Random walks that avoid a bounded set

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1. The exit problem for random walks

Let $X_1, X_2, \dots$ be i.i.d. r.v.'s so $S_n := S_0 + X_1 + \dots + X_n$ is a random walk in $\mathbb{R}$. Denote $\mathbb{P}_x(\cdot) := \mathbb{P}(\cdot | S_0 = x)$, $\mathbb{E}_x := \int d\mathbb{P}_x$.

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Let $\tau_B := \inf\{n \geq 1 : S_n \in B\}$ be the hitting time for a Borel set B. A huge number of works studies the asymptotic of $\mathbb{P}_x(\tau_B > n)$ under different assumptions of S_n and B. For example, in $\mathbb{R}$ a rather complete theory had been developed for the half-line $B = (-\infty, 0)$ (from Sparre-Andersen '50s to Rogozin '72). Some recent advances include cones in $\mathbb{R}^d$ (Denisov & Wachtel '12+).

We will assume that B is bounded. Fewer results are available here.

Kesten, Spitzer '63: For *any* aperiodic RW in $\mathbb{Z}^{1,2}$ and any *finite* $B \subset \mathbb{Z}^{1,2}$, there exists

$$\lim_{n\to\infty} \frac{\mathbb{P}_x(\tau_B > n)}{\mathbb{P}_0(\tau_{\{0\}} > n)} := g_B(x).$$

Remark: if S_n is asymptotically α-stable with $1 < \alpha \le 2$, then $\mathbb{P}_0(\tau_{\{0\}} > n) \sim cn^{1/\alpha - 1} L(n)$. Moreover, $L(n) = const$ if $\alpha = 2$.

Remark: $g_B(x)$ is harmonic for the RW S_n killed as it hits B, that is $g_B(x) = \mathbb{E}_x g_B(S_1) \mathbb{1}_{\{\tau_B > 1\}}$ for $x \notin B$.

2. The current setup and a first approach

We assume that $\mathbb{E}X_1 = 0$, $Var(X_1) := \sigma^2 \in (0, \infty)$. Consider the basic case $B = (-d, d)$ for a $d > 0$. Put $p_n(x) := \mathbb{P}_x(\tau_{(-d,d)} > n)$.

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$$p_n(x) \geq \mathbb{P}_x(T_1 > n) \sim \sqrt{\frac{2}{\pi}} \frac{U_d(x)}{\sigma\sqrt{n}}, \quad U_d(x) := x - \mathbb{E}_x S_{T_1}.$$

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Note that $U_d(x) = U_+(x - d)$ for $x \geq d$, where U_+ is the usual renewal function, which is harmonic for the walk killed as it enters $(-\infty, 0)$. The same holds for $U_d(x) = U_-(x + d)$ for $x \leq -d$.

Then

$$
\begin{aligned}
p_n(x) &= \mathbb{P}_x(T_1 > n) + \mathbb{P}_x(T_1 \le \varepsilon n, \tau_{(-d,d)} > n) \\
&+ \mathbb{P}_x(\varepsilon n < T_1 \le (1-\varepsilon)n, \tau > n) + \mathbb{P}_x((1-\varepsilon)n < T_1 \le n, \tau >
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Denoting $H_1 := S_{T_1}$ the first overshoot over $(-d, d)$, we control the 2nd term by

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\mathbb{E}_x p_{(1-\varepsilon)n}(H_1)\mathbb{1}_{\{|H_1|\ge d\}} &\ge \mathbb{P}_x(T_1 \le \varepsilon n, \tau_{(-d,d)} > n) \\
&\ge \mathbb{E}_x p_n(H_1)\mathbb{1}_{\{|H_1|\ge d, T_1 \le \varepsilon n\}}
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The 3rd term is negligible by

$$\mathbb{P}_x(\varepsilon n < T_1 \le (1-\varepsilon)n, \tau > n) \le C\mathbb{P}_x(\varepsilon n < T_1 \le (1-\varepsilon)n) \cdot \mathbb{E}_x p_{\varepsilon n}(H_\infty)$$

given we have a good control of $p_n(x)$ in x.

3. The result for the basic case

Let T_k be the moment of the kth jump over the $(-d, d)$ *from the outside*; let $H_k := S_{T_k}, k \geq 0$ be the overshoots; denote the # of jumps over $(-d, d)$ before it is hit as $\kappa := \min(k \geq 1 : |H_k| < d)$.

