The probabilistic approximation of the one-dimensional initial boundary value problem solution.

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We consider the equation

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + f(x)u,$$

where σ is a complex-valued parameter, such that $\mathrm{Re}\sigma^2 \geq 0$.
When σ is a real number this equation corresponds to the Heat equation while when $\mathrm{Re}\sigma^2 = 0$ it corresponds to the Schrödinger equation. For this equation we consider the initial boundary value problem with Dirichlet condition

$$u(0,x) = \varphi(x),\ \ u(a,t) = 0,\ \ u(t,b) = 0,$$

where $[a,b] \subset \mathbb{R}$.
For simplicity we suppose that our interval $[a,b]$ is an interval $[0,\pi]$. The general case can be reduced to this case by a linear change of argument.

In the case when σ is a real number there exists probabilistic representation of the solution in a form of the mathematical expectation (so called Feynman - Kac formula), namely

$$u(t,x) = \mathrm{E}\left\{\varphi(\widetilde{\xi}_x(t\wedge\tau))e^{\int_0^{t\wedge\tau} f(\widetilde{\xi}_x(v))dv}\right\}, \tag{1}$$

where $\widetilde{\xi}_x(t)$ is a Brownian motion with a parameter σ, killed at the exit time τ from the interval $[a,b]$. This approach doesn't work if $\mathrm{Im}\sigma \neq 0$.

It is known that when σ is not a real number there exists no analogue of the Wiener measure and hence one can not present the Feynman - Kac formula as an integral with respect to a σ-additive measure in a trajectory space.

When $\mathrm{Re}\sigma^2 = 0$ (that corresponds to the Schrödinger equation) one can apply an integral with respect to the so called Feynman measure that is a finitely-additive complex measure in the trajectory space which is defined as a limit over a sequence of partitions of an interval $[0, T]$. It should be mentioned that this approach is not a probabilistic approach in the usual sense since the very notion of a probability space does not appear in it. There exists a number of papers and books devoted to the strict mathematical background of the Feynman integral. One of them has received new attention in works by Doss , Albeverio and Mazzucchi, and Thaler is based on a construction of an analytic prolongation of (1) (in σ).

Unfortunately this approach demands a potential analyticity which is a rather restrictive condition. In addition one can not use this approach as a basis for a probabilistic construction of an approximation of the Feynman integral and a construction of trajectories for a complex-valued process $\sigma w(t)$ since it demands a notion of an analytical function with an argument which is a continuous function itself.

An alternative approach to the mathematical background of the Feynman integral is due to Albeverio, Høegh-Krohn and Maslov and is based on a construction of a complex measure corresponding to the Fourier transform of (4) rather then to (4) itself (that is a measure in momentum space rather then in coordinate representation).

To get the stochastic approximation of the solution we use another approach based on the generalized function theory. Roughly speaking, we consider the Wiener process as a jump Lèvy process with a Lèvy measure Λ of the form $\Lambda = \frac{\delta^{(2)}}{2}$, where $\delta^{(2)}$ is a generalized function (not a measure) that acts on a test function φ as $(\delta^{(2)}, \varphi) = \varphi''(0)$.
Namely, on a special probability space we define the sequence of probability measures $\{\mathrm{P}_n\}$ (each measure $\{\mathrm{P}_n\}$ is generated by some compound Poisson process) and a limit object $\mathrm{L} = \lim_{n\to\infty} \mathrm{P}_n$ but this limit object is not a measure it is only generalized function. That means that the convergence $\int f d\mathrm{P}_n \to (\mathrm{L}, f)$ is valid only if f belongs to the class of test functions. Now we describe this construction.

Let Ω_0 denote the space of all discrete signed measures on $[0, T]$ with a finite spectrum (finite number of atoms). Each element ν of this space can be represented in the form $\nu = \sum_{k=1}^{n} x_k \delta_{t_k}$, where δ_{t_k} denotes a unit mass ($\delta-$measure) at a point t_k.
Let $f : \Omega_0 \to \mathbb{C}$ be a Borel function, for every $k \in \mathbb{N}_0,\ \mathbb{N}_0 = \{0, 1, 2 \dots\}$; we use a notation f_k for a symmetric function of k two-dimensional variables defined by

$$f_k((t_1, x_1), (t_2, x_2), \dots, (t_k, x_k)) = f(\sum_{j=1}^{k} x_j \delta_{t_j}),\ \ t_j \in [0, T], x_j \in \mathbb{R}.$$

On Ω_0 we define a sequence of probability measures P_n.

