Potential analysis for positive recurrent Markov chains with asymptotically zero drift: Power-type asymptotics.

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Let $\{X_n,\ n \geq 0\}$ be a time homogeneous Markov chains with values in $\mathbb{R}_+$.

Denote by $\xi(x)$ a random variable corresponding to the jump of the chain at point x, that is,

$$\mathbf{P}(\xi(x) \in B) = \mathbf{P}(X_{n+1} - X_n \in B | X_n = x) = \mathbf{P}_x(X_1 \in x + B).$$

Let $m_k(x)$ denote the kth moment of the chain at x, i.e.,

$$m_k(x) := \mathbf{E}\xi^k(x).$$

We consider the case of the **asymptotically zero drift**:

$$m_1(x) \to 0 \quad \text{as } x \to \infty.$$

The existence of invariant distribution (positive recurrence) was studied by

Lamperti(1960, 1963):

If $2xm_1(x) + m_2(x) \leq -\varepsilon$, then the chain is positive recurrent.

Was can be said about this stationary distribution?

Examples of chains with calculable stationary distributions.

- **Diffusion with the drift $m_1(x)$ and the diffusion coefficient $m_2(x)$.**

  The invariant density function solves the Kolmogorov forward equation

$$0 = -\frac{d}{dx}(m_1(x)p(x)) + \frac{1}{2}\frac{d^2}{dx^2}(m_2(x)p(x)),$$

  which has the following solution:

$$p(x) = \frac{C}{m_2(x)} \exp\left\{ \int_0^x \frac{2m_1(y)}{m_2(y)} dy \right\}.$$

- **Markov chains on $\mathbb{Z}_+$ with $|\xi(x)| \le 1$.**

  Set $p_-(x) = \mathbf{P}(\xi(x) = -1)$, $p_+(x) = \mathbf{P}(\xi(x) = 1)$ and $1 - p_-(x) - p_+(x) = \mathbf{P}(\xi(x) = 0)$.

  Then the stationary probabilities $\pi(x)$, $x \in \mathbb{Z}_+$ satisfy

  $$\pi(x) = \pi(x-1)p_+(x-1) + \pi(x)(1-p_-(x)-p_+(x)) + \pi(x+1)p_-(x+1).$$

  Consequently,

  $$\pi(x) = \pi(0)\prod_{k=1}^{x}\frac{p_+(k-1)}{p_-(k)}.$$

Theorem 1. Suppose that, as $x \to \infty$,

$$m_1(x) \sim -\frac{\mu}{x}, \ \ m_2(x) \sim b \text{ and } 2\mu > b.$$

Suppose also that there exists a differentiable function $r(x) > 0$ such that $r'(x) \sim -\frac{2\mu}{bx^2}$ and

$$\frac{2m_1(x)}{m_2(x)} = -r(x) + O(x^{-2-\delta}).$$

Suppose also that

$$\sup_x \mathbf{E}|\xi(x)|^{3+\delta} < \infty, \quad \mathbf{E}[\xi^{2\mu/b+3+\delta}(x); \xi(x) \geq Ax] = O(x^{2\mu/b})$$

and

$$m_3(x) \to m_3 \in (-\infty, \infty)$$

Then there exists a constant $c > 0$ such that

$$\pi(x, \infty) \sim cxe^{-\int_0^x r(y)dy} = cx^{-2\mu/b+1}\ell(x).$$

Menshikov and Popov (1995) investigated Markov chains on $\mathbb{Z}_+$ with bounded jumps: For every $\varepsilon > 0$ there exist constants $c_\pm(\varepsilon)$ such that

$$c_-(\varepsilon)x^{-2\mu/b-\varepsilon} \le \pi(\{x\}) \le c_+(\varepsilon)x^{-2\mu/b+\varepsilon}.$$

Korshunov (2011) has shown that if $\{(\xi^+(x))^{2+\gamma}, x \ge 0\}$ and $\{(\xi^-(x))^2, x \ge 0\}$ are uniformly integrable, then the moment of order γ of the distribution π is finite for $\gamma < 2\mu/b - 1$, and infinite for $\gamma > 2\mu/b - 1$. Consequently, for every $\varepsilon > 0$ there exists $c(\varepsilon)$ such that

$$\pi(x, \infty) \le c(\varepsilon)x^{-2\mu/b+1+\varepsilon}.$$

Based on this result on the existence/nonexistence of moments one can expect that the statement of Theorem 1 should be true under less restrictive moment conditions.

First, we conjecture that $\mathbf{E}[\xi^{2\mu/b+3+\delta}(x);\xi(x)\ge Ax]=O(x^{2\mu/b})$ can be replaced by $\mathbf{E}[\xi^{2\mu/b+1+\delta}(x);\xi(x)\ge Ax]=O(x^{2\mu/b})$.

