Entropy, from Boltzmann H-theorem to Perelman's W-formula for Ricci flow

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Outline

1. Perelman's W-entropy formula for Ricci flow
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3. Perelman's W-entropy for the Witten Laplacian
4. Some open problems

The Poincaré conjecture

Conjecture (H. Poincaré 1904)

Every compact and simply connected 3-dimensional (smooth) manifold is homeomorphic (diffeomorphic) to S^3.

Higher dimensional Poincaré conjecture:

- S. Smale proved the generalized Poincaré conjecture for $n \geq 5$. He was awarded the 1966 Fields Medal.
- M. Freedman proved the generalized Poincaré conjecture for $n = 4$. He was awarded the 1986 Fields Medal.

To prove the Poincaré conjecture for 3-dimensional manifolds, R. Hamilton (JDG1982) initiated the method of using the Ricci flow equation to deform the Riemannian metric on Riemannian manifolds.

Hamilton's Ricci flow

Let M be a compact manifold with a Riemannian metric g. The Ricci flow (RF) is a nonlinear heat equation for the evolution of Riemannian metric on M defined as follows:

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2\mathrm{Ric}_{g(t)}, \\ g(0) &= g. \end{aligned}$$

More precisely, for all $i, j = 1, \ldots, n$,

$$\begin{aligned} \frac{\partial g_{ij}(t)}{\partial t} &= -2\mathrm{R_{ij}}(g(t)), \\ g_{ij}(0) &= g_{ij}. \end{aligned}$$

which is a system of nonlinear 2nd order weak parabolic equations.

Hamilton's theorems

Theorem (Hamilton 1982)

Given a compact Riemannian manifold (M, g_0), there exists $T > 0$ such that the Ricci flow equation

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}_{g(t)}, \quad t \geq 0$$

has a unique solution $g(t, x)$ in $[0, T) \times M$ such that

$$g(0) = g_0.$$

Hamilton's theorems

Theorem (Hamilton 1982)

Let M be a 3-dimensional compact manifold, g_0 a Riemannian metric on M with positive Ricci curvature . Then the normalized (i.e., the volume-preserving) Ricci flow equation

$$\frac{\partial}{\partial t} g(t) = \frac{2r}{n} g(t) - 2\mathrm{Ric}_{g(t)},$$

where

$$r = \frac{\int_M R dv}{V(M)},$$

has a global solution $g(t)$ on $[0, \infty) \times M$ such that

$$g(0) = g_0.$$

Moreover, $g(t)$ converges to a Riemannian metric of constant positive Ricci (and hence sectional) curvature.

Perelman's W-entropy on Poincaré conjecture

- In 2002-2003, G. Perelman posted three papers on Arxiv for the proof of Poincaré conjecture and Thurston's geometrization conjecture.
- In August 2006, Perelman was awarded the Fields medal at ICM 2006 Madrid for for "his contributions to geometry and his revolutionary insights into the analytical and geometric structure of the Ricci flow." However, Perelman declined to accept the award.
- On 18 March 2010, it was announced that he had met the criteria to receive the first Clay Millennium Prize for resolution of the Poincaré conjecture.
- On 1 July 2010, he turned down the prize of one million dollars, saying that he considers his contribution to proving the Poincaré conjecture to be no greater than that of Richard Hamilton, who introduced the theory of Ricci flow with the aim of attacking the geometrization conjecture

Perelman's modified Ricci flow

G. Perelman: The entropy formula for the Ricci flow and its geometric applications. http://arXiv.org/abs/maths0211159.

Let

$$\mathcal{M} = \{g : \text{Riemannian metrics on } M\}.$$

Define

$$\mathcal{F} : \mathcal{M} \times C^\infty(M) \longrightarrow \mathbb{R}$$

$$\mathcal{F}(g, f) := \int_M (R + |\nabla f|^2) e^{-f} dv,$$

$$R = \text{the scalar curvature of } g.$$

Theorem (Perelman2002)

The gradient flow of $\mathcal{F}$ on $\mathcal{M} \times C^\infty(M)$, with the constraint condition that

$$\frac{dm}{dx} = e^{-f}\sqrt{detg} \text{ is fixed,}$$

is given by the modified Ricci flow (MRF)

$$\begin{aligned} \frac{\partial}{\partial t}g &= -2(Ric_g + \nabla^2 f), \\ \frac{\partial}{\partial t}f &= -\Delta f - R. \end{aligned}$$

Remark The quantity $Ric_g + \nabla^2 f$ is called the Bakry-Emery Ricci curvature on $(M, g, e^{-f}dv)$. It was introduced by Bakry and Emery in 1984 when they studied logarithmic Sobolev inequalities and Poincaré inequalities for diffusion processes on Riemannian manifolds with weighted volume measures.