Namely, let ν_n be a Poisson random measure on $[0, T]$ with intensity measure ndt, so that $\mathrm{E}\nu_n(dt) = ndt$, and $\{\xi_j\}_{j=1}^{\infty}$ be an orthogonal gaussian sequence of random variables independent of ν_n.

For every $n \in \mathbb{N}$ we construct a random signed measure ζ_n (random element of Ω_0) by

$$\zeta_n = \sum_{j=1}^{k} \frac{\xi_j}{\sqrt{n}} \delta_{t_j} \in \Omega_0,$$

where $\nu_n = \sum_{j=1}^{k} \delta_{t_j}$, $\quad t_1 < t_2 < \cdots < t_k$ is a realization of the Poisson random measure ν_n, and $k = \nu_n([0, T])$.

Denote by P_n the distribution of ζ_n in Ω_0, and by E_n the mathematical expectation with respect to the measure P_n.

By g_n we denote a generalized function that acts on a test function $\psi : \mathbb{R} \to \mathbb{C}$ as

$$(g_n, \psi) = \frac{n}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\psi\left(\frac{y}{\sqrt{n}}\right) - \psi(0)\right) e^{-\frac{y^2}{2}} dy.$$

For every $k \in \mathbb{N}$ by m^k we denote the Lebesgue measure on $[0, T]^k$.

Theorem

For every bounded $f : \Omega_0 \to \mathbb{C}$ we have

$$\int_{\Omega_0} f d\mathrm{P}_n = \mathrm{E}_n f = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{[0,T]^k} \left(g_n^{\otimes k}, f_k\right) dm^k.$$

Here we integrate with respect to time variables $t_j \in [0, T]$, and generalized function $(g_n)^{\otimes k}$ acts with respect to space variables $x_j \in \mathbb{R}$.

Now consider the limit $\lim_{n\to\infty} P_n$. The limit object is not a measure, it is only a linear functional (generalized function). First note that for every sufficiently smooth function ψ we have

$$\lim_{n\to\infty}(g_n,\psi) = \frac{\psi^{(2)}(0)}{2} = (\frac{\delta^{(2)}}{2},\psi),$$

so that $\frac{\delta^{(2)}}{2} = \lim_{n\to\infty} g_n$ (in generalized functions sense).
Now under some class of test functions $\mathcal{G}$ we define a generalized function L. For every $f \in \mathcal{G}$, $f : \Omega_0 \to \mathbb{C}$ we put

$$\mathrm{L}f = (\mathrm{L}, f) = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{[0,T]^k} \left(\left(\frac{\delta^{(2)}}{2} \right)^{\otimes k}, f_k \right) dm^k. \qquad (6)$$

In (6) we suppose that the generalized function $\left(\frac{\delta^{(2)}}{2}\right)^{\otimes k}$ acts with respect to $x_1, \ldots, x_k$, m^k denotes the Lebesgue measure on $[0, T]^k$ and we integrate with respect to $(t_1, \ldots, t_k)$.

Theorem

For every $f \in \mathcal{G}$

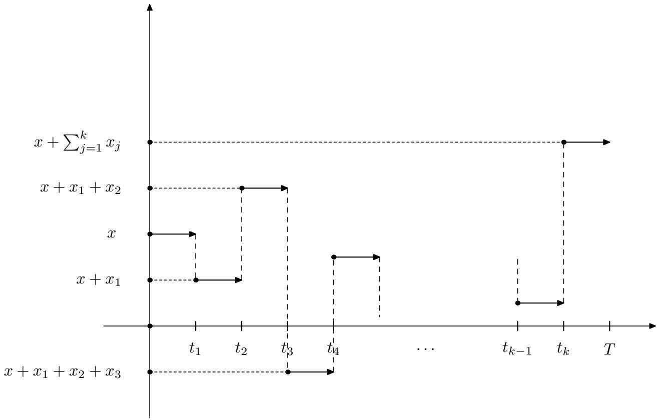
$$\lim_{n\to\infty} \mathrm{E}_n f = \mathrm{L} f.$$

L — is not a measure.

For every $x \in \mathbb{R}$ we define a random process $\xi_x^\sigma(t), \ t \in [0, T]$ by

$$\xi_x^\sigma(t) = \xi_x^\sigma(t, \omega) = x + \sigma\omega([0, t]), \quad \omega \in \Omega_0$$

For every fixed n the process $\xi_x^\sigma(t), \ t \in [0, T]$ is a complex-valued piecewise constant process on the probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathrm{P}_n)$. But on the the space $(\Omega_0, \mathcal{F}, \mathrm{L})$ the process $\xi_x^\sigma(t)$ is not a random process in usual sense because instead of a probability measure on Ω_0 we have only a generalized function L.


