ch.f.'s as well (Lukacs, 1970).

Definition (see also (Goldstein, Reinert, 1997))

Let X be a r.v. with the ch.f. $f(t)$ and $\mathrm{E}X = 0$, $\mathrm{E}X^2 = \sigma^2 > 0$. Then the distribution of any r.v. $X^{(z)}$ with the ch.f.

$$\frac{f'(t) - f'(0)}{tf''(0)} = -\frac{f'(t)}{\sigma^2 t}$$

is called the X-**zero biased distribution**.

Definition (see also (Goldstein, 2007))

Let X be a r.v with the ch.f. $f(t)$ and $\mathrm{E}X^2 = \sigma^2 > 0$. Then the distribution of any r.v. $X^{\square}$ with the ch.f.

$$f^{\square}(t) = \frac{f''(t)}{f''(0)} = -\frac{f''(t)}{\sigma^2}$$

is called the X-**square biased distribution**.

Square bias transformation: properties

Elementary properties of $X^\square$:

- $d\,\mathsf{P}(X^\square < x) = \dfrac{x^2}{\sigma^2} d\,\mathsf{P}(X < x)$, $x \in \mathbf{R}$.
- $\mathsf{E}X^2 G(X) = \sigma^2 \mathsf{E} G(X^\square)\ \forall G \ \Leftrightarrow\ X^\square$ has the X-square biased distribution (see also (Goldstein, Reinert, 2005)).
- $X^\square \stackrel{d}{=} X \ \Leftrightarrow \mathsf{P}(|X| = \sigma) = 1$.
- $(cX)^\square \stackrel{d}{=} cX^\square$.
- If $\mathsf{E}|X|^3 < \infty$, then $\sigma^2 \mathsf{E}X^\square = \mathsf{E}X^3$, $\sigma^2 \mathsf{E}|X^\square| = \mathsf{E}|X|^3$.

Theorem

If $\mathsf{E}X = 0$ *and* $\mathsf{E}|X|^3 < \infty$, *then*

$$L_1(X, X^\square) \leqslant \mathsf{E}|X|^3,$$

with the equality attained at any symmetric three-point distribution with an atom at zero, $L_1(X, Y)$ *being the* L_1*-distance:*

$$L_1(X, Y) \equiv \inf\left\{ \mathsf{E}|X' - Y'| \colon X' \stackrel{d}{=} X,\ \ Y' \stackrel{d}{=} Y \right\}.$$

Square bias transformation: applications

Theorem

For any r.v.'s X, Y *with* $\mathsf{E}|X| \vee \mathsf{E}|Y| < \infty$

$$|\mathsf{E}e^{itX} - \mathsf{E}e^{itY}| \leqslant 2\sin\left(L_1(X,Y)\frac{|t|}{2} \wedge \frac{\pi}{2}\right), \quad t \in \mathbf{R}.$$

$$L_1(X, X^{(z)}) \leqslant \frac{1}{2}\mathsf{E}|X|^3 \quad \text{(Tyurin, 2009), (Goldstein, 2009).}$$

$$L_1(X, X^{\square}) \leqslant \mathsf{E}|X|^3 \quad \text{(Sh., 2012).}$$

Corollary

For any r.v. X *with* $\mathsf{E}X = 0$, $\mathsf{E}X^2 = 1$ *and* $\mathsf{E}|X|^3 = b \geqslant 1$ *for all* $t \in \mathbf{R}$

$$\left|\frac{f'(t)}{t} + f(t)\right| \leqslant 2\sin\left(\frac{b|t|}{4} \wedge \frac{\pi}{2}\right),$$

$$|f''(t) + f(t)| \leqslant 2\sin\left(\frac{b|t|}{2} \wedge \frac{\pi}{2}\right).$$

Exact estimates for the real part of ch.f.