Perelman's modified Ricci flow

Theorem (Perelman2002)

Let $(g(t), f(t))$ be the solution of the Ricci flow (obtained via a time-dependent change of diffeomorphism on M)

$$\begin{aligned} \partial_t g &= -2Ric_g, \\ \partial_t f &= -\Delta f + |\nabla f|^2 - R. \end{aligned}$$

Then

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_M |Ric + \nabla^2 f|^2 e^{-f} dv.$$

In particular, $\mathcal{F}(g(t), f(t))$ is nondecreasing in time and the monotonicity is strict except that

$$Ric + \nabla^2 f = 0 \ \ (\text{steady Ricci soliton}).$$

Perelman's modified Ricci flow

(M, g) is called a Ricci soliton if there exist a function $f \in C^\infty(M)$ and some $\lambda \in \mathbb{R}$ such that

$$Ric + \nabla^2 f = \lambda g.$$

$$\begin{aligned} \lambda &> 0, && \text{shrinking Ricci soliton} \\ \lambda &= 0, && \text{steady Ricci soliton} \\ \lambda &< 0, && \text{expanding Ricci soliton.} \end{aligned}$$

Theorem (Hamilton1995, Ivey1993)

Every compact Riemannian Ricci steady or expanding soliton must be Einstein.

Perelman's $\mathcal{W}$-entropy

To study shrinking solitons, Perelman introduced the following important functional

$$\mathcal{W}(g, f, \tau) = \int_M \left[\tau(R + |\nabla f|^2) + f - n\right] \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv$$

and called it the W-entropy associated with the Ricci flow.

Perelman's W-entropy formula for Ricci flow

Theorem (Perelman2002)

Let M be a compact manifold. Let $g(t), f(t), \tau(t), t \in [0, T)$ be the solution of

$$\begin{aligned} \partial_t g &= -2Ric, \\ \partial_t f &= -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \\ \partial_t \tau &= -1. \end{aligned}$$

Then

$$\frac{d}{dt}\mathcal{W}(g, f, \tau) = 2\tau \int_M \left| Ric + \nabla^2 f - \frac{g}{2\tau}\right|^2 \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv.$$

In particular, $\mathcal{W}(g, f, \tau)$ is nondecreasing in time and the monotonicity is strict unless that (M, g) is a shrinking Ricci soliton

$$Ric + \nabla^2 f = \frac{g}{2\tau}.$$

Application to the resolution of Poincaré conjecture

- The monotonicity of the W-entropy plays a crucial role in the proof of the no local collapsing theorem, which is equivalent to the long time standing Hamilton's Little Loop Conjecture.
- As pointed out by Perelman (2002), this ”removes the major stumbling block in Hamilton's approach to geometrization”.

How to understand Perelman's $\mathcal{W}$-entropy

- What is the hidden insight for Perelman when he introduced his W-entropy functional?

$$\mathcal{W}(g, f, \tau) = \int_M \left[\tau(R + |\nabla f|^2) + f - n\right] \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv.$$

- What is the role of the Gaussian heat kernel in Perelman's W-entropy functional?
- What is the role of the dimension $n = \mathrm{dim} M$ in Perelman's W-entropy functional?
- Is there some relationship between Perelman's entropy and Boltzmann's entropy?