$x + \sum_{j=1}^{k} x_j$
$x + x_1 + x_2$
x
$x + x_1$
t_1
t_2
t_3
t_4
$\cdots$
t_{k-1}
t_k
T
$x + x_1 + x_2 + x_3$

Let φ be a continuous function $\varphi : [0, \pi] \to \mathbb{C}$. Decompose the function φ into a sum

$$\varphi(x) = \varphi(0) + \frac{\varphi(\pi) - \varphi(0)}{\pi} x + \varphi_0(x),$$

so that $\varphi_0(0) = \varphi_0(\pi) = 0$. For every φ_0 by φ_0^{odd} denote its odd continuation from $[0, \pi]$ to $[-\pi, \pi]$.

Denote by $\mathcal{H}_D$ the set of functions $\varphi : [0, \pi] \to \mathbb{C}$, such that the function φ_0^{odd} is of the form

$$\varphi_0^{\mathrm{odd}}(x) = \sum_{m=-\infty}^{\infty} B_m e^{imx},$$

where $B_0 = 0$, and for all m $B_{-m} = -B_m$ and only finite number of indices B_m are not equal to 0 (that is φ^{odd} is a trigonometrical polynomial).

For every function $\varphi \in \mathcal{H}_D$ define its continuation $\widetilde{\varphi}_D : \mathbb{C} \to \mathbb{C}$, setting for $z \in \mathbb{C}$

$$\widetilde{\varphi}_D(z) = \varphi(0) + \frac{\varphi(\pi) - \varphi(0)}{\pi} z + \sum_{m=-\infty}^{\infty} B_m e^{imz}.$$

On the domain $\mathcal{H}_D$ for $t \in [0, T]$ define linear operators P_D^t and $P_{D,n}^t$ by

$$P_D^t \varphi(x) = \mathrm{L}\widetilde{\varphi}_D(\xi_x^\sigma(t)) = \mathrm{L}\widetilde{\varphi}_D(x + \sigma\omega[0,t])$$

and

$$P_{D,n}^t \varphi(x) = \mathrm{E}_n\widetilde{\varphi}_D(\xi_x^\sigma(t)) = \mathrm{E}_n\widetilde{\varphi}_D(x + \sigma\omega[0,t]).$$

Note that $P_D^t, P_{D,n}^t$ are semigroups of operators, so that $P_D^{t+s} = P_D^t P_D^s$ and $P_{D,n}^{t+s} = P_{D,n}^t P_{D,n}^s$.

Theorem

1.

$$P_D^t\varphi(x) = \varphi(0) + \frac{\varphi(\pi) - \varphi(0)}{\pi}x + \sum_{m=-\infty}^{\infty} B_m e^{-\frac{t\sigma^2 m^2}{2}} e^{imx},$$

2.

$$P_{D,n}^t\varphi(x) = \varphi(0) + \frac{\varphi(\pi) - \varphi(0)}{\pi}x + \sum_{m=-\infty}^{\infty} B_m e^{\left(nt\left(e^{-\frac{m^2\sigma^2}{2n}} - 1\right)\right)} e^{imx}$$

3.For every σ, $\mathrm{Re}\sigma^2 \geq 0$ $|e^{-\frac{t\sigma^2 m^2}{2}}| \leq 1$ *and* $|e^{\left(nt\left(e^{-\frac{m^2\sigma^2}{2n}} - 1\right)\right)}| \leq 1.$

Initially operators $P_D^t, P_{D,n}^t$ are defined on the space $\mathcal{H}_D$, but using this theorem one can extend the operators $P_{D,n}^t$ and P_D^t on the set of functions φ, such that the series $\sum_{m=-\infty}^{\infty} |B_m|$ converges.
For every φ such that $\sum_{m=-\infty}^{\infty} m^2|B_m| < \infty$ define functions $u = u(t,x),\ t \geq 0, x \in \mathbb{R},\ u_n = u_n(t,x)$ by

$$u(t,x) = P_D^t\varphi(x) = \mathrm{L}\widetilde{\varphi}_D(\xi_x^\sigma(t)),$$

$$u_n(t,x) = P_{D,n}^t\varphi(x) = \mathrm{E}_n\widetilde{\varphi}_D(\xi_x^\sigma(t)).$$

Theorem

1.The function $u(t,x)$ is a solution of the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}, \quad u(0,x)=\varphi(x), \quad u(t,0)=\varphi(0), \; u(t,\pi)=\varphi(\pi).$$