Theorem 1 (V., 13)

Let S_n be a random walk with $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 := \sigma^2 \in (0, \infty)$. Then for any $d > 0$ and any $x \in \mathbb{R}$,

$$p_n(x) \sim \sqrt{\frac{2}{\pi}} \frac{V_d(x)}{\sigma\sqrt{n}}, \quad V_d(x) := \sum_{i=1}^{\kappa} |H_i - H_{i-1}|.$$

Moreover, this holds uniformly for $x = o(\sqrt{n})$. Further,

- *$V_d(x)$ is harmonic for the walk killed as it enters $(-d, d)$;*
- *$0 < U_d(x) \leq V_d(x) < \infty$ for $|x| \geq d$;*
- *$V_d(x) \sim |x|$ as $|x| \to \infty$;*
- *$V_d(\pm(d + y)) - U_d(\pm(d + y)) = \mathbb{P}_x(T_1 < n | \tau_{(-d,d)} > n) \to 0$ as $d \to \infty$ for any fixed $y > 0$.*

4. Ideas beyond the proof

1. It costs to jump over:

There exists a $\gamma \in (0, 1)$ such that

$$\mathbb{P}_x(|H_1| \geq d) \leq \gamma.$$

This follows since H_1 converge weakly as $x \to \pm\infty$ to the overshoots over "infinitely remote" levels.

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2. Some regularity of $p_n(x)$ is both x and n is needed.

Lemma

$p_n(x) \leq C|x|n^{-1/2}$ for some $C > 0$ and any $x \in \mathbb{R}$ and $n \geq 1$.

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3. The mechanism of stabilisation:

For any $\alpha \in (0,1)$ it holds that

$$\mathbb{E}_x|H_1| \leq \alpha|x| + K(\alpha), \quad |x| \geq d.$$

This follows from the known $\mathbb{E}_x|H_1| = o(|x|)$ as $|x| \to \infty$.

5. Motivation: the largest gap in the range of a RW

Define the range of RW S_n as

$$G_n := \max_{1 \le k \le n-1} S_{(k+1,n)} - S_{(k,n)},$$

where $S_{(1,n)} \le S_{(2,n)} \le \cdots \le S_{(n,n)} =: M_n$ denote the elements of $S_1, \ldots, S_n$ arranged in the weakly ascending order.

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The following is suggested by Yuval Peres and Jian Ding:

$$\begin{aligned} \mathbb{P}(G_n \ge 2d) &\le 2\sum_{k=1}^{n} \mathbb{P}_0(\tau_{(0,2d)} > k-1, T_1 \le k-1)\mathbb{P}_0(\tau_{(0,2d)} > n-k \\ &= \frac{4}{\sigma^2 \pi} \sum_{k=1}^{n-1} \frac{V_d(-d) - U_d(-d) + o(1)}{\sqrt{k}} \cdot \frac{V_d(-d) + o(1)}{\sqrt{n-k}} \end{aligned}$$

implying that

$$\limsup_n \mathbb{P}(G_n \ge 2d) \le \frac{4}{\sigma^2}\big(V_h(-d) - U_d(-d)\big)V_d(-d).$$

The RHS $\to 0$ as $d \to \infty$ implying that G_n is tight.

6. Generalization for an arbitrary set

Denote $r := \sup\{x : x \in B\}$ and $l := \inf\{x : x \in B\}$ the "edges" of B. Let T'_k be the moments of jumps over the r or l, and put $H'_k := S'_{T_k}, k \geq 0$. Further, denote $\kappa' := \min\{k \geq 1 : T'_k \geq \tau_B\}$. Let M be the state space, that is $M := \lambda\mathbb{Z}$ in the arithmetic case and $M := \mathbb{R}$ if otherwise.

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Theorem 2 (V., 13)

Let S_n be a random walk with $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 := \sigma^2 \in (0, \infty)$, and let $B \subset M$ a bounded non-empty Borel set. In the non-arithmetic case assume that there exist $a, b \in Int(B) \neq \varnothing$ such that

$$\mathbb{P}(X_1 < a - r) > 0, \quad \mathbb{P}(X_1 > b - l) > 0.$$

Then for any $x \in M$,

$$p_n(x) \sim \frac{\sqrt{2} V_B(x)}{\sigma\sqrt{\pi n}}, \quad V_B(x) := \mathbb{E}_x \sum_{i=1}^{\kappa'} \left| H'_i - H'_{i-1} \right| \mathbb{1}_{\{H'_{i-1} \notin Conv(B)\}}.$$

Moreover, this holds uniformly for $x = o(\sqrt{n})$. *Further,*

- $V_{(-d,d)}(x) = V_d(x)$;
- $0 < V_B(x) < \infty$ *for* $|x| \notin Conv(B)$
- $V_B(x) \sim |x|$ *as* $|x| \to \infty$.

7. Heuristics

1. It costs uniformly to jump over $[l, r]$ or hit $[l, r]$ and exit it avoiding B.
2. It costs exponentially in time to stay within $[l, r]$ so the time spent there is negligible.
3. The rest is as in the basic case.