Boltzmann equation and H-theorem

In 1872, Boltzmann introduced the kinetic equation of idea gas, now called the Boltzmann equation. More precisely, let $f(x, v, t)$ be the probabilistic distribution of the ideal gas at time t, at position x and with velocity v, then f satisfies the Bolztmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f, f)$$

where the collision term is defined by

$$Q(f, f) = \int_{\mathbb{R}^3} \int_{S^2} [f(x, v')f(x, v'_*) - f(x, v)f(x, v_*)]B(v - v_*, \theta)dv_* dS(u),$$

where dS denotes the surface measure on S^2, v', v'_* are defined in terms of v, v_*, u by

$$v' = v - [(v - v_*) \cdot u]u, \quad v'_* = v_* + [(v - v_*) \cdot u]u,$$

and $B : \mathbb{R}^3 \times S^2 \to (0, \infty)$ is the collision kernel and is assumed to be rotationally invariant, i.e., $B(z, u) = B(|z|, |z \cdot u|), \forall z \in \mathbb{R}^3$ and $u \in S^2$.

Boltzmann equation and H-theorem

In the same paper, Boltzmann introduced the H-quantity (now called the Bolztmann H-entropy)

$$H(t) = -\int_{\mathbb{R}^6} f(x,v,t)\log f(x,v,t)dxdv.$$

The Bolztmann H-Theorem states that: if $f(x,v,t)$ is the solution of the Boltzmann equation, which is "sufficiently well-behaved", then the following formula holds

$$\frac{dH}{dt} = \frac{1}{4}\int_{S^2}\int_{\mathbb{R}^3}(f'f'_* - ff_*)[\log(f'f'_*) - \log(ff_*)]B(v-v_*,\theta)dv_*dS(u),$$

where $f = f(\cdot,v,\cdot)$, $f' = f(\cdot,v',\cdot)$, $f_* = f(\cdot,v_*,\cdot)$, $f'_* = f(\cdot,v'_*,\cdot)$.

Boltzmann entropy and H-theorem

By the Boltzmann formula for the H-entropy formula and using the elementary inequality

$$(x - y)(\log x - \log y) \geq 0, \quad \forall x, y \in \mathbb{R}^+,$$

one can conclude that H is always nondecreasing in time, i.e.,

$$\frac{dH}{dt} \geq 0, \quad \forall t > 0,$$

and the equality holds if and only if

$$f' f'_* = f f_*, \quad \forall v, v_* \in \mathbb{R}^3, u \in S^2,$$

which implies that f is a local Maxwell distribution, i.e.,

$$f(x, v, t) = n(x, t) \left(\frac{m}{2\pi k T(x, t)} \right)^{3/2} \exp \left(- \frac{|v - \bar{v}(x, t)|^2}{2kmT(x, t)} \right),$$

where the parameters m is the mass of particle, $n(x, t) \in \mathbb{R}^3$ is the particle density at (x, t), $\bar{v}(x, t) \in \mathbb{R}^3$ and $T(x, t) > 0$ are the mean velocity and the local temperature at (x, t).

Boltzmann's statistical interpretation of entropy

In 1877, L. Boltzmann gave the statistical definition of entropy S by the formula

$$S = k \log W,$$

where k is the so-called Boltzmann constant, and W is the number of possible microstates corresponding to the macroscopic state of a system.

The Bolztmann entropy formula gives the logarithmic connection between Clausius' thermodynamic entropy S and the number W of the most probable microstates consistent with the given macrostate.

Max Planck 1901

Canonical ensemble with density of state measure

Let us consider a canonical ensemble with the Maxwell-Boltzmann distribution

$$d\mu_\beta(E) = \frac{1}{Z_\beta} g(E) e^{-\beta E} dE,$$

where dE denotes the Lebesgue measure on $\mathbb{R}^+$, $g(E)$ is called the "density of states" so that $g(E)dE$ represents the number of states with energy between E and $E + dE$ per unit volume, and the partition function at the temperature β^{-1} is defined by

$$Z_\beta = \int_{\mathbb{R}^+} g(E) e^{-\beta E} dE$$

.

