Asymptotic error distributions of numerical methods for SDEs with jumps

Hanchao Wang, Zhengyan Lin

Zhejiang University

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1. Introduction

In this talk, we consider the following stochastic differential equations (SDE)

$$X_t = x_0 + \int_0^t f(X_{s-})dY_s \tag{1}$$

where f is a continuous function, Y is a semimartingale with jumps.

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According to the formula

$$X_0^n = x_0, \quad X_{i/n}^n = X_{(i-1)/n}^n + f(X_{(i-1)/n}^n)(Y_{i/n}^n - Y_{(i-1)/n}^n),$$

the approximated solution of (1) is defined at the time i/n by induction on the integer i.

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The main gap is to find a rate u_n, a sequence going to ∞, such that

$$u_n U_t^n = u_n(X_{[nt]/n}^n - X_{[nt]/n})$$

admits nondegenerate limiting process.

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They gave the sharp rate $u_n = \sqrt{n}$ when the continuous martingale part presented in Lévy processes. When continuous martingale part vanish, the results are quite different.

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He employed the independent and stationary properties of increments of Lévy processes to obtain the results. However, when we study same problem for more general Itô semimartingale, it is more difficult to get the similar results following the same line.

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According to the formula

$$\begin{aligned} \widetilde{X}_0^n &= x_0, \\ \widetilde{X}_{i/n}^n &= \widetilde{X}_{(i-1)/n}^n + f(\widetilde{X}_{(i-1)/n}^n)(Y_{i/n}^n - Y_{(i-1)/n}^n) \\ &\quad + f'f(\widetilde{X}_{(i-1)/n}^n)\int_{(i-1)/n}^{i/n}(Y_s - Y_{(i-1)/n})dY_s \end{aligned}$$

Milstein scheme is defined.

Yan (2005) studied the asymptotic error distribution of SDEs driven by continuous process. The second work in this talk is to study the case of SDEs driven by semimartingale with jump, i.e., to study the weak convergence of

$$\widetilde{U}_t^n = u_n(\widetilde{X}_{[nt]/n}^n - X_{[nt]/n})$$

2. Euler scheme for stochastic differential equation driven by pure jump semimartingales

Assumption 1:

$$Y_t = \int_0^t \sigma_{s-} dZ_s$$

where

a) Z is a non-homogeneous Lévy process with spot characteristics $(b_t^Z, 0, G_t)$, $b_t'^Z = b_t^Z - \int_{\{|x|\leq 1\}} xG_t(dx)$ there are constant $\alpha \in (0,2)$ and two functions θ_t^+, $\theta_t^- \geq 0$ on $\mathbb{R}_+$ such that,

$$\lim_{x \downarrow 0} \sup_{0 \le t \le 1} |x^{\alpha} \overline{G}_t^{\pm}(x) - \theta_t^{\pm}| = 0,$$

where $\overline{G}_t^{+}(x) = G_t((x, \infty))$, $\overline{G}_t^{-}(x) = G_t((-\infty, -x))$and b_t^Z is locally bounded, θ_t^{+}, θ_t^{-} are Riemann integrable over each finite interval.

b) The process σ is an Itô semimartingale with spot characteristics $(b_t^\sigma, c_t^\sigma, F_t^\sigma)$, which are such that the processes b_t^σ, c_t^σ and $\int(x^2 \wedge 1)F_t^\sigma(dx)$ are locally bounded.

Assumption 2 f is a C^3 (three times differentiable) function.

Theorem 1: Under Assumption 2 and in the following cases, the sequence $(Y, u_n U^n)$ converges in law to (Y, U), where U is the unique solution of the linear equation

$$U_t = \int_0^t f'(X_{s-}) U_{s-} dY_s - W_t$$

and where W can be described as follows:

