Discrete Fractal Dimensions of the Ranges of Random Walks Associate with Random Conductances

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Russian-Chinese Seminar on Asymptotic Methods in Probability Theory and Mathematical Statistics, June 2013

Based on Joint Work with Yimin Xiao

Outline

The Random Conductance Model

- $\mathbb{Z}^d$ = d-dimension integer lattice; E_d = {non-oriented nearest neighbor bonds}
- **Environment**: for a given distribution $\mathbb{Q}$ on $[0, \infty)$,

$$\mu_e \sim_{i.i.d.} \mathbb{Q}, \qquad \text{for all } e \in E_d;$$

- Given a realization $\omega = \{\mu_e : e \in E_d\}$, two random walks:
  1. Variable speed random walk (VSRW), (X_t), waits at x for an exponential time with mean $1/\mu_x$;
  2. Constant speed random walk (CSRW), (Y_t), waits at x for an exponential time with mean 1;

  and then jumps to a neighboring site y with probability

$$P_{xy}(\omega) = \frac{\mu_{xy}}{\mu_x} \quad \text{where } \mu_x = \sum_{y \sim x} \mu_{xy}.$$

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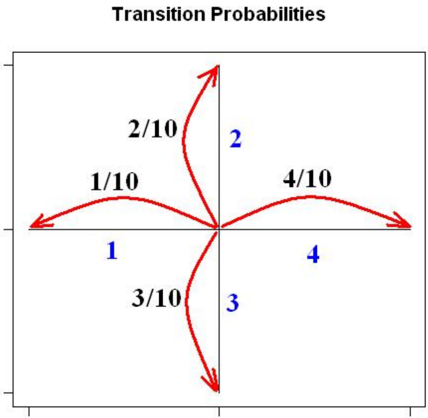
Transition Probabilities



Examples

Eg 1:

- $\mathbb{Q} = \delta_{\{1\}}$, then μ_e are constantly 1, and Y_t is just the usual nearest neighbor random walk
- Functional CLT (FCLT):

$$\frac{Y_{nt}}{\sqrt{n}} \Rightarrow B_t.$$

Eg 2:

- $\mathbb{Q} =$ Bernoulli(p), then Y_t is a simple random walk on the connected component of percolation

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Two laws

- Two laws:
  1. **Quenched Law**: For any given realization ω, study the law P_ω of $(X_t)/(Y_t)$ under this realization
  2. **Averaged (or Annealed) Law**: the law by taking expectation of the quenched law P_ω w.r.t. $\mathbb{P}$
- Focus on quenched law P_ω
- Basic Questions: the long run behavior of $(X_t)/(Y_t)$, e.g.,
  1. does the quenched FCLT (QFCLT) hold?
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QFCLT

- [Barlow and Deuschel(2010)] For the VSRW X, when $d \geq 2$, for $\mathbb{P}$-a.a. ω, under P_0^ω, $X_{n^2 t}/n \Rightarrow \sigma_V B_t$, where σ_V is non-random, and B_t is a standard d-dimensional Brownian-motion.
- [Barlow and Deuschel(2010)] For the CSRW Y, when $d \geq 2$, for $\mathbb{P}$-a.a. ω, under P_0^ω, $Y_{n^2 t}/n \Rightarrow \sigma_C B_t$,
$$\text{where } \sigma_C = \begin{cases} \sigma_V/\sqrt{2d\mathbb{E}\mu_e}, & \text{if } \mathbb{E}\mu_e < \infty, \\ 0, & \text{if } \mathbb{E}\mu_e = \infty. \end{cases}$$
- [Barlow and Černý(2011)], [Černý(2011)] For the CSRW Y, when $d \geq 2$ and $\mathbb{Q}(\mu_e \geq u) \sim C/u^\alpha$ for some $\alpha \in (0,1)$, then for $\mathbb{P}$-a.a. ω, under P_0^ω, $Y_{n^{2/\alpha} t}/n$ converges to a multiple of the fractional kinetics process;
- [Barlow and Zheng(2010)] For the CSRW Y, when $d \geq 3$ and $\mathbb{Q}$ is Cauchy tailed, then for $\mathbb{P}$-a.a. ω, under P_0^ω, $Y_{n^2 (\log n)\, t}/n$ converges to a multiple of a d-dimensional Brownian-motion.

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Discrete Hausdorff Dimension

- For any $n \in \mathbb{N}$, let $V_n = V(0, 2^n)$ be the cube of side length 2^n centered at $0 \in \mathbb{Z}^d$, and $S_n := V_n \setminus V_{n-1}$
- For any set $B \subseteq \mathbb{Z}^d$, let $s(B)$ be its side length
- [Barlow and Taylor(1992)] For any measure function h and any set $A \subseteq \mathbb{Z}^d$, the **discrete Hausdorff measure** of A w.r.t h is

$$m_h(A) = \sum_{n=1}^{\infty} \nu_h(A, S_n).$$

where

$$\nu_h(A, S_n) = \min\left\{ \sum_{i=1}^{k} h\left(\frac{s(B_i)}{2^n}\right) : A \cap S_n \subset \bigcup_{i=1}^{k} B_i \right\}.$$

- For $\alpha > 0$, define $h(r) = r^{\alpha}$, and let $m_\alpha(A) = m_h(A)$. Then the **discrete Hausdorff dimension** of A is given by

$$\dim_{\mathrm{H}} A = \inf\left\{ \alpha > 0 : m_\alpha(A) < \infty \right\}$$

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Discrete Packing Dimension

- [Barlow and Taylor(1992)] For any measure function h, $\varepsilon > 0$, and any set $A \subseteq \mathbb{Z}^d$, the **discrete packing measure** of A w.r.t h is

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Discrete Dimensions of the Range of RCM

Theorem

[Xiao and Zheng(2013)] Let

$$\mathrm{R} = \{x \in \mathbb{Z}^d : X_t = x \text{ for some } t \geq 0\}$$

be the range of VSRW X (as well as that of CSRW Y). Assume that $d \geq 2$ and $\mathbb{Q}(\mu_e \geq 1) = 1$. Then for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\dim_{\mathrm{H}} \mathrm{R} = \dim_{\mathrm{P}} \mathrm{R} = 2, \quad \mathrm{P}_0^\omega\text{-a.s.}.$$

where $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{P}}$ denote respectively the discrete Hausdorff and packing dimension.

Recurrent/Transient Sets for RCM

Theorem

[Xiao and Zheng(2013)] Assume that $d \geq 3$ and $\mathbb{P}(\mu_e \geq 1) = 1$. Let $A \subset \mathbb{Z}^d$ be any (infinite) set. Then for $\mathbb{P}$-almost every $\omega \in \Omega$, the following statements hold.

(i) *If $\dim_{\mathrm{H}} A < d - 2$, then*

$$\mathrm{P}_0^\omega \big(X_t \in A \ \text{ for arbitrarily large } t > 0 \big) = 0.$$

(ii) *If $\dim_{\mathrm{H}} A > d - 2$, then*

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Remark

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- Basic idea: derive various estimates for ordinary random walks used in [Barlow and Taylor(1992)], by using general Markov chain techniques
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  1. Gaussian heat kernel bounds for the VSRW ([Barlow and Deuschel(2010)]);
  2. Hitting probability estimates;
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- $\dim_{\mathrm{P}} \mathrm{R} \le 2$ P_0^ω-a.s.: first moment argument;
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Thank you!

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