Second, the convergence of third moments is a technical condition, since the corresponding limit does not appear in the answer.

Theorem 2. Suppose that all conditions of Theorem 1 hold except probably the convergence of third moments. If X_n lives on $\mathbb{Z}_+$ and $\xi(x)\ge -1$, then the statement of Theorem 1 remains valid.

Random walks with delay. Consider a Markov chain given by the recursion

$$W_{n+1} = (W_n + \eta_n)^+, n \geq 0,$$

where η_n are independent identically distributed random variables with $\mathbf{E}\eta_1 < 0$.

This Markov chain is ergodic and, furthermore,

$$W_n \Rightarrow \sup_{n \geq 0} \sum_{k=1}^{n} \eta_k,$$

i.e., the stationary measure of W_n is the distribution of the supremum of the random walk $S_n = \sum_{k=1}^{n} \eta_k$.

If there exists $h > 0$ such that $\mathbf{E}e^{h\eta_1} = 1$, then one can determine π using the following program:

1. Exponential change of measure: One considers a new random walk $\hat{S}_n$ with increments
$$\mathbf{P}(\hat{\eta}_i \in dx) = e^{hx}\mathbf{P}(\eta \in dx);$$
2. Limit theorem for the overshoot of $\hat{S}_n$.

As a result one gets, under the additional assumption $\mathbf{E}\eta_1 e^{h\eta_1} < \infty$,

$$\pi(x, \infty) \sim ce^{-hx}.$$

Asymptotically homogeneous Markov chains.

Assume that

$$\xi(x) \Rightarrow \xi \quad \text{as } x \to \infty$$

and

$$\mathbf{E}e^{h\xi} = 1, \quad \mathbf{E}\xi e^{h\xi} < \infty.$$

Borovkov and Korshunov (1996) have shown that if

$$\sup_x \mathbf{E}e^{h\xi(x)} < \infty \text{ and } \int_0^\infty \left(\int_{\mathbb{R}} e^{ht} |\mathbf{P}(\xi(x) < t) - \mathbf{P}(\xi < t)| dt \right) dx < \infty,$$

then

$$\pi(x, \infty) \sim ce^{-hx}.$$

Korshunov (2004) derived limit theorems for pre-limiting distributions. His method is based on the exponential change of measure and subsequent analysis of non-probabilistic transition kernels.

Let $B = [0, x_0]$ be such that $\pi(B) > 0$ and set $\tau_B := \min\{k \geq 1 : X_k \in B\}$. For the measure π we have

$$\pi(dx) = \int_B \pi(dz) \sum_{n=1}^{\infty} \mathbf{P}_z(X_n \in dx, \tau_B > n),\ x > x_0.$$

If we find a positive function $V(x)$ such that $V(x) = \mathbf{E}_x[V(X_1), \tau_B > 1]$, then we may change the measure:

$$\pi(dx) = \frac{1}{V(x)} \int_B \pi(dz) V(z) \sum_{n=1}^{\infty} \mathbf{P}_z(\widehat{X}_n \in dx) = c_0 \frac{H(dx)}{V(x)},$$

where $\widehat{X}_n$ is a Markov chain with the following transition kernel

$$\mathbf{P}_x(\widehat{X}_1 \in dy) = \frac{V(y)}{V(x)} \mathbf{P}_x(X_1 \in dy, \tau_B > 1)$$

and initial distribution

$$\mathbf{P}(\widehat{X}_0 \in dz) = \frac{1}{c_0} \pi(dz) V(z), \ z \in B.$$

Harmonic function (1).

Let $U(x)$ be positive on (x_0,∞) and zero on $[0,x_0]$. Define

$$u(x)=\mathbf{E}_x[U(X_1)]-U(x).$$

If $\mathbf{E}_x\sum_{n=0}^{\tau_B-1}(u(X_n))^+<\infty$ for all $x>0$, then

$$V(x):=U(x)+\mathbf{E}_x\sum_{n=0}^{\tau_B-1}u(X_n)$$

is well-defined, non-negative and harmonic for X_n:

$$V(x)=\mathbf{E}_x[V(X_1),\tau_B>1].$$

If $U_1(x)\sim U_2(x)$ as $x\to\infty$, then $V_1\equiv V_2$. But it remains uncler, whether $V(x)$ is unique.

Harmonic function (2).

Taylor expansion:

$$\begin{aligned} u(x) &= \mathbf{E}U(x+\xi(x)) - U(x) \\ &= U'(x)m_1(x) + \frac{1}{2}U''(x)m_2(x) + R(x). \end{aligned}$$

All individual properties of $\xi(x)$ are hidden in $R(x)$, and first two terms are “universal”, i.e., they depend on m_1 and m_2 only.