Case I. Under Assumption 1 with $\alpha > 1$, then $u_n = (\frac{n}{\log n})^{1/\alpha}$, and

$$W_t = \int_0^t f(X_{s-})f'(X_{s-})\sigma_{s-}^2 dV_s$$

where V is another Lévy process, independent of Z, with spot characteristics $(b_t^V, 0, G_t^V)$ given by

$$b_t^V = \frac{-\alpha(\theta_t')^2}{2^{(1-\alpha)}(\alpha-1)},$$

$$G_t^V(dx) = \frac{\alpha(\theta_t')^2}{2^{1-\alpha}}[((\theta_t^+)^2 + (\theta_t^-)^2)1_{\{x>0\}} + 2\theta_t^+\theta_t^- 1_{\{x>0\}}]\frac{1}{|x|^{1+\alpha}}dx.$$

Case II. Under Assumption 1 with $\alpha = 1$, then $u_n = \frac{n}{(\log n)^2}$, and

$$W_t = \frac{-\theta_t'^2}{4} \int_0^t f(X_{s-}) f'(X_{s-}) \sigma_{s-}^2 \, ds.$$

Case III. Under Assumption 1, $\alpha < 1$ with $b_t'^Z = 0$ then $u_n = (\frac{n}{\log n})^{1/\alpha}$, and

$$W_t = \int_0^t f(X_{s-})f'(X_{s-})\sigma_{s-}^2 dV_s$$

where V is another Lévy process, independent of Z, with spot characteristics $(b_t^V, 0, G_t^V)$ given by

$$b_t^V = 0, \quad G_t^V(dx) = \frac{\theta_t^2 \alpha}{4|x|^{1+\alpha}} dx.$$

3. Milstein scheme for SDEs driven by semimartingale with jumps

Assumption 3:

$$Y_t = \int_0^t \sigma_{s-} d\widetilde{Z}_s$$

where

(a) $\widetilde{Z}$ is a Lévy process with spot characteristics (b, c, F), where $b \in \mathbb{R}$, $c > 0$, F is a positive measure on $\mathbb{R}$ with $\int (x^2 \wedge 1) F(dx) < \infty$. The continuous local martingale part of $\widetilde{Z}$ is $\sqrt{c} W'$, W' is a standard Brownian motion.

b) The process σ is an Itô semimartingale with spot characteristics $(b_t^\sigma, c_t^\sigma, F_t^\sigma)$, which are such that the processes b_t^σ, c_t^σ and $\int (x^2 \wedge 1) F_t^\sigma(dx)$ are locally bounded, and $\int_0^1 \sigma_{t-}^6 dt < \infty$ a.s..

Theorem 2: Under Assumptions 2 and 3, the sequence $(Y, n\widetilde{U}^n)$ converges in law to $(Y, \widetilde{U})$, where $\widetilde{U}$ is the unique solution of the linear equation

$$\widetilde{U}_t = \int_0^t f'(X_{s-})\widetilde{U}_{s-}dY_s - M_t$$

and where M can be described as follows:

$$\begin{aligned} M_t &= \frac{\sqrt{6}cc}{6}\int_0^t f^2(X_{s-})f'(X_{s-})\sigma_{s-}^3 dB_s \\ &+\frac{\sqrt{3}cc}{6}\int_0^t f^2(X_{s-})f''(X_{s-})\sigma_{s-}^3 dW_s \\ &+\frac{\sqrt{c}}{2}\sum_{n:S_n\leq t}[\sqrt{\chi_n}N_n'(f^2f')(X_{S_n-}) \\ &+\sqrt{1-\chi_n}N_n''f^2(X_{S_n-})\int_0^1 f'(X_{S_n-}+u\Delta X_{S_n})du](\Delta Y_{S_n})^2 \\ &+\frac{\sqrt{c}}{2}\sum_{n:S_n\leq t}[\sqrt{\chi_n}N_n'(f^2f'')(X_{S_n-}) \\ &+\sqrt{1-\chi_n}N_n''f^2(X_{S_n-})\int_0^1 f''(X_{S_n-}+u\Delta X_{S_n})du](\Delta Y_{S_n})^2 \end{aligned}$$

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$(N'_n)_{n\geq 1}$ and $(N''_n)_{n\geq 1}$ are two sequence of standard normal variables.

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$(S_n)_{n\geq 1}$ is an arbitrary ordering of all jump times of $\widetilde{Z}$.