Theorem

For any r.v. X with $\mathsf{E}X^2 = 1$ *and* $\beta_{2+\delta} \equiv \mathsf{E}|X|^{2+\delta} < \infty$, $0 < \delta \leqslant 1$,

$$\mathsf{E}\cos tX \leqslant 1 - \psi_\delta(t, \beta_{2+\delta}) \leqslant 1 - t^2/2 + \varkappa_\delta \beta_{2+\delta} |t|^{2+\delta}, \quad t \in \mathbf{R}, \qquad (1)$$

where

$$\psi_\delta(t,\varepsilon) = \begin{cases} t^2/2 - \varkappa_\delta \varepsilon |t|^{2+\delta}, & \varepsilon^{1/\delta}|t| < \theta, \\ \varepsilon^{-2/\delta}\left(1 - \cos\left(\varepsilon^{1/\delta} t\right)\right), & \theta \leqslant \varepsilon^{1/\delta}|t| \leqslant 2\pi, \quad \varepsilon > 0, \\ 0, & \varepsilon^{1/\delta}|t| > 2\pi, \end{cases}$$

$\theta = \theta(\delta)$ *is the unique root of the equation*

$$\delta\theta^2 + 2\theta\sin\theta = 2(2+\delta)(1-\cos\theta), \quad 0 < \theta < 2\pi,$$

$$\varkappa_\delta = \sup_{x>0}\left(\cos x - 1 + x^2/2\right)x^{-2-\delta} = (\cos\theta - 1 + \theta^2/2)\theta^{-2-\delta}.$$

Equality in (1) is attained at a symmetric three-point distribution.

(Prawitz, 1972): $\delta = 1$.
(Ushakov, 1999): $\mathsf{E}\cos tX \leqslant 1 - t^2/2 + \varkappa_\delta \beta_{2+\delta}|t|^{2+\delta}$, $t \in \mathbf{R}$, $0 < \delta \leqslant 1$.

Sharpening of the von Mises inequality

Corollary

For any r.v. X *with* $\mathsf{E}X = 0$, $\mathsf{E}X^2 = 1$, $\beta_\delta \equiv \mathsf{E}|X|^\delta$ *and* $\beta_{2+\delta} \equiv \mathsf{E}|X|^{2+\delta} < \infty$, $0 < \delta \leqslant 1$,

$$\left|\mathsf{E}e^{itX}\right| \leqslant \sqrt{1 - 2\psi_\delta(t, \beta_{2+\delta} + \beta_\delta)}, \quad t \in \mathbf{R},$$

$$\Rightarrow \ \left|\mathsf{E}e^{itX}\right| < 1, \quad |t|(\beta_{2+\delta} + \beta_\delta)^{1/\delta} < 2\pi.$$

Corollary

For any lattice r.v. X *with span* h *and* $\mathsf{E}X = 0$, $\mathsf{E}X^2 = 1$, $\beta_\delta \equiv \mathsf{E}|X|^\delta$, $\beta_{2+\delta} \equiv \mathsf{E}|X|^{2+\delta} < \infty$ *for some* $0 < \delta \leqslant 1$,

$$h \leqslant (\beta_{2+\delta} + \beta_\delta)^{1/\delta}; \quad \textit{in particular, for } \delta = 1\colon\ h \leqslant \beta_3 + \beta_1.$$

If X *has a symmetric distribution, then*

$$h \leqslant \max\left\{\beta_{2+\delta}^{1/\delta}, 2\right\}; \quad \textit{in particular, for } \delta = 1\colon\ h \leqslant \max\left\{\beta_3, 2\right\}.$$

(Mises, 1939): $h \leqslant 2\beta_3$.

Centering inequality for the third moments

Theorem

For all $a \in \mathbf{R}$ and any r.v. X with $\mathsf{E}X = a$ and $\mathsf{E}|X|^3 < \infty$

$$\mathsf{E}|X-a|^3 \leqslant \frac{17+7\sqrt{7}}{27}\mathsf{E}|X|^3 < 1.3156 \cdot \mathsf{E}|X|^3,$$

with the equality attained at the two-point distribution of the form

$$\mathsf{P}\left(X = \frac{6a}{4-\sqrt{7}\pm\sqrt{1+2\sqrt{7}}}\right) = \frac{3\pm\sqrt{1+2\sqrt{7}}}{6}.$$

Applications: method of truncation.