Then we can take U such that

$$U'(x)m_1(x) + \frac{1}{2}U''(x)m_2(x) = 0.$$

Harmonic function (3).

$$m_1(x)U'(x) + \frac{m_2(x)}{2}U''(x) = 0,\ U(x_0) = 0.$$

Consequently,

$$U(x) = \int_{x_0}^{x} e^{R(y)}dy,\ x > x_0$$

where $R(x) := \int_{x_0}^{x} \frac{-2m_1(x)}{m_2(x)}dz.$

If $m_1(x) \sim -\frac{\mu}{x}$ and $m_2(x) \sim b$, then

$$U(x) = x^{2\mu/b+1}\ell(x).$$

$U(x)$ is harmonic for a diffusion with the drift m_1 and the diffusion coefficient $m_2(x)$, which is in the same “universality class” as the original Markov chain.

Harmonic function (4).

Under the conditions of Theorem 1, V generated by U from the previous slide, satisfies

(1) $$V(x) = U(x) + Ce^{R(x)} + o(e^{R(x)}),\ \text{as} x \to \infty.$$

Without the convergence of third moments we have less information on V:

$$V(x) = U(x) + o(U(x)).$$

Using (1) we obtain

$$\mathbf{E}\widehat{\xi}(x) \sim \frac{\mu + b}{x}$$

and

$$\mathbf{E}\widehat{\xi}^2(x) \sim b.$$

Consequently,

$\widehat{X}_n$ is transient.

Renewal theorem (1).

Let $\widehat{X}_n$ be a transient chain with

$$\mathbf{E}\widehat{\xi}(x) \sim \frac{\widehat{\mu}}{x}, \quad \mathbf{E}\widehat{\xi}^2(x) \sim \widehat{b}.$$

Then we have

$$\frac{\widehat{X}_n^2}{n} \Rightarrow \gamma,$$

where γ has the Γ-distribution with mean $2\widehat{\mu} + \widehat{b}$ and variance $2\widehat{b}(2\widehat{\mu} + \widehat{b})$.

Renewal theorem (2).

$$\begin{aligned} H(x) :&= \sum_{n=1}^{\infty} \mathbf{P}(\widehat{X}_n \le x) \ge \sum_{n=1}^{Tx^2} \mathbf{P}(\widehat{X}_n^2/n \le x^2/n) \\ &= \sum_{n=1}^{Tx^2} \mathbf{P}(\gamma \le x^2/n) + o(x^2) \\ &= x^2 \int_0^T \mathbf{P}(\gamma \le 1/z)dz + o(x^2). \end{aligned}$$

Noting that

$$\lim_{T\to\infty} \int_0^T \mathbf{P}(\gamma \le 1/z)dz = \frac{1}{2\widehat{\mu} - \widehat{b}},$$

we obtain

$$\liminf_{x\to\infty} \frac{H(x)}{x^2} \ge \frac{1}{2\widehat{\mu} - \widehat{b}}.$$

Renewal theorem (3).

We can generalise the lower bound to the asymptotics:

$$H(x) \sim \frac{x^2}{2\widehat{\mu} - \widehat{b}} \quad \text{as } x \to \infty.$$

Proof of Theorem 1:

$$
\begin{aligned}
\pi(x,\infty) &= c_0 \int_x^\infty \frac{H(dy)}{V(y)} \\
&\sim c_0 \int_x^\infty \frac{H(dy)}{U(y)} \\
&= c_0 \left(-\frac{H(x)}{U(x)} + \int_x^\infty \frac{H(y)U'(y)}{U^2(y)} dy \right) \\
&\sim c \frac{x^2}{U(x)}.
\end{aligned}
$$

Harmonic functions vs Lyapunov functions.

Lyapunov functions. We choose an explicit function: x^a, e^{hx}, $x^2 \log x$. Therefore, there are no problems with regularity properties. Usually one hopes to get either submartingale or supermartingale, which can be used to obtain lower and upper bounds.

Harmonic functions. Explicit expressions are known in special cases only. One has to derive all needed properties. Harmonic functions lead to martingales and, therefore, can be used by deriving asymptotics.

References:

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- Denisov, D. and Wachtel, V. Exit times for integrated random walks. ArXiv: 1207.2270.

Further developments.

We are going to consider a problem with the following moment conditions:

$$m_1(x) \sim -\frac{1}{x^\beta},\ \beta \in (0,1) \quad \text{and} \quad m_2(x) \sim b.$$

Here one expects that

$$\pi(x,\infty) \approx \exp\{-cx^{1-\beta}\}$$

and, respectively,

$$V(x) \approx \exp\{cx^{1-\beta}\}.$$