Entropy of the Maxwell-Boltzmann distribution

By Boltzmann's statistical mechanics interpretation of the entropy,

$$S = \lim_{N\to\infty} \frac{\log W}{N} = \log Z_\beta + \beta\langle E\rangle,$$

where

$$\langle E\rangle = -\frac{\partial}{\partial\beta} \log Z_\beta.$$

In other words, the entropy of the Maxwell-Boltzmann distribution is given by

$$S = \log Z_\beta - \beta\frac{\partial}{\partial\beta} \log Z_\beta.$$

Moreover, the fluctuation of the energy E with respect to the Maxwell-Boltzmann distribution μ_β is given by

$$\langle(E - \langle E\rangle)^2\rangle = \frac{\partial^2}{\partial\beta^2} \log z_\beta.$$

Perelman's statistical interpretation of W-entropy

Let M be a closed manifold, (g_{ij}) and f be the solution of the modified Ricci flow equation and the conjugate heat equation. Suppose that there exists a canonical ensemble with a "density of states" measure for which the partition function is given by

$$\log Z_\beta = \int_M \left(\frac{n}{2} - f\right) dm,$$

where

$$dm = udV, \quad u = (4\pi\tau)^{-n/2}e^{-f}, \quad \beta^{-1} = \tau = T - t, \quad t \in (0, T).$$

By the Boltzmann entropy formula

$$S = \log Z_\beta - \beta \frac{\partial}{\partial \beta} \log Z_\beta,$$

Perelman proved that

$$S = -\int_M \left(\tau(R + |\nabla f|^2) + f - n\right) dm.$$

Therefore

$$W = -S.$$

H-theorem vs W-entropy formula

The Boltzmann H-entropy formula and the Perelman W-entropy formula are in the same spirit in the following three points.

- The Boltzmann H-entropy formula gives the time derivative formula of the Boltzmann H-entropy along the solutions of the Boltzmann equation, and the Perelman W- entropy formula gives the time derivative formula of the Perelman W -entropy along the solutions of the Ricci flow equation and the conjugate heat equation.
- The Boltzmann H-entropy is monotone along the Boltzmann equation, and the Perelman W-entropy is monotone along the Ricci flow and the conjugate heat equation.
- From the Boltzmann H-entropy formula and the Perelman W-entropy formula, we can derive that the equilibrium state of the Boltzmann H-entropy is the local Maxwell distribution, and the equilibrium state of the Perelman W-entropy is the shrinking Ricci solitons.

H-theorem vs W-entropy formula

- The problem of the convergence rate of the solutions of the Boltzmann equation towards the equilibrium state is the famous Cercignani conjecture: .... C. Villani (2010 Fields Medal).
- The longtime behavior of the Ricci flow via Perelman's W-entropy.

Problem (Perelman2002)

If the flow is defined for all sufficiently large τ (that is, we have an ancient solution to the Ricci flow, in Hamilton's terminology), we may be interested in the behavior of the entropy S as $\tau \to \infty$. A natural question is whether we have a gradient shrinking soliton whenever S stays bounded.

Ni's entropy formula

Theorem (Ni 2005)

Let (M, g) be a compact Riemannian manifold with a fixed metric. Let

$$u = \frac{e^{-f}}{(4\pi t)^{n/2}}$$

be a positive solution of

$$\partial_t u = \Delta u.$$

Let

$$W(u,t) = \int_M \left(t|\nabla f|^2 + f - n\right) \frac{e^{-f}}{(4\pi t)^{n/2}} dv.$$

Then

$$\frac{d}{dt} W(u,t) = -2 \int_M t \left(\left| \nabla^2 f - \frac{g}{2t} \right|^2 + Ric(\nabla f, \nabla f) \right) \frac{e^{-f}}{(4\pi t)^{n/2}} dv.$$

In particular, if $Ric \geq 0$, then $W(u,t)$ is decreasing in time t.

Witten Laplacian and Bakry-Emery Ricci curvature

Let (M, g) be a complete Riemannian manifold, $\phi \in C^2(M)$. Let us consider a weighted volume measure on M:

$$d\mu = e^{-\phi} dvol.$$

Then the integration by parts formula holds: $\forall u, v \in C_0^\infty(M)$

$$\int_M \langle \nabla u, \nabla v \rangle d\mu = \int_M (-Lu) v d\mu = \int_M u(-Lv) d\mu.$$

where L is the weighted Laplacian (called the Witte Laplacian) with respect to μ, i.e.,

$$L = \Delta - \nabla\phi \cdot \nabla.$$

W-entropy for the Witten Laplacian

Inspired by the works of Perelman (2002) and Ni (2005), we have

Theorem (L. Math Ann2012)