Theorem

For any r.v. X *with* $\mathsf{E}X=0$, $\mathsf{E}X^2=1$, $\mathsf{E}X^3=\alpha_3$, $\mathsf{E}|X|^k=\beta_k$, $k=1,3$,

$$|\alpha_3|+3\beta_1\leqslant\inf_{\lambda\geqslant1}\left\{\lambda\beta_3+M(p(\lambda),\lambda)\right\},$$

where

$$p(\lambda)=\frac{1}{2}-\sqrt{\frac{\lambda+1}{\lambda+3}}\sin\left(\frac{\pi}{6}-\frac{1}{3}\arctan\sqrt{\lambda^2+2\frac{\lambda-1}{\lambda+3}}\right),$$

$$M(p,\lambda)=\frac{1-\lambda+2(\lambda+2)p-2(\lambda+3)p^2}{\sqrt{p(1-p)}},\quad 0<p\leqslant\frac{1}{2},\ \lambda\geqslant1,$$

(Esseen, 1945, 1956): $X,X_1,X_2,\ldots$ — i.i.d., $\mathsf{E}X=0$, $\mathsf{E}X^2=1$, $\Rightarrow$

$$\limsup_{n\to\infty}\sqrt{n}\sup_x|\mathsf{P}(X_1+\ldots+X_n<x\sqrt{n})-\Phi(x)|=\frac{|\alpha_3|+3h}{6\sqrt{2\pi}}\leqslant\frac{\sqrt{10}+3}{6\sqrt{2\pi}}\beta_3,$$

h being the span, if X is lattice, and $h=0$ otherwise.
(Sh., 2009, 2012) $\Rightarrow$

$$|\alpha_3|+3h\leqslant|\alpha_3|+3(\beta_3+\beta_1)\leqslant\inf_{\lambda\geqslant1}\{(\lambda+3)\beta_3+M(p(\lambda),\lambda)\}\leqslant(\sqrt{10}+3)\beta_3.$$

Applications

Applications: CLT for sums of independent r.v.'s

By $\mathcal{F}_3$ denote the set of all d.f.'s of a r.v. X such that

$$\mathsf{E}X = 0, \quad \mathsf{E}|X|^3 < \infty.$$

Let $X_1, \ldots, X_n$ be independent r.v.'s with d.f.'s $F_1, \ldots, F_n \in \mathcal{F}_3$. Let

$$\sigma_j^2 = \mathsf{E}X_j^2, \quad \beta_{3,j} = \mathsf{E}|X_j|^3, \quad j = 1, 2, \ldots, n,$$

$$B_n^2 = \sum_{j=1}^n \sigma_j^2 > 0, \quad \ell_n = \frac{1}{B_n^3} \sum_{j=1}^n \beta_{3,j}, \quad \tau_n = \frac{1}{B_n^3} \sum_{j=1}^n \sigma_j^3,$$

$$\overline{F}_n(x) = \mathsf{P}(X_1 + \ldots + X_n < xB_n) = (F_1 * \ldots * F_n)(xB_n),$$

$$\Delta_n = \Delta_n(F_1, \ldots, F_n) = \sup_x |\overline{F}_n(x) - \Phi(x)|,$$

$$\Delta_n(F) = \Delta_n(F, \ldots, F), \quad n = 1, 2, \ldots,$$

$\Phi(\cdot)$ being the standard normal d.f. It can be made sure that

$$\ell_n \geqslant \tau_n \geqslant n^{-1/2}.$$

Moment-type estimates with optimal structure. I

Theorem

Let $\beta_{1,j} = \mathsf{E}|X_j|$, $j = 1, \ldots, n$. *Then for all* $F_1, \ldots, F_n \in \mathcal{F}_3$, $n \geqslant 1$