2.The function $u_n(t,x)$ is a solution of the initial boundary value problem

$$\frac{\partial u_n}{\partial t} = A^{\sigma}_{D,n}u_n, \; u_n(0,x)=\varphi(x), \; u_n(t,0)=\varphi(0), \; u_n(t,\pi)=\varphi(\pi),$$

where the operator $A^{\sigma}_{D,n}$ acts on $\psi \in \mathcal{H}_D$ as

$$A^{\sigma}_{D,n}\psi(x) = \frac{n}{\sqrt{2\pi}}\int_{\mathbb{R}}(\widetilde{\psi}_D(x+\frac{\sigma y}{\sqrt{n}}) - \widetilde{\psi}_D(x))e^{-\frac{y^2}{2}}dy.$$

Let $\varphi : [0,\pi] \to \mathbb{C}$ be a continuous function such that $\varphi(x) = a + bx + \varphi_0(x)$, $\varphi_0(0) = \varphi_0(\pi) = 0$ and $\sum_{m=-\infty}^{\infty} |B_m| < \infty$. Define a norm by

$$\|\varphi\|_{\mathcal{R}^{\mathrm{odd}}} = |a| + |b|\pi + \sum_{m=-\infty}^{\infty} |B_m|.$$

It is clear that $\|\varphi\|_\infty \leqslant \|\varphi\|_{\mathcal{R}^{\mathrm{odd}}}$.

Theorem

Suppose that $\mathrm{Re}\sigma^2 \geq 0$. *Then uniform in* $t \in [0, T]$

$$\lim_{n\to\infty} \|P^t_{D,n}\varphi - P^t_D\varphi\|_{\mathcal{R}^{\mathrm{odd}}} = 0.$$

We define semigroups of operators F_n^t and F^t ($t \geq 0$) setting for $\varphi \in \mathcal{H}_D$

$$F^t\varphi(x) = \mathrm{L}\left[\widetilde{\varphi}_D(\xi_x^\sigma(t)) \exp\left(\int_0^t \widetilde{f}^{\mathrm{even}}(\xi_x^\sigma(v))dv \right)\right]$$

and

$$F_n^t\varphi(x) = \mathrm{E}_n\left[\widetilde{\varphi}_D(\xi_x^\sigma(t)) \exp\left(\int_0^t \widetilde{f}^{\mathrm{even}}(\xi_x^\sigma(v))dv \right)\right].$$

Here by $\widetilde{f}^{\mathrm{even}}$ we denote an even continuation of f and suppose that $\widetilde{f}^{\mathrm{even}}$ is a trigonometrical polynomial.
One can extend the operators F_n^t and F^t on the spaces $\mathcal{R}_0 = \{g(x) = \sum B_m e^{imx} : \sum |B_m| < \infty\}$ $(\varphi^{\mathrm{odd}}, f^{\mathrm{even}} \in \mathcal{R}_0)$.

Suppose that $f^{\mathrm{even}} \in \mathcal{R}_0$. For every φ such that $\varphi(0) = \varphi(\pi) = 0$ $\sum_{m \in Z} m^2 |B_m| < \infty$ define functions $u = u(t, x),\ t \geq 0, x \in \mathbb{R}$, $u_n = u_n(t, x)$ by

$$u(t, x) = F^t \varphi(x), \quad u_n(t, x) = F_n^t \varphi(x).$$

Theorem

1. The function $u(t, x)$ is a solution of the initial boundary value problem ($u(0, x) = \varphi(x), u(t, 0) = u(t, \pi) = 0$)
$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + f(x) u$.
2. The function $u_n(t, x)$ is a solution of the initial boundary value problem ($u_n(0, x) = \varphi(x), u_n(t, 0) = u_n(t, \pi) = 0$)
$\frac{\partial u_n}{\partial t} = \frac{n}{\sqrt{2\pi}} \int_{\mathbb{R}} (u_n(t, x + \frac{\sigma y}{\sqrt{n}}) - u_n(t, x)) e^{-\frac{y^2}{2}} dy + f(x) u$.

Theorem

Suppose that $\mathrm{Re}\sigma^2 \geq 0$, $\varphi(0) = \varphi(\pi) = 0$ *and* $\varphi^{\mathrm{odd}}, f^{\mathrm{even}} \in \mathcal{R}_0$. *Then uniform in* $t \in [0, T]$

$$\lim_{n\to\infty} \|F_n^t\varphi - F^t\varphi\|_{\mathcal{R}_0} = 0.$$

Thank you for your attention!