Let $u = \frac{e^{-f}}{(4\pi t)^{m/2}}$ *be a positive solution of* $\partial_t u = Lu$. *Let*

$$H_m(u,t) := -\int_M u \log u d\mu - \frac{m}{2}\left(\log(4\pi t) + 1\right),$$

and

$$W(u,t) := \frac{d}{dt}(tH_m(u,t)).$$

Then

$$\frac{d}{dt}H_m(u,t) = -\int_M \left(L \log u + \frac{m}{2t}\right) u d\mu,$$

and

$$W(u,t) = \int_M \left(t|\nabla f|^2 + f - m\right) \frac{e^{-f}}{(4\pi t)^{m/2}} d\mu.$$

Li-Yau Harnack inequality

Theorem (L. JMPA2005, Math Ann2012)

Let u *be a positive solution of the heat equation*

$$\left(\frac{\partial}{\partial t} - L\right) u = 0.$$

Suppose that

$$Ric_{m,n}(L) := Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n} \geq 0.$$

Then the Li-Yau Harnack inequality holds

$$L \log u + \frac{m}{2t} \geq 0.$$

Equivalently

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{m}{2t}.$$

Probabilistic interpretation of Perelman's W-entropy

The Boltzmann H-entropy of the Gaussian heat kernel measure on $\mathbb{R}^m$

$$d\mu_t(x) = f(x,t)dx = \frac{e^{-\frac{\|x\|^2}{4t}}}{(4\pi t)^{m/2}}dx, \quad t > 0$$

is given by

$$\begin{aligned} H(\mu_t) &= -\int_{\mathbb{R}^m} f(x,t)\log f(x,t)dx \\ &= \frac{m}{2}(1+\log(4\pi t)). \end{aligned}$$

Probabilistic interpretation of Perelman's W-entropy

Hence

$$H_m(u,t) = -\int_M u\log u d\mu - \frac{m}{2}(\log(4\pi t)+1)$$

is the difference of the Boltzmann H-entropy

$$H(u,t) = -\int_M u\log u d\mu$$

of the heat equation $\partial_t u = Lu$ on M and the Boltzmann H-entropy $H(\mu_t)$ of the Gaussian heat kernel on $\mathbb{R}^m$.

Moreover, the definition formula

$$W(u,t) = \frac{d}{dt}(tH_m(u,t))$$

gives the probabilistic interpretation of Perelman's W-entropy for the Witten Laplacian. See Li (Math Ann2012).

Probabilistic interpretation of Perelman's W-entropy

Let

$$H_n(u,\tau) = -\int_M u(\tau)\log u(\tau)dvol_{g(\tau)} - \frac{n}{2}(\log(4\pi\tau)+1)$$

be the difference of the Boltzmann H-entropy

$$H(u,t) = -\int_M u\log u dvol_{g(\tau)}$$

of the conjugate heat equation on M with Ricci flow metric $g(\tau)$

$$\partial_\tau u = \Delta u + Ru,$$

and the Boltzmann H-entropy $H(\mu_t)$ on $\mathbb{R}^n$. Then

$$W(u,\tau) = \frac{d}{d\tau}(\tau H_n(u,\tau)).$$

Probabilistic interpretation of Perelman's W-entropy

The above probabilistic interpretation is equivalent to Perelman's statistical interpretation using the Boltzmann formula in statistical mechanics:
Indeed, let $\beta = \tau^{-1}$ and notice that $\log Z_\beta = -H_n(u, t)$. Now

$$\frac{d}{d\tau} = \frac{d}{d\beta}\frac{d\beta}{d\tau} = -\frac{1}{\tau^2}\frac{d}{d\beta} = -\beta^2\frac{d}{d\beta}.$$

Hence

$$\begin{aligned} W(u, \tau) &= -\frac{d}{d\tau}(\tau H_n(u, \tau)) \\ &= \beta^2 \frac{d}{d\beta}\left(\frac{1}{\beta}\log Z_\beta\right) \\ &= -\log Z_\beta + \beta\frac{\partial}{\partial\beta}\log Z_\beta \\ &= -S. \end{aligned}$$

Perelman's W-entropy formula for Witten Laplacian

Theorem (L. Math Ann2012)