$$\Delta_n \leqslant \frac{2\ell_n}{3\sqrt{2\pi}} + \frac{1}{2\sqrt{2\pi}} \sum_{j=1}^{n} \frac{\beta_{1,j}\,\sigma_j^2}{B_n^3} + \begin{cases} 6\ell_n^{5/3}, & \text{non-i.i.d. case,} \\ 3\ell_n^2, & \text{i.i.d. case,} \end{cases}$$

$$\Delta_n \leqslant \frac{2\ell_n}{3\sqrt{2\pi}} + \sqrt{\frac{2\sqrt{3}-3}{6\pi}} \sum_{j=1}^{n} \frac{\sigma_j^3}{B_n^3} + \begin{cases} 3\ell_n^{7/6}, & \text{non-i.i.d. case,} \\ 2\ell_n^{3/2}, & \text{i.i.d. case.} \end{cases}$$

$$(2\sqrt{2\pi})^{-1} = 0.1994\ldots, \qquad \sqrt{(2\sqrt{3}-3)/(6\pi)} = 0.1569\ldots$$

Theorem

$$\underline{C_{\mathrm{AE}}}(\mathcal{F}_3) = \limsup_{\ell\to 0}\limsup_{n\to\infty} \sup_{F\in\mathcal{F}_3:\,\ell_n=\ell} \frac{\Delta_n(F)}{\ell} = \frac{2}{3\sqrt{2\pi}} = 0.2659\ldots$$

$$\Delta_n \leqslant \frac{2\ell_n}{3\sqrt{2\pi}} + \frac{1}{2\sqrt{2\pi}} \sum_{j=1}^{n} \frac{\sigma_j^3}{B_n^3} + \begin{cases} O(\ell_n^{4/3}), & \text{non-i.i.d. case (Bentkus, 1991),} \\ O(\ell_n^2), & \text{i.i.d. case (Prawitz, 1975).} \end{cases}$$

Moment-type estimates with optimal structure. II

Theorem

For all $c \geqslant 2/(3\sqrt{2\pi})$, $n \geqslant 1$, $F_1, \ldots, F_n \in \mathcal{F}_3$

$$\Delta_n \leqslant c\ell_n + K(c)\sum_{j=1}^n \frac{\sigma_j^3}{B_n^3} + \begin{cases} 3\ell_n^{7/6} \wedge A(c)\ell_n^{4/3}, & \textit{non-i.i.d. case,} \\ 2\ell_n^{3/2} \wedge A(c)\ell_n^2, & \textit{i.i.d. case,} \end{cases}$$

where

$$K(c) = \frac{1-\theta+2(\theta+2)p(\theta)-2(\theta+3)p^2(\theta)}{6\sqrt{2\pi p(\theta)(1-p(\theta))}}\Bigg|_{\theta=6\sqrt{2\pi}c-3},$$

$$p(\theta) = \frac{1}{2} - \sqrt{\frac{\theta+1}{\theta+3}}\sin\left(\frac{\pi}{6} - \frac{1}{3}\arctan\sqrt{\theta^2+2\frac{\theta-1}{\theta+3}}\right), \quad \theta \geqslant 1,$$

$A(c) \to \infty$ *as* $c \to 2/(3\sqrt{2\pi})$, $A(c)$ *decreases monotonically and is given in the explicit form. Value of* $K(c)$ *can be made less for no* $c \geqslant \frac{2}{3\sqrt{2\pi}}$.