4. The outline of proof

For Theorem 1: note that,

$$U_t^n = \sum_{i=1}^{[nt]} \int_{(i-1)/n}^{i/n} (f(X_{(i-1)/n}^n) - f(X_{(i-1)/n}))dY_s - \sum_{i=1}^{[nt]} \int_{(i-1)/n}^{i/n} (f(X_{s-}) - f(X_{(i-1)/n}))dY_s$$

and set $\overline{Y}_t^n = Y_{[nt]/n}$, $\overline{X}_t^n = X_{[nt]/n}$, and

$$W_t^n = \sum_{i=1}^{[nt]} \int_{(i-1)/n}^{i/n} (f(X_{s-}) - f(X_{(i-1)/n}))dY_s.$$

We obtain

$$U_t^n = \int_0^t (f(\overline{X}_{s-}^n + U_{s-}^n) - f(\overline{X}_{s-}^n))d\overline{Y}_s^n - W_t^n.$$

By Theorem 2.2 in Jacod (2004), the convergence of $(Y, u_n W^n)$ can implies weak convergence of $(Y, u_n U^n)$.
we need to introduce a sequence $v_n \to 0$,
Case I $v_n = \frac{\log n}{n^{1/(2\alpha)}}$, Case II $v_n = \frac{\log n}{n}$, Case III $v_n = (\frac{\log n}{n})^{1/\alpha}$.

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Denote the successive jump times of Z, after $\frac{i-1}{n}$ and of size bigger than or equal to v_n by $T(n,i)_p$. Let $K(n,i)$ be the integer such that $T(n,i)_{K(n,i)} \leq \frac{i}{n} < T(n,i)_{K(n,i)+1}$.

$$I : u_n f f'(X_{(i-1)/n})\Delta Y_{T(n,i)_1} \widetilde{M}^{n,i}_{i/n} 1_{\{K(n,i)\geq 1\}},$$
$$II : u_n f f'(X_{(i-1)/n}) \int_{I(n,i)} (\widetilde{A}^{n,i}_{s-} + \Delta Y_{T(n,i)_1} 1_{\{K(n,i)\geq 1\}}) d\widetilde{A}^{\upsilon_n}_s,$$
$$III : u_n f f'(X_{(i-1)/n})\Delta Y_{T(n,i)_1} \Delta Y_{T(n,i)_2} 1_{\{K(n,i)\geq 2\}}.$$

are the main convergence parts of W^n, where

$$A^{\upsilon_n} = b^Z - x1_{\{\upsilon \le |x| \le 1\}} * \nu^Z, \; M^{\upsilon_n} = x1_{\{|x| \le \upsilon\}} * (\mu^Z - \nu^Z),$$

$$\widetilde{A}_t^{\upsilon_n} = \int_0^t \sigma_{s-} dA_s^{\upsilon_n}, \; \widetilde{M}_{i/n}^{n,i} = \int_{(i-1)/n}^{i/n} \sigma_{s-} dM_s^{\upsilon}.$$

For Theorem 2:

$$d\widetilde{U}^n_t = \widetilde{U}^n_{t-} f'(X_{t-}) dY_t$$
$$-f^2 f'(X_{t-}) d(nG^n_t) - \frac{1}{2} f^2 f''(X_{t-}) d(nH^n_t)$$

is the key to prove, where

$$G^n_t = \int_0^t \int_{n(s)}^s (Y_r - Y_{n(r)}) dY_r dY_s, H^n_t = \int_0^t (Y_s - Y_{n(s)})^2 dY_s,$$

$n(s) = k/n$ if $k/n < s \leq (k+1)/n$.

For the jump part, the main convergence parts are

$$\int_0^t \int_{n(s)}^s (A_r^\varepsilon - A_{n(r)}^\varepsilon) dA_r^\varepsilon d\widetilde{Z}_s^c, \int_0^t (A_r^\varepsilon - A_{n(r)}^\varepsilon)^2 d\widetilde{Z}_s^c,$$

where A^ε is the jump part of $\widetilde{Z}$ and of size bigger than or equal to ε, $\widetilde{Z}^c$ is the continuous local martingale part of $\widetilde{Z}$. For the continuous part, Yan (2005) have already proved.

5. Main references

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Thank you!