Let M be a complete Riemannian manifold with bounded geometry condition, $\phi \in C^4(M)$ with $\nabla\phi \in C_b^3(M)$. Let

$$u(t,x) = \frac{e^{-f}}{(4\pi\tau)^{m/2}}$$

be the fundamental solution of the heat equation

$$\partial_t u = Lu.$$

Then

$$\begin{aligned}
\frac{dW(u,t)}{dt} &= -2\int_M \left(\tau\left|\nabla^2 f - \frac{g}{2\tau}\right|^2 + Ric_{m,n}(L)(\nabla f, \nabla f)\right) u d\mu \\
&\quad - \frac{2}{m-n}\int_M \tau\left(\nabla\phi \cdot \nabla f + \frac{m-n}{2\tau}\right)^2 u d\mu.
\end{aligned}$$

Perelman's W-entropy formula for Witten Laplacian

Corollary (L. Math Ann. 2012)

Suppose that there exists a constant $m \geq n$ such that

$$Ric_{m,n}(L) := Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n} \geq 0.$$

Then $W(u,t)$ is monotone decreasing along the heat equation

$$(\partial_t - L)u = 0.$$

That is

$$\frac{dW(u,t)}{dt} \leq 0.$$

A rigidity theorem for Perelman's W-entropy

Note that

$$\frac{d\mathcal{W}}{dt} = 0 \quad \Longleftrightarrow \quad \begin{cases} \nabla^2_{ij} f = \frac{g_{ij}}{2t}, \quad \forall i, j = 1, \ldots, n, \\ Ric_{m,n}(L)(\nabla f, \nabla f) = 0, \\ \nabla\phi \cdot \nabla f + \frac{m-n}{2t} = 0, \end{cases}$$

$$\Longrightarrow \quad \begin{cases} Ric_{m,n}(L)(\log u, \log u) = 0, \\ L \log u + \frac{m}{2t} = 0. \end{cases}$$

This is the case when

$$M = \mathbb{R}^n, \quad m = n, \quad \phi(x) = C, \quad u(x,t) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}.$$

Question

Can we prove a rigidity theorem for the W-entropy under the condition $Ric_{m,n}(L) \geq 0$ on n-dimensional complete Riemannian manifolds?

The following result gives an affirmative answer to this question.

Theorem (L. Math Ann2012)

Let (M, g) be an n-dimensional complete Riemannian manifold with bounded Riemannian curvature as well as its derivatives, and $\phi \in C_b^3(M)$. Suppose that there exist a constant $m \geq n$ and a point $o \in M$ such that

$$Ric_{m,n}(L) \geq 0.$$

Then $\frac{d\mathcal{W}}{dt} = 0$ for some $t > 0$ if and only if

$$M = \mathbb{R}^n, \quad m = n, \quad \phi(x) = C, \quad u(x, t) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}.$$

W-entropy for time dependent Witten Laplacian

Theorem (S.-Z. Li, L. 2012)

Let M be a compact manifold, $\{g(t), \phi(t), t \in [0, T]\}$ be such that

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\mathrm{Tr}\frac{\partial g}{\partial t}.$$

Let

$$u(x,t) = \frac{e^{-f(x,t)}}{(4\pi t)^{m/2}}$$

be the solution of the heat equation $\partial_t u = Lu$. Then

$$\frac{d\mathcal{W}(u,t)}{dt} = -2\int_M t\left[\left|\nabla^2 f - \frac{g}{2t}\right|^2 + \left(\frac{1}{2}\frac{\partial g}{\partial t} + Ric_{m,n}(L)\right)(\nabla f, \nabla f)\right] u d\mu$$
$$- \frac{2}{m-n}\int_M t\left(\nabla\phi \cdot \nabla f + \frac{m-n}{2t}\right)^2 u d\mu.$$

W-entropy formula on Perelman's super m-Ricci flow

Theorem (S.-Z. Li, L. 2012)

Let M be a compact manifold. Suppose that $\{g(t), \phi(t), t \in [0, T]\}$ satisfies

$$\begin{aligned} \frac{\partial g}{\partial t} &\geq -2\left(Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n}\right), \\ \frac{\partial \phi}{\partial t} &= \frac{1}{2}\mathrm{Tr}\frac{\partial g}{\partial t}. \end{aligned}$$

Let $u(x, t) = \frac{e^{-f(x,t)}}{(4\pi t)^{m/2}}$ be the solution of the heat equation $\partial_t u = Lu$. Then

$$\frac{dW(u, t)}{dt} \leq 0.$$

Remark: We can further extend the above results to complete Riemannian manifolds with bounded geometry condition, and prove a rigidity theorem for the W-entropy on complete Riemannian manifold with time dependent Witten Laplacian.