Remark. $K(c)$ decreases monotonically for $c \geqslant 2/(3\sqrt{2\pi})$ and

$$K\left(\frac{\sqrt{10}+3}{6\sqrt{2\pi}}\right) = 0.$$

Corollaries

Estimates in Kolmogorov's form:

Corollary

For all $n \geqslant 1$ *and* $F_1, \ldots, F_n \in \mathcal{F}_3$

$$\Delta_n \leqslant \frac{\sqrt{10}+3}{6\sqrt{2\pi}}\,\ell_n + \begin{cases} 4\ell_n^{4/3}, & \text{non-i.i.d. case,} \\ 3\ell_n^2, & \text{i.i.d. case.} \end{cases}$$

(Chistyakov, 2001): $\quad \Delta_n \leqslant \dfrac{\sqrt{10}+3}{6\sqrt{2\pi}}\,\ell_n + O\big(\ell_n^{40/39}|\log \ell_n|^{7/6}\big).$

For symmetric Bernoulli distributions:

Corollary

For all $n \geqslant 1$ *and* $F_1, \ldots, F_n \in \mathcal{F}_3$ *such that* $\beta_{3,j} = \sigma_j^3$, $j = 1, \ldots, n$,

$$\Delta_n \leqslant \frac{\ell_n}{\sqrt{2\pi}} + \begin{cases} 4\ell_n^{4/3}, & \text{non-i.i.d. case,} \\ 3\ell_n^2, & \text{i.i.d. case.} \end{cases}$$

Applications: Poisson random sums

Let $X, X_1, X_2, \ldots$ be i.i.d. r.v.'s with common d.f. $F(x)$, and such that

$$\mathsf{E}X = a, \quad \mathsf{E}X^2 = a^2 + \sigma^2 > 0, \quad \mathsf{E}|X|^3 = \beta_3 < \infty.$$

By $\mathcal{F}_3$ denote the set of all d.f.'s of the r.v. X, satisfying the above conditions for some $a, \sigma > 0$ and β_3.
Let N_λ, $\lambda > 0$, have the Poisson distribution:

$$\mathsf{P}(N_\lambda = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots,$$

and be independent of $X_1, X_2, \ldots$. Denote

$$S_\lambda = X_1 + \ldots + X_{N_\lambda} \quad (\text{for } N_\lambda = 0 \text{ define } S_\lambda = 0),$$

$$\Delta_\lambda = \Delta_\lambda(F) = \sup_{x \in \mathbf{R}} \left| \mathsf{P}\left(\frac{S_\lambda - \lambda a}{\sqrt{\lambda(a^2 + \sigma^2)}} < x \right) - \Phi(x) \right|, \quad \lambda > 0, \ x \in \mathbf{R},$$

$$\ell_\lambda = \frac{\beta_3}{(a^2 + \sigma^2)^{3/2}\sqrt{\lambda}}.$$

Convergence rate estimates for Poisson random sums

Theorem

For all $\lambda > 0$ *and* $F \in \mathcal{F}_3$

$$\Delta_\lambda \leqslant \frac{2\ell_\lambda}{3\sqrt{2\pi}} + \frac{\ell_\lambda^2}{2} \quad \textit{and} \quad \Delta_\lambda \leqslant \begin{cases} 0.3031 \cdot \ell_\lambda, & \forall \ell_\lambda, \\ 0.2929 \cdot \ell_\lambda, & \ell_\lambda \leqslant 0.1, \\ 0.2660 \cdot \ell_\lambda, & \ell_\lambda \leqslant 10^{-4}. \end{cases}$$

The **lower asymptotically exact** and **asymptotically exact** constants:

$$\begin{aligned} \underline{M}_{\mathrm{AE}}(\mathcal{F}_3) &= \limsup_{\ell\to 0} \limsup_{\lambda\to\infty} \sup_{F\in\mathcal{F}_3:\ \ell_\lambda=\ell} \Delta_\lambda(F)/\ell, \\ M_{\mathrm{AE}}(\mathcal{F}_3) &= \limsup_{\ell\to 0} \sup_{\lambda, F\in\mathcal{F}_3:\ \ell_\lambda=\ell} \Delta_\lambda(F)/\ell. \end{aligned}$$

Theorem

$$\underline{M}_{\mathrm{AE}}(\mathcal{F}_3) = M_{\mathrm{AE}}(\mathcal{F}_3) = \frac{2}{3\sqrt{2\pi}} = 0.2659\ldots\ .$$

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