Gaussian Processes and Intrinsic Volumes

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Notations

1. H is a Hilbert space
2. K is a convex set in $\mathbb{R}^n$ or in H
3. B_k is the unit ball of dimension k
4. $\kappa_k = \mathrm{Vol}_k(B_K)$
5. Minkowski sum: $A + B = \{x + y \,|\, x \in A, y \in B\}$

Intrinsic volumes in $\mathbb{R}^n$

Steiner's formula

$$\mathrm{Vol}_n(K + rB_n) = \sum_{k=0}^{n} r^{n-k} \kappa_{n-k} V_i(K), \quad r > 0.$$

Alternative definition

$$V_k(K) = \frac{\binom{n}{k}\kappa_n}{\kappa_k \kappa_{n-k}} \int_{L_k^n} \mathrm{Vol}_k(K|L)\, \mu_k(dL).$$

In particular, $V_0 = 1$, $V_n = \mathrm{Vol}$, $2V_{n-1} =$ the surface area, $(2\kappa_{n-1}/n\kappa_n)V_1 =$ the mean width.

Intrinsic volumes in Hilbert space

Definition (Sudakov '71, Chevet '76)

The supremum of the intrinsic volumes of inscribed finitely dimensional convex sets.

Intrinsic volumes and Gaussian Processes

Theorem (Sudakov '71)

Let $X_t, t \in T$, be a centered Gaussian process and $K = \mathrm{conv}\{X_t \in H \mid t \in T\}$. Then

$$V_1(K) = \sqrt{2\pi}\,\mathbb{E} \sup_{t \in T} X_t.$$

Theorem (Tsirelson '85)

Let $X_t^1, \ldots, X_t^k$ be independent copies of X_t. Then

$$V_k(K) = \frac{(2\pi)^{k/2}}{k!\kappa_k} \mathbb{E}\,\mathrm{Vol}_k(\mathrm{conv}\{(X_t^1, \ldots, X_t^k) \in \mathbb{R}^k | t \in T\})$$

Finite dimension

Sudakov-Tsirelson's result in Euclidian space

Let $X, X_1, \dots, X_k \in \mathbb{R}^n$ be independent standard Gaussian vectors. For any convex $K \subset \mathbb{R}^n$

$$V_k(K) = \frac{(2\pi)^{k/2}}{k!\kappa_k} \mathbb{E}\, \mathrm{Vol}_k((\langle X_1, x\rangle, \dots, \langle X_n, x\rangle)|x \in K\})$$

In particular,

$$V_1(K) = \sqrt{2\pi}\, \mathbb{E} \sup_{x \in K} \langle X, x\rangle.$$

Mean width of a regular simplex

Theorem (Schneider '92)

Let $e_1, \dots, e_n$ be an orthonormal basis in $\mathbb{R}^n$. Let K be a regular simplex:

$$K = \mathrm{conv}\{e_1, \dots, e_n\}.$$

Then

$$V_1(K) = \sqrt{2\pi}\sqrt{2\ln n} \cdot (1 + o(1)), \quad n \to \infty.$$

Theorem (very old)

Let $\xi_1, \dots, \xi_n \in \mathbb{R}^1$ be independent standard Gaussian variables. Then

$$\mathbb{E} \max_{i=1,\dots,n} \xi_i = \sqrt{2\ln n} \cdot (1 + o(1)), \quad n \to \infty.$$

Gaussian polytope

Definition

Let $X_1, \ldots, X_n \in \mathbb{R}^k$ be independent standard Gaussian vectors. Their convex hull is called a Gaussian polytope:

$$P_{k,n} = \mathrm{conv}\{X_1, \ldots, X_n\}.$$

Theorem (Affentranger '91)

$$\mathbb{E}\mathrm{Vol}_k(P_{k,n}) = \kappa_k (\ln n)^{k/2} \cdot (1 + o(1)), \quad , n \to \infty.$$

Corollary

If K is a regular simplex in $\mathbb{R}^n$, then

$$V_k(K) = \frac{(2\pi)^{k/2}}{k!} (\ln n)^{k/2} \cdot (1 + o(1)), \quad , n \to \infty.$$

Regular crosspolytope

Definition

Let $X_1, \dots, X_n \in \mathbb{R}^k$ be independent standard Gaussian vectors. Their symmetrized convex hull is called a symmetric Gaussian polytope:

$$P^s_{k,n} = \mathrm{conv}\{\pm X_1, \dots, \pm X_n\}.$$

Theorem (Hug, Munsonius, Reitzner '04)

$$\mathbb{E}\mathrm{Vol}_k(P^s_{k,n}) = C_{k,n}(\ln n)^{k/2} \cdot (1 + o(1)), \quad n \to \infty.$$

Corollary (Finch '11 for k=1)

If K is a regular crosspolytope in $\mathbb{R}^n$ defined by $\mathrm{conv}\{\pm e_1, \dots, \pm e_n\}$, then

$$V_k(K) = C'_{k,n}(\ln n)^{k/2} \cdot (1 + o(1)), \quad n \to \infty.$$

Wiener spiral

Definition (Kolmogorov '40)

Let $H = L([0,1])$. Wiener spiral is a curve in H defined as

$$w = \{1_{[0,t]} \in H \mid t \in [0,1]\}.$$

Theorem (Gao, Vitale '03)

$$V_k(\mathrm{conv}(w)) = \frac{\kappa_k}{k!}.$$

Theorem (Eldan, '12)

Let W_t, $t \in [0,1]$, be a Brownian motion in R^k. Then

$$\mathbb{E}\,\mathrm{Vol}_k(\mathrm{conv}\{W_t, t \in [0,1]\}) = \left(\frac{\pi}{2}\right)^{k/2} \frac{1}{\Gamma(k/2+1)^2}.$$

Spiral for Brownian bridge

Notation

Consider a curve in $H = L([0,1])$

$$b = \{1_{[0,t]} - t \in H \mid t \in [0,1]\}.$$

Theorem (Gao '03)

$$V_k(\mathrm{conv}(b)) = \frac{2\kappa_k^2}{k!\kappa_{k+1}}.$$

Corollary (Randon-Furling, Majumdar, Comtet '09 for k=2)

Let $B_t, t \in [0,1]$, be a Brownian bridge in $\mathbb{R}^k$. Then

$$\mathbb{E}\,\mathrm{Vol}_k(\mathrm{conv}\{B_t, t \in [0,1]\}) = \frac{2\kappa_k^3}{(2\pi)^{k/2}\kappa_{k+1}}.$$

Final slide

THANK YOU!