Open problem

In his statistical interpretation of W-entropy, Perelman assumed that there exists a canonical ensemble with a "density of states" measure such that its partition function is given by

$$\begin{aligned}\log Z_\beta &= \int_M \left(\frac{n}{2} - f\right) dm \\ &= \frac{n}{2}(1 + \log(4\pi\tau)) + \int_M u \log u dv \\ &= -H_n(u, \tau),\end{aligned}$$

where $g(\tau)$ satisfies the modified forward Ricci flow equation

$$\partial_t g = 2(Ric + \nabla^2 f),$$

and

$$dm = \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv.$$

Open problem

The following problem is naturally arising from Perelman's statistical interpretation of the W-entropy for Ricci flow.

Problem

Is there a canonical ensemble with a density of state measure $g(E)dE$ whose Bolztmann S-entropy is indeed the negative Perelman's W-entropy? If such a canonical ensemble exists, how to explicitly construct it?

This problem is closely related to the constructive conformal field theory (CFT). If such a canonical ensemble exists, it will naturally indicate which Riemannian metric is the canonical metric.

Open problem

Suppose that there is a certain canonical ensemble Ω and an energy function $E : \Omega \to \mathbb{R}$ such that the density of state measure $g(E)dE$ exists and satisfies

$$\log \int_{\mathbb{R}} e^{-\beta E} g(E) dE = \frac{n}{2}(1 + \log(4\pi\tau)) + \int_M u \log u dv.$$

The left hand side is the logarithmic Laplace transform of the density of state measure $g(E)dE$.

Observation

Under some technical conditions which need to be verified, if such $g(E)dE$ exists, then by the analytic extension of the partition function

$$\beta \to Z(\beta) := \int_{\mathbb{R}} e^{-\beta E} g(E) dE,$$

and using the inverse Fourier transformation, we have (Li arxiv2013)

$$g(E) = \frac{1}{2\pi i} \int_{\mathrm{Re}\beta - \sqrt{-1}\infty}^{\mathrm{Re}\beta + \sqrt{-1}\infty} e^{\beta E} Z(\beta) d\beta,$$

where $\beta = \mathrm{Re}\beta + \sqrt{-1}\mathrm{Im}\beta$, and for $\tau = \beta^{-1}$

$$\log Z(\beta) = \frac{n}{2}(1 + \log(4\pi\tau)) + \int_M u(\tau) \log u(\tau) dvol_{g(\tau)}.$$

To complete the above program, the first step is to prove that the solution of the Ricci flow and the conjugate equation

$$\begin{aligned}
\partial_\tau g(\tau) &= 2Ric_{g(\tau)}, \\
\partial_\tau u(\tau) &= \Delta_{g(\tau)} u(\tau) + R_{g(\tau)} u(\tau)
\end{aligned}$$

admit a unique analytic continuation from $\tau \in [0, T)$ to $\tau \in \mathbb{C}$. Then to prove the integral

$$g(E) = \frac{1}{2\pi i} \int_{\mathrm{Re}\beta - \sqrt{-1}\infty}^{\mathrm{Re}\beta + \sqrt{-1}\infty} e^{\beta E} Z(\beta) d\beta,$$

is convergent.

The difficulty for proving the existence is due to the nonlinearity of the Ricci flow. However, the above discussion proves the uniqueness of the density of state measure $g(E)dE$ whose Boltzmann S-entropy is the negative Perelman's W-entropy for Ricci flow.

Perelman's W-entropy and LDP via CFT

To end this talk, let us mention that, there is (or there might be) some deep connection between Perelman's W-entropy and the Sanov type theorem in the large deviation theory via the conformal field theory (CFT). We will develop this in future.